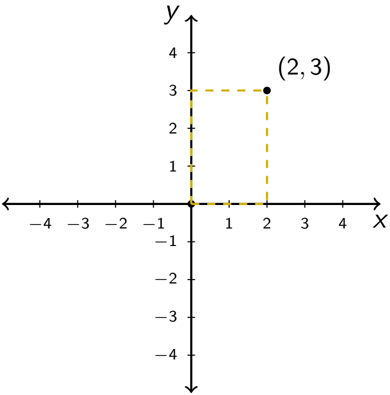
Section 12.1

Three-Dimensional Coordinate Systems

Goals:

- Plot points in a three-dimensional coordinate system.
- Use the distance formula.
- Recognize the equation of a sphere and find its radius and center.
- Graph an implicit function with a free variable.

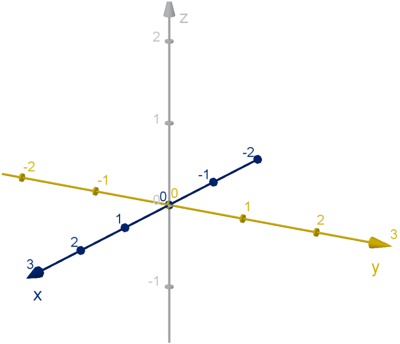
Key Observations from Two-Dimensional Space



1. Assign origin and two directions (x, y).
2. y is 90 degrees anticlockwise from x.
3. Axes consist of the points displaced in only one direction.
4. Coordinates refer to displacement from the origin in each direction.
5. Either displacement can happen first.
6. The possible coordinates are in bijection with the points in the plane.

Directions and Axes in Three-Dimensional Space (Three-Space)

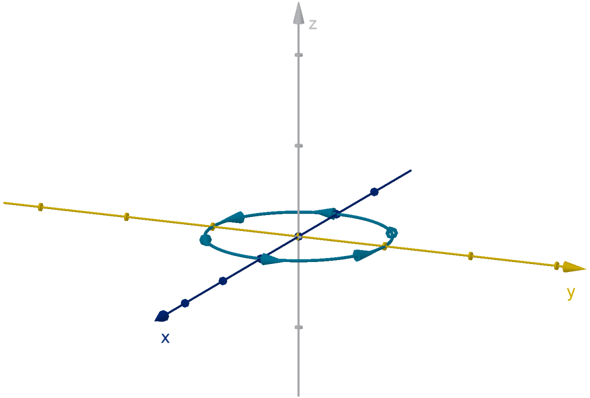
In a three-dimensional Cartesian coordinate system. We can extrapolate from two dimensions.



1. Assign origin and two three directions (x, y, z).
2. Each axis makes a 90 degree angle with the other two.
3. The z direction is determined by the right-hand rule.

The Right-Hand Rule

The right hand rule says that if you make the fingers of your right hand follow the (counterclockwise) unit circle in the *xy*-plane, then your thumb indicates the direction of the positive *z*-axis.



Drawing a Location in Three-Dimensional Coordinates

The point $(2, 3, 5)$ is the point displaced from the origin by

- 2 in the x direction
- 3 in the y direction
- 5 in the z direction.

How do we draw a reasonable diagram of where this point lies?

Negative Coordinate Values

How can we draw a reasonable diagram of $(-5, 1, -4)$?

Exercise

Draw diagrams of points with the following coordinates.

1. $(6, 1, 2)$

2. $(-3, 0, 0)$

3. $(2, -1, 4)$

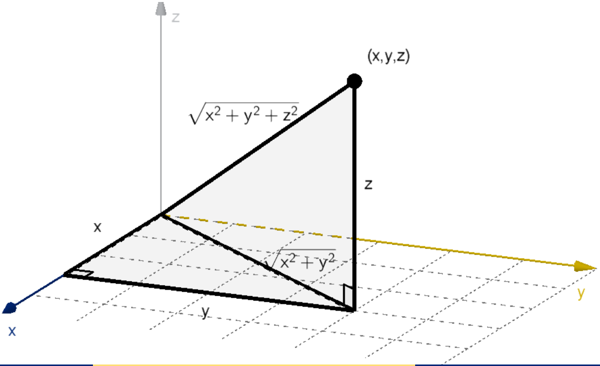
4. $(0, 3, 5)$

Distance Formula in Three-Space

Theorem

The distance from the origin to the point (x, y, z) is given by the Pythagorean Theorem

$$D = \sqrt{x^2 + y^2 + z^2}$$



General Formula

Theorem

The distance from the point (x_1, y_1, z_1) *to the point* (x_2, y_2, z_2) *is given by*

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Graphs in 3 Dimensions

Definition

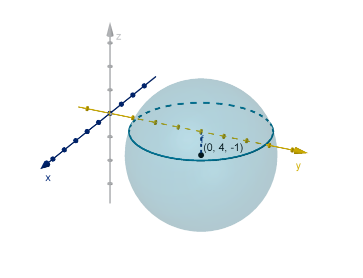
The **graph** of an implicit equation is the set of points whose coordinates satisfy that equation. In other words, the two sides are equal when we plug the coordinates in for x, y and z.

