Section 13.1

Vector Functions and Space Curves

Goals:

- Graph certain plane curves.
- Compute limits and verify the continuity of vector functions.

Equation of a Line

The equation of a line was our first example of a vector valued function. For example

$$\mathbf{r}(t) = \langle 3 - 2t, 5 + t, 2 + 4t \rangle .$$

Observe

1. The input of this function is a number: t.
2. The output of this function is a vector: $\mathbf{r}(t)$.
3. $\mathbf{r}$ is really defined by three real-valued functions:

$$x = 3 - 2t \qquad y = 5 + t \qquad z = 2 + 4t.$$

Vector Valued Functions

Definition

A general vector valued function $\mathbf{r}(t)$ has a number as an input and a vector of some fixed dimension as its output.
If the outputs are two-dimensional, then there are **component functions** $f(t)$ and $g(t)$ such that

$$\mathbf{r}(t) = \langle f(t), g(t) \rangle$$

or

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}.$$

The **domain** of $\mathbf{r}$ is the set of all t for which both component functions are defined.

The Plane Curve Associated to a Vector Function

If we view the outputs of a vector function

$$\mathbf{r}(t) = \langle f(t), g(t) \rangle$$

as position vectors, then the points they define trace out a shape in two-dimensional space.

Definition

A **plane curve** is the set of points defined by two **parametric equations**

$$x = f(t) \qquad y = g(t).$$

The variable t is called the **parameter**. We might restrict t to some interval, or let t range over the entire real number line.

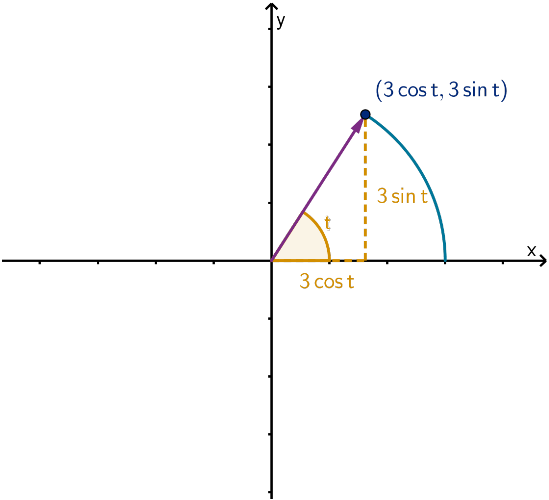
Example 1

Graph the plane curves associated to the following vector functions:

1. $\mathbf{r}(t) = \langle 4 + 2t, 3 - 3t \rangle$
2. $\mathbf{r}(t) = \langle 4 + 2t, 3 - 3t \rangle \qquad 0 \leq t \leq 1$
3. $\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t \rangle$
4. $\mathbf{r}(t) = \left\langle t, t^3 \right\rangle$

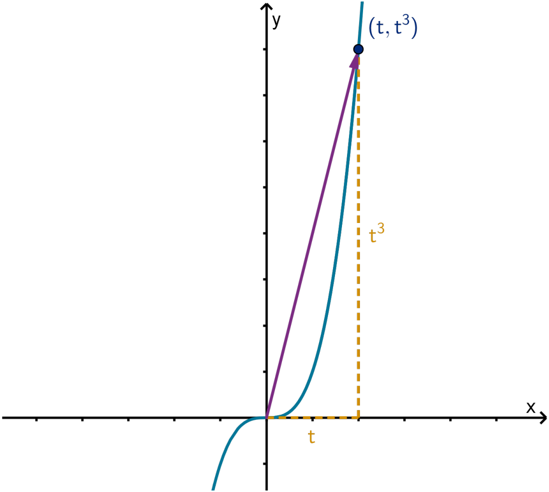
Example 1

3. $\mathbf{r}(t) = \langle 3\cos t, 3\sin t \rangle$



Example 1

4. $\mathbf{r}(t) = \langle t, t^3 \rangle$



Exercises

1. Sketch the plane curve of $\mathbf{r}(t) = (3+t)\mathbf{i} + (5-4t)\mathbf{j} \quad 0 \leq t \leq 1$.

2. Sketch the plane curve of $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)\rangle \quad 0 \leq t \leq 2\pi$.

3. How would $\mathbf{r}(t) = \langle 2\cos(t), 2\sin(t)+4\rangle \quad 0 \leq t \leq 2\pi$ differ from 2? Plot some points if you need to.

4. How would $\mathbf{r}(t) = \langle 6\cos(t), 2\sin(t)\rangle \quad 0 \leq t \leq 2\pi$ differ from 2? Does this plane curve have a shape you recognize?

5. What graph is defined by $\mathbf{r}(t) = (t^3 - 4t)\mathbf{i} + t\mathbf{j}$?

The Space Curve Associated to a Vector Function

If the outputs of $\mathbf{r}(t)$ are three-dimensional vectors then we have three component functions and $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$.