Example 1

The graph of

$$x^2 + (y-4)^2 + (z+1)^2 = 9$$

is the set of points that are distance 3 from the point $(0, 4, -1)$



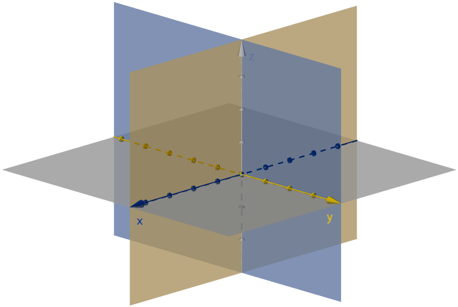
Example 2

Sketch the graph of the equation $y = 3$.

Coordinate Planes

In addition to coordinate axes, 3 dimensional space has 3 coordinate planes.

1. The graph of $z = 0$ is the xy-plane.
2. The graph of $x = 0$ is the yz-plane.
3. The graph of $y = 0$ is the xz-plane.

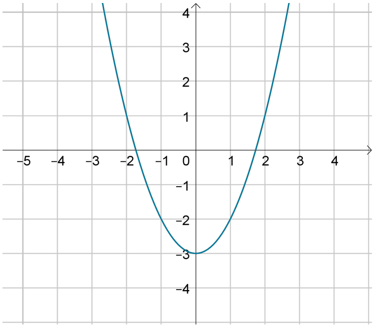


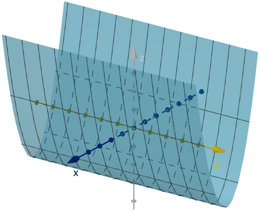
Example 3 - Free Variable Method

Sketch the graph of the equation $z = x^2 - 3$.

Implicit Equations

Notice that the graph of an implicit equation in the plane is generally one-dimensional (a curve), whereas the graph of an implicit equation in three-space is generally two-dimensional (a surface).





$y = x^2 - 3$

$z = x^2 - 3$

Exercise

Sketch the graph of each equation.

1. $x = -4$
2. $x^2 + y^2 = 9$
3. $x^2 + 4x + y^2 + z^2 - 2z = 4$

Summary Questions

- What is the right hand rule and what does it tell you about a three-dimensional coordinate system?
- In three-space, what is the y-axis? What are the coordinates of a general point on it?
- In three space, what is the xz plane? What are the coordinates of a general point on it? What is its equation?
- How do we use a free variable to sketch a graph?
- How do we recognize the equation of a sphere?

Section 12.2

Vectors

Goals:

- Distinguish vectors from scalars (real numbers) and points.
- Add and subtract vectors, multiply by scalars.
- Express real world vectors in terms of their components.

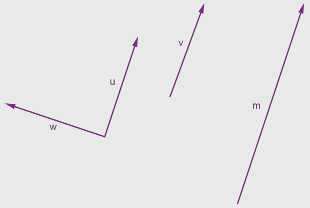
What is a Vector?

Definition

A **vector** in the plane or in three-space consists of a magnitude (length) and a direction. Two vectors with the same direction are **parallel**. Two vectors with the same magnitude and the same direction are **equal**.

Example

Here are four vectors represented by arrows. Two of them are equal.



Examples of Vectors

Here are some vectors

- 3 miles south
- The force that a magnetic field applies to a charged particle
- The velocity of an airplane

Here are some non-vectors

- 17
- The mass of an automobile
- 3:15 PM
- Atlanta, GA

Endpoint Notation

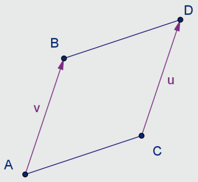
The vector **v** from point A to point B can be represented by the notation

$$\overrightarrow{AB}.$$

A is the **initial point** and B is the **terminal point**.

Theorem

$\overrightarrow{AB} = \overrightarrow{CD}$ if and only if $ABDC$ is a parallelogram (perhaps a squished one).



Coordinate Notation

We can represent a vector in Cartesian 3-space by the x, y and z components of its displacement. If $A = (2, 3, 7)$ and $B = (5, 3, 6)$ then we can represent

$$\overrightarrow{AB} = \langle 3, 0, -1 \rangle$$

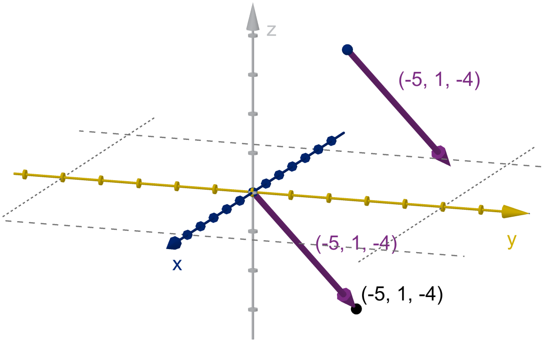
Theorem

$\mathbf{v} = \mathbf{u}$ *if and only if their coordinate representations match in each component.*

A vector in the Cartesian plane only has two components.

The Position Vector

Every point in a Cartesian coordinate system has a **position vector**, which gives the displacement of that point from the origin. The components of the vector are simply the coordinates of the point.

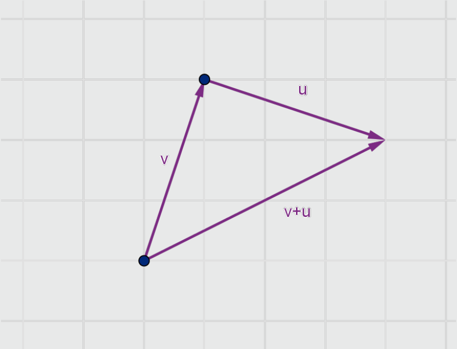


There is only one point $(-5, 1, -4)$ but there are many vectors $\langle -5, 1, -4 \rangle$.

Vector Sums

Definition

The **sum** of two vectors $\mathbf{v} + \mathbf{u}$ is calculated by positioning $\mathbf{v}$ and $\mathbf{u}$ head to tail. The sum is the vector from the initial point of one to the terminal point of the other. In coordinate notation, we just add each component numerically.



$$\begin{array}{r} \langle\ \ 1, \quad 3\rangle \\ +\langle\ \ 3, \ -1\rangle \\ \hline \langle\ \ 4, \quad 2\rangle \end{array}$$

Scalar Multiples

Definition

Given a number (called a scalar) λ and a vector $\mathbf{v}$ we can produce the **scalar multiple** $\lambda\mathbf{v}$, which is the vector in the same direction as $\mathbf{v}$ but λ times as long.

If λ is negative then $\lambda\mathbf{v}$ extends in the opposite direction. Either way, we say $\lambda\mathbf{v}$ is **parallel** to $\mathbf{v}$.

In coordinates scalar multiplication is distributed to each component.

$$2.5\,\langle 6, 4\rangle = \langle 15, 10\rangle$$

Example 1

Given diagrams of two vectors $\mathbf{u}$ and $\mathbf{v}$, how would we calculate $\frac{1}{2}\mathbf{u} + \mathbf{v}$?

What if we are instead given the coordinates $\mathbf{u} = \langle a, b \rangle$ and $\mathbf{v} = \langle c, d \rangle$?

Exercise

Given diagrams of two vectors $\mathbf{u}$ and $\mathbf{v}$, how would we draw $\mathbf{u} - \mathbf{v}$? What it its significance?

Standard Basis Notation

We can represent any vector in the plane or 3-space as a sum of scalar multiples of the following **standard basis vectors**

Plane

$\mathbf{i} = \langle 1, 0 \rangle$
$\mathbf{j} = \langle 0, 1 \rangle$

3-Space

$\mathbf{i} = \langle 1, 0, 0 \rangle$
$\mathbf{j} = \langle 0, 1, 0 \rangle$
$\mathbf{k} = \langle 0, 0, 1 \rangle$

The vector $\langle 3, 5, -2 \rangle$ can be written as $3\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$. You can check yourself that the sum on the right gives the correct vector.

The Length of a Vector

The **length** or **magnitude** of a vector is calculated using the distance formula and notated $|\mathbf{v}|$. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then

$$|\mathbf{v}| = \sqrt{a^2 + b^2 + c^2}$$

Example 2

If $\mathbf{v} = \langle 3, 5, -2 \rangle$ calculate $|\mathbf{v}|$

Unit Vectors

A **unit vector** is a vector of length 1. Given a vector $\mathbf{v}$ the scalar multiple

$$\frac{1}{|\mathbf{v}|}\mathbf{v}$$

is a unit vector parallel to $\mathbf{v}$.