Definition

A **space curve** is the set of points defined by three **parametric equations**

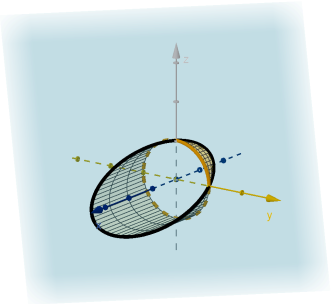
$$x = f(t) \qquad y = g(t) \qquad z = h(t).$$

Visualizing Space Curves

The space curve defined by

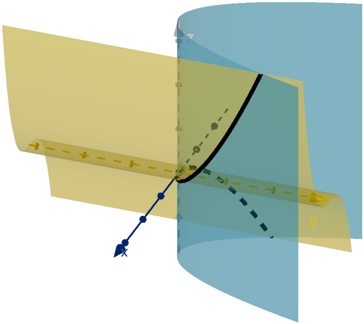
$$\mathbf{r}(t) = (1 - \cos(t) - \sin(t))\mathbf{i} + \cos(t)\mathbf{j} + \sin(t)\mathbf{k}$$

is best understood by projecting a yz unit circle onto a plane $x = 1 - y - z$.



Visualizing Space Curves

The space curve defined by $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ can be understood as the intersection of two surfaces:

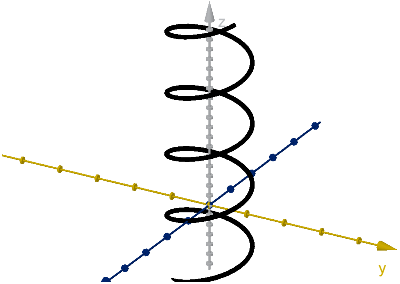


Visualizing Space Curves

The space curve defined by

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + \frac{t}{4}\mathbf{k}$$

can be understood by a projectile motion argument.



Limits

Definition

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ then

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

Provided the limits of all three component functions exist.

Continuity

Definition

A vector function $\mathbf{r}$ is **continuous** at a if

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a).$$

This is the case if and only if the component functions $f(t), g(t)$ and $h(t)$ are continuous at a.

Example 2

Is

$$\mathbf{r}(t) = t^2\mathbf{i} + e^t\mathbf{j} + \frac{\sin t}{t}\mathbf{k}$$

continuous at $t = 0$? Justify your answer using the definition of continuity.

Summary Questions

- What is the difference between a vector function and a plane or space curve?
- Describe 4 different types of plane curves you'll be expected to know.

Section 13.2

Derivatives and Integrals of Vector Functions

Goals:

- Compute derivatives and integrals of vector functions.
- Calculate tangent vectors and tangent lines.
- Interpret derivatives as velocity and acceleration.

The Derivative of a Vector Function

Definition

We define the **derivative** of $\mathbf{r}(t)$ by

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

Notice since the numerator is a vector and the denominator is a scalar, we are taking the limit of a vector function.

Computing the Derivative

If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ then what is $\mathbf{r}'(t)$?

Computing the Derivative

Theorem

If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ *then*

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j},$$

Provided these derivatives exist.

Similarly, if $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ *then*

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k},$$

Provided these derivatives exist.

Properties of the Derivative

The following properties follow from applying the derivative rules you learned in single-variable calculus to each component of a vector function.

Theorem

For any differentiable vector functions $u(t), v(t)$, differentiable real-valued function $f(t)$ and constant c we have

1. $(\mathbf{u}+\mathbf{v})' = \mathbf{u}' + \mathbf{v}'$
2. $(c\mathbf{u})' = c\mathbf{u}'$
3. $(f\mathbf{u})' = f'\mathbf{u} + f\mathbf{u}'$
4. $(\mathbf{u}\cdot\mathbf{v})' = \mathbf{u}'\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v}'$
5. $(\mathbf{u}\times\mathbf{v})' = \mathbf{u}'\times\mathbf{v} + \mathbf{u}\times\mathbf{v}'$

The Tangent Vector

Definition

The vector $\mathbf{r}'(t_0)$ is called a **tangent vector** to the curve defined by $\mathbf{r}(t)$. If $\mathbf{r}(t_0)$ defines the point P, then we call $\mathbf{r}'(t_0)$ the tangent vector at P.

We can construct the **unit tangent vector** at P by diving by the length of $\mathbf{r}(t)$. It is denoted $\mathbf{T}(t_0)$.

$$\mathbf{T}(t_0) = \frac{\mathbf{r}'(t_0)}{|\mathbf{r}'(t_0)|}$$

By replacing t_0 with a variable t, we can define the **derivative function** $\mathbf{r}'(t)$.