Angles Between Vectors

Angles are a good way of comparing directions. In general, two vectors will not intersect to form an angle, so we use the following definition:

Definition

The angle between two vectors is the angle they make when they are placed so their initial points are the same.

If they make a right angle, we call them **orthogonal**. If they make an angle of 0 or π, they are parallel.

Note that there is no good way to measure clockwise in 3 or more dimensions, so the angle between two vectors is never negative, nor more than π.

Summary Questions

- How is a vector similar to a point? To a number?
- How is a vector different from a point? From a number?
- How can you tell if two vectors point in the same direction? Opposite directions?
- If $\mathbf{u}$ and $\mathbf{v}$ are position vectors of the points P and Q, how are $\mathbf{u}$ and $\mathbf{v}$ related to $\overrightarrow{PQ}$?

Section 12.3

The Dot Product

Goals:

- Calculate the dot product of two vectors.
- Determine the geometric relationship between two vectors based on their dot product.
- Calculate vector and scalar projections of one vector onto another.

Definition of the Dot Product

Definition

The **dot product** of two vectors is a number.
For two dimensional vectors $\mathbf{v} = \langle v_1, v_2 \rangle$ and $\mathbf{u} = \langle u_1, u_2 \rangle$ we define

$$\mathbf{v} \cdot \mathbf{u} = v_1 u_1 + v_2 u_2$$

For three dimensional vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ we define

$$\mathbf{v} \cdot \mathbf{u} = v_1 u_1 + v_2 u_2 + v_3 u_3$$

Example 1

1. Calculate $\langle 2, 3, -1 \rangle \cdot \langle 4, 1, 5 \rangle$
2. Calculate $(-2\mathbf{i} + 4\mathbf{k}) \cdot (\mathbf{i} + 2\mathbf{j} - \mathbf{k})$

Questions

1. How does the dot product behave algebraically? Why is it called a “product?”
2. How does the dot product behave geometrically? Does knowing the dot product of two vectors tell us anything about them?

Exercise

Let $\mathbf{u} = \langle 2, 3 \rangle$, $\mathbf{v} = \langle 4, -1 \rangle$ and $\mathbf{w} = \langle -5, 2 \rangle$.

1. Compute $\mathbf{u} \cdot \mathbf{u}$ and $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{w}$.
2. Compute $\mathbf{v} \cdot \mathbf{u}$. How does it compare to $\mathbf{u} \cdot \mathbf{v}$?
3. How is $\mathbf{u} \cdot \mathbf{u}$ related to $|\mathbf{u}|$?
4. Compute $3\mathbf{u}$ and $3\mathbf{v}$ then take their dot product. How is it related to $\mathbf{u} \cdot \mathbf{v}$?
5. Compute $\mathbf{v} + \mathbf{w}$ then compute $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w})$. How is it related to $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{w}$?
6. Why do you think we call this operation a “dot product” and not a “dot sum?”
7. If you wanted to prove that relationships your noticed in **2**–**5** work for all possible vectors, how would you do that?

Algebraic Properties of the Dot Product

The following properties hold for any vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ and scalars m and n.

Commutative $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

Distributive $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

Associative $m\mathbf{u} \cdot n\mathbf{v} = mn(\mathbf{u} \cdot \mathbf{v})$

Dot Products of Parallel Vectors

Theorem

If $\mathbf{u}$ *and* $\mathbf{v}$ *are parallel then*

$$\mathbf{u} \cdot \mathbf{v} = \begin{cases} |\mathbf{u}||\mathbf{v}| & \textit{if } \mathbf{u} \textit{ and } \mathbf{v} \textit{ have the same direction} \\ -|\mathbf{u}||\mathbf{v}| & \textit{if } \mathbf{u} \textit{ and } \mathbf{v} \textit{ have opposite directions} \end{cases}$$

Dot Products of Orthogonal Vectors

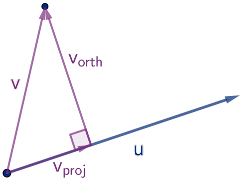
Theorem

If $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal then*

$$\mathbf{u} \cdot \mathbf{v} = 0.$$

Vector Projections and Scalar Projections

Two vectors need not be parallel or orthogonal, but given vectors **u** and **v** we can always write $\mathbf{v} = \mathbf{v}_{\text{proj}} + \mathbf{v}_{\text{orth}}$.



The properties of the dot product tell us that

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= \mathbf{u} \cdot (\mathbf{v}_{\text{proj}} + \mathbf{v}_{\text{orth}}) \\ &= \pm\, |\mathbf{u}||\mathbf{v}_{\text{proj}}| + 0\end{aligned}$$

Definition

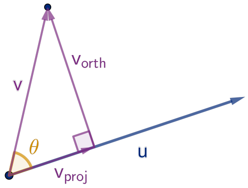
The number $\dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|}$ is called the **scalar projection** of **v** onto **u**.

The Cosine Formula

Theorem

Let $\mathbf{u}$ *and* $\mathbf{v}$ *have the same initial point and meet at angle* θ. *The following formula holds in any dimension:*

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta$$



Recall that $\cos\theta$ is

- positive when $\theta < \pi/2$
- negative when $\theta > \pi/2$
- zero when $\theta = \pi/2$.

So the sign of $\mathbf{u} \cdot \mathbf{v}$ tells us whether θ is acute, obtuse or right.

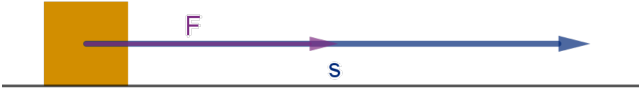
Example 2

What is the angle between $\langle 1, 0, 1 \rangle$ and $\langle 1, 1, 0 \rangle$?

Application - Work

In physics, we say a force **works** on an object if it moves the object in the direction of the force. Given a force F and a displacement d, the formula for work is:

$$W = Fs$$

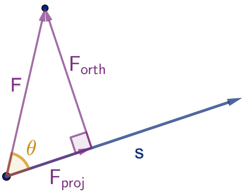


Work in More Dimensions

In higher dimensions, displacement and force are vectors.

If the force and the displacement are not in the same direction, then only $\mathbf{F}_{\text{proj}}$ contributes to work.

$$W = \mathbf{F}_{\text{proj}} \cdot \mathbf{s} = \mathbf{F} \cdot \mathbf{s}$$



Summary Questions

- What algebraic properties does a dot product share with real number multiplication?
- How is the angle between two vectors related to their dot product?
- What is a scalar projection, and how do you compute it?

Section 12.4

The Cross Product

Goals:

- Calculate the determinant of a 2×2 or 3×3 matrix.
- Calculate the cross product of two vectors.
- Understand the geometric relationship between two vectors and their cross product.

Matrices

Definition

A **matrix** is a rectangular array of values (usually numbers). An $m \times n$ matrix has m rows and n columns. If a matrix has the same number of rows and columns, it is **sqaure**.

Examples

a 2×4 matrix

$$\begin{bmatrix} 3 & 0 & 4 & -2 \\ 4 & 2 & 0 & 1 \end{bmatrix}$$

a 3×1 matrix

$$\begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}$$

a square 3×3 matrix

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 2 \\ 3 & 1 & 1 \end{bmatrix}$$

The Determinant of a Matrix

A **determinant** is a number that we can compute and associate to a square matrix. If the matrix has a name (like M), we use the notation $\det M$ or $|M|$. We can also write

$$\det \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 2 \\ 3 & 1 & 1 \end{bmatrix} \text{ or } \begin{vmatrix} 1 & 3 & 0 \\ 0 & 2 & 2 \\ 3 & 1 & 1 \end{vmatrix}$$

Computing the Determinant

The determinant of a 2×2 matrix is calculated by the formula

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The formulas for larger matrices are derived from those of smaller **minor** matrices.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

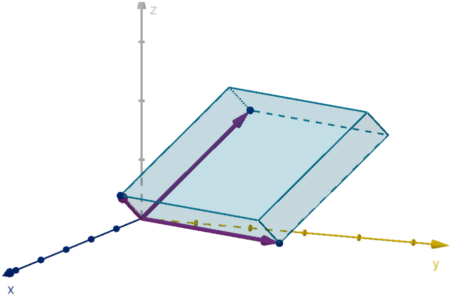
Example 1

Calculate $\begin{vmatrix} 1 & 3 & 0 \\ 0 & 2 & 2 \\ 3 & 1 & 1 \end{vmatrix}$

The Geometric Meaning of the Determinant

The absolute value of the determinant of a matrix is the volume of the **parallelepiped** constructed from the row (or column) vectors.

$$\begin{vmatrix} 1 & 3 & 0 \\ 0 & 2 & 2 \\ 3 & 1 & 1 \end{vmatrix} = 18$$



The Cross Product

Definition

The **cross product** is a product of three-dimensional vectors $\mathbf{u}$ and $\mathbf{v}$, whose output is also a three dimensional vector denoted

$$\mathbf{u} \times \mathbf{v}.$$

The cross product is defined as follows on the standard basis vectors:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i} \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k} \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i} \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$$

Notice that the cross product of two vectors is a vector, whereas the dot product is a number.