The Tangent Line and Linearization

Definition

The **tangent line** or **linearization** to $\mathbf{r}(t)$ at P is the line through $P = \mathbf{r}(t_0)$ in the direction of $\mathbf{r}'(t)$. Its equation is

$$\mathbf{L}(t) = \mathbf{r}(t_0) + \mathbf{r}'(t_0)(t - t_0).$$

Remarks:

1. $\mathbf{L}(t)$ is similar to other linearizations we've learned in calculus.
2. $\mathbf{L}(t)$ is a line. Its direction vector is $\mathbf{r}'(t_0)$.
3. Like other linearizations, if we plug in t close to t_0, then $\mathbf{L}(t)$ gives a good approximation of $\mathbf{r}(t)$.
4. We can make $\mathbf{L}(t)$ look more like the vector equation of a line by replacing $t - t_0$ with a parameter s.

$$\mathbf{L}(s) = \mathbf{r}(t_0) + s\mathbf{r}'(t_0).$$

Example 1

Let $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

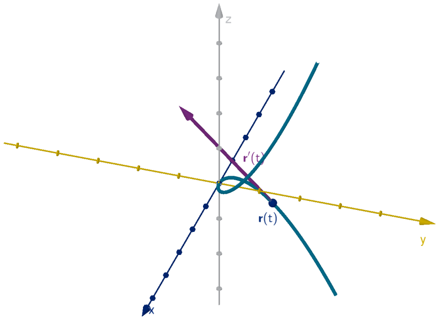
1. Compute $\mathbf{r}'(-1)$.

2. Compute $\mathbf{T}(-1)$.

3. Compute an equation of the tangent line to $\mathbf{r}(t)$ at $t = -1$.

Interpreting the Tangent Vector as a Velocity

If we imagine that $\mathbf{r}(t)$ describes the position of an object at time t, then $\mathbf{r}'(t)$ tells us the velocity (direction and magnitude) of the object.



The Definite Integral of a Vector Function

Definition

The **definite integral** from a to b of a vector function $\mathbf{r}(t)$ is denoted and defined:

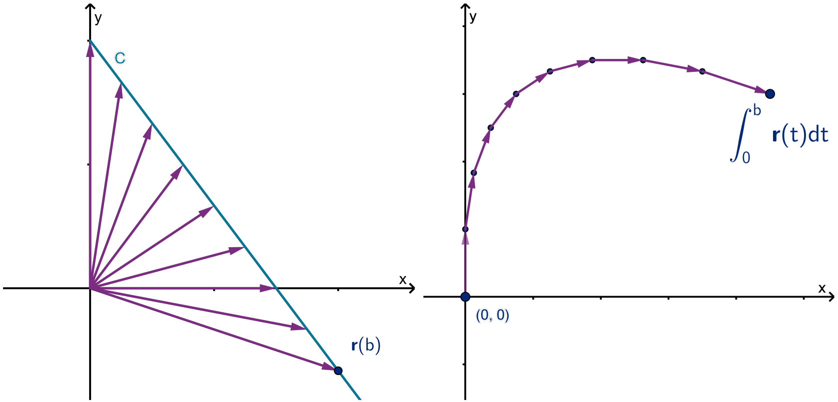
$$\int_a^b \mathbf{r}(t)dt = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{r}(t_i^*)\Delta t$$

Where $\Delta t = \frac{b-a}{n}$ and t_i^* is any representative of the i^{th} subinterval of $[a, b]$.

Besides the name of the function, this is identical to how we defined a real-valued integral.

Visualizing a Definite Integral

The vectors of $\mathbf{r}(t)$ are added together to estimate $\int_a^b \mathbf{r}(t)dt$. We can visualize $\int_a^b \mathbf{r}(t)dt$ as the change in position of a particle that has traveled with velocity $\mathbf{r}(t)$ from time $t = a$ until $t = b$.



Computing the Definite Integral

Since limits and sums can be broken down by components, we have the following formula:

Theorem

If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ *then*

$$\int_a^b \mathbf{r}(t)dt = \left(\int_a^b f(t)dt\right)\mathbf{i} + \left(\int_a^b g(t)dt\right)\mathbf{j} + \left(\int_a^b h(t)dt\right)\mathbf{k}$$

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus applies just as well to vector functions as real-valued functions.

Theorem

If $\mathbf{r}(t)$ *is continuous, then*

$$\frac{d}{dx}\left(\int_0^x \mathbf{r}(t)dt\right) = \mathbf{r}(x)$$

If $\mathbf{R}(t)$ *is an* **antiderivative** *of* $\mathbf{r}(t)$ *then*

$$\int_a^b \mathbf{r}(t)dt = \mathbf{R}(b) - \mathbf{R}(a).$$

Example 2 - Physics Application

Suppose we are given that the velocity of a particle at time t is given by $\mathbf{v}(t) = 2t\mathbf{i} + t^2\mathbf{j}$. If the particle is at the position $(2, -3)$ at $t = 0$, what is its position at $t = 3$?

Conclusions from “Trigonometric Space Curves”

Summary Questions

- How do you interpret the derivatives of a vector function in terms of motion?
- What is the relationship between a tangent vector, a unit tangent vector and a tangent line?
- What type of object is $\int \mathbf{r}(t)dt$? What type of object is $\int_a^b \mathbf{r}(t)dt$?