The Cross Product - Algebraic Definition

In order to finish defining the cross product, we need the following algebraic properties:

1. The cross product is associative with scalar multiplication:

$$(a\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (a\mathbf{v}) = a(\mathbf{u} \times \mathbf{v})$$

2. The cross product distributes across vector sums:

$$(\mathbf{u}_1 + \mathbf{u}_2) \times \mathbf{v} = \mathbf{u}_1 \times \mathbf{v} + \mathbf{u}_2 \times \mathbf{v}$$

$$\mathbf{u} \times (\mathbf{v}_1 + \mathbf{v}_2) = \mathbf{u} \times \mathbf{v}_1 + \mathbf{u} \times \mathbf{v}_2$$

Example 2

If $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$, compute $\mathbf{u} \times \mathbf{v}$.

The Determinant Formula

Formula

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}$$

If we're a bit sloppy and allow our matrix to have vectors as entries, we can write more compactly:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

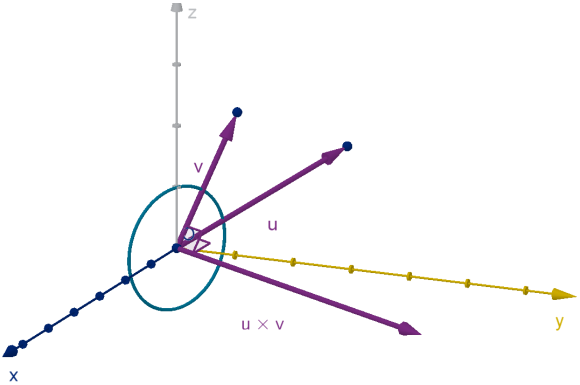
Example 3

Calculate $\langle 2, 0, 3 \rangle \times \langle 3, 1, 1 \rangle$.

Direction of the Cross Product

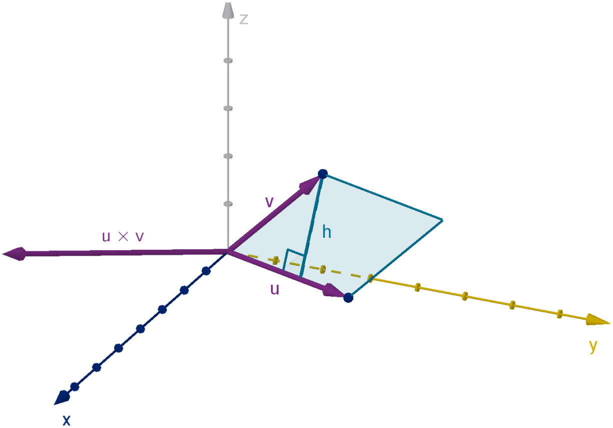
The direction of $\mathbf{u} \times \mathbf{v}$ is given by the following facts:

- $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
- If your right hand traces a circle from $\mathbf{u}$ through $\mathbf{v}$, then your thumb points in the direction of $\mathbf{u} \times \mathbf{v}$.



Magnitude of the Cross Product

- If θ is the angle between **u** and **v**, the length satisfies the formula
$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}| \sin\theta.$$
- $|\mathbf{u} \times \mathbf{v}|$ is also the area of the parallelogram defined by **u** and **v**.



Example 4

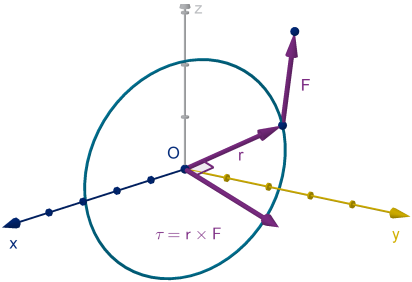
If $\mathbf{u} = 4\mathbf{k}$ and $\mathbf{v}$ is in the xy-plane, then what can we say about $\mathbf{u} \times \mathbf{v}$?

Application - Torque

In physics, **torque** measures the tendency of a rigid body to rotate around a fixed origin. If we apply the force $\mathbf{F}$ at the position $\mathbf{r}$ from the origin, the torque is

$$\tau = \mathbf{r} \times \mathbf{F}.$$

Viewing torque as a vector is very useful. For example, if more than one force is applied, the torques can be added to compute a total torque on the object.



Summary Questions

- What do the cross product and dot product have in common? How are they different?
- Would you rather use the minor matrices or the distributive method to compute a cross product? Why?
- Can a cross product be used to compute the angle between two vectors? Would you prefer to use the dot product? Why?

Section 12.5

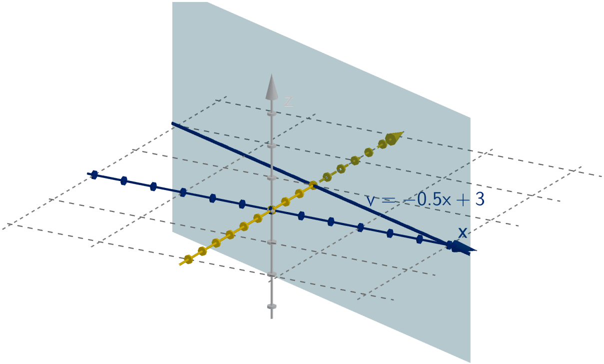
Equations of Lines and Planes

Goals:

- Give equations of lines in both vector and parametric form.
- Give equations of planes in both vector and normal forms.
- Use equations to find intersections of lines and planes.

Equation of a Line, First Attempt

In two dimensions, lines have equations like $y = -0.5x + 3$. If we used this equation in three dimensions, z would be a free variable, and we'd get a plane.



Parametric and Vector Equations

Definition

The graph of a **vector equation**

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

or a **parametric equation**

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

is the set of points $(f(t), g(t), h(t))$ obtained when all possible real numbers t are plugged into the equations.

Generally the graph of a parametric equation is one-dimensional, like a line or a curve.

Equation of a Line, Vector Version

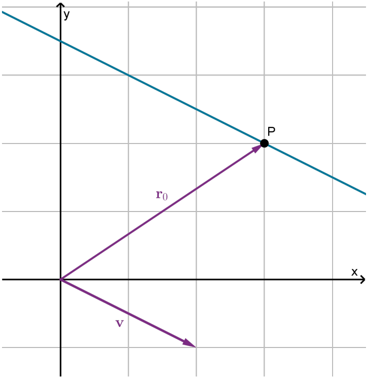
Here is a way to describe a line by vector equation:

Equation

If $\mathbf{r}_0$ is the position vector of an **known point**, and $\mathbf{v}$ is a **direction vector** parallel to the line, then the line is described by

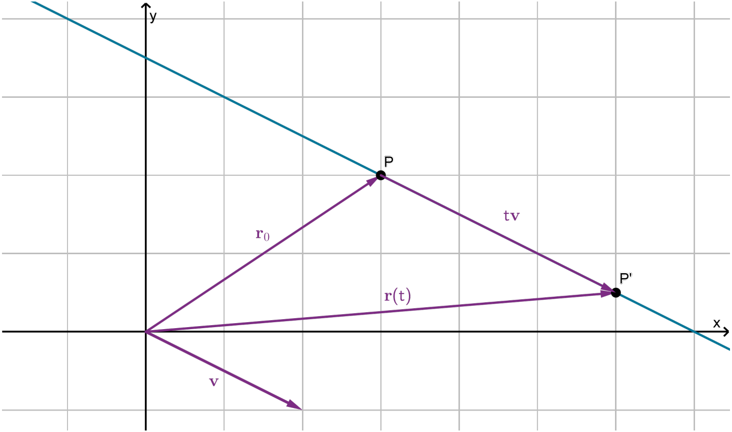
$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}$$

where t can be any real number.



Visualizing the Vector Equation

The endpoints of the vectors $\mathbf{r}(t)$ trace out the line as t ranges over all real numbers.



Exercise

Suppose you want to give the vector equation of a line whose known point is $(3, 2)$ and which also passes through $(5, 1)$.

1. Compute a direction vector $\mathbf{v}$ of this line.
2. Write a vector equation for it.
3. What is the slope of this line? How is it related to $\mathbf{v}$?
4. Is the point $(-1, 4)$ on this line? What t does it correspond to?

Equation of a Line, Parametric Version

Equation

If $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ and $\mathbf{v} = \langle a, b, c \rangle$ then the vector equation resolves as

$$\begin{aligned} \mathbf{r}(t) &= \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle \\ &= \langle x_0, y_0, z_0 \rangle + \langle ta, tb, tc \rangle \\ &= \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle . \end{aligned}$$

This gives the following parametric equations.

$$\begin{aligned} x &= x_0 + ta \\ y &= y_0 + tb \\ z &= z_0 + tc \end{aligned}$$

Equation of a Line, Symmetric Equations Version

If we want we can solve for t in the parametric equations,

$$x = x_0 + ta$$
$$y = y_0 + tb$$
$$z = z_0 + tc$$

and get three expressions that all equal t, and hence all equal each other.

Equation

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Exercise

1. Rewrite the vector equation $\mathbf{r}(t) = \langle 2, 5, 1 \rangle + t \langle 2, -1, -4 \rangle$ as a triple equation.

2. Use the triple equation to determine whether this line passes through $(7, 3, -9)$.

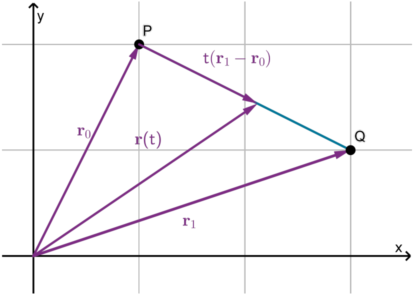
The Equation of a Line Segment

If we restrict the values of t to a finite interval, we get a segment instead of a line.

Formula

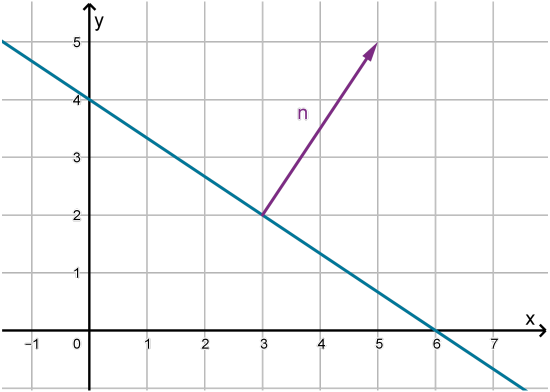
A vector equation of the segment from the endpoint $\mathbf{r}_0$ to the endpoint $\mathbf{r}_1$ is

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1$$



Review - Normal Equation of a Line

In algebra, you learned the **normal equation** of a line: e.g. $2x + 3y = 12$. Why is it called this?

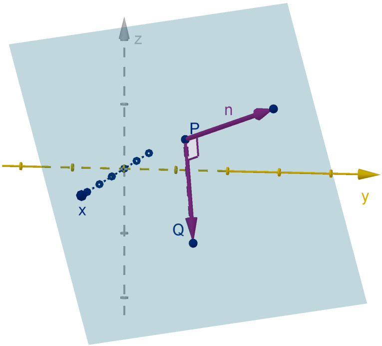


Normal Vectors to a Plane

A **normal vector** to a plane is orthogonal to every vector in the plane.

Theorem

In three-dimensional space, every plane has normal vectors. They are all parallel to each other.



Equation of a Plane, Vector Version

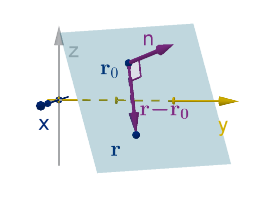
Theorem

If $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$ *describes an* **known point** *on a plane, and* $\mathbf{n} = \langle a, b, c \rangle$ *is a normal vector. Then the equation of the plane is*

$$(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{n} = 0$$

or

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$



Notice that since x_0, y_0 and z_0 are constants, we can distribute and collect them into a single term: d.

$$ax + by + cz - ax_0 - by_0 - cz_0 = 0$$
$$ax + by + cz + d = 0$$

Example 1

Find the equation of the plane that contains the points $(2, 1, 1)$, $(3, 4, -1)$ and $(0, 5, 2)$.

Example 2

Find the equation of the plane that contains the point $(0, 0, 4)$ and the line $\mathbf{r}(t) = \langle 2, 0, 2\rangle + t\,\langle 3, 1, 0\rangle$.

Example 3

Find the equation of the plane with intercepts $(4, 0, 0)$, $(0, 3, 0)$ and $(0, 0, 8)$.

Exercises

Consider the plane that contains $(0, 0, 0)$, $(4, 0, 3)$ and $(4, 5, 3)$

1. Give the equation of this plane.

2. Where does this plane intersect the line $\mathbf{r}(t) = \langle 2 + 3t, 4 - t, 3t\rangle$?

3. Where does this plane intersect the line $\mathbf{r}(t) = \langle 2 - 4t, 2 + t, 3 - 3t\rangle$?

4. Given any plane and any line, what are the possible numbers of intersection points that they can have? Can you justify your answer with algebra?

Summary Questions

- Why do we use the vector equation of a line instead of slope-intercept form?
- What two pieces of information do you need to write the vector equation of a line?
- What information do you need in order to write the equation of a plane?
- How do you find the intersection of a plane with a line?
- How are the normal vectors of a plane related to each other?