Solutions to Odd-Numbered Problems

Q1

$\left(5^2 5^4\right)^3 = \left(5^6\right)^3 = 5^{18}$.

Q3

$2\log_5 x + \log_5 y - 3\log_5 z = \log_5 x^2 + \log_5 y + \log_5 \frac{1}{z^3} = \log_5 \frac{x^2 y}{z^3}$.

Q5

$$\begin{aligned} 2e^x - 7 &= 22 \\ 2e^x &= 15 \\ e^x &= \frac{15}{2} \\ x &= \ln \frac{15}{2} \end{aligned}$$

Q7

$$\begin{aligned} 2\sin^2 x - 1 &= 0 \\ 2\sin^2 x &= 1 \\ \sin^2 x &= \frac{1}{2} \\ \sin x &= \pm\frac{\sqrt{2}}{2} \\ x &= \frac{\pi}{4} + 2n\pi \text{ or } x = \frac{3\pi}{4} + 2n\pi \end{aligned}$$

Q9

$$\begin{aligned} 4^{3x-2} &= 15 \\ 3x - 2 &= \log_4 15 \\ 3x &= \log_4 15 + 2 \\ 3x &= \log_4 15 + \log_4 16 \\ 3x &= \log_4 240 \\ x &= \frac{\log_4 240}{3} \end{aligned}$$

Q17

$$\begin{aligned} \frac{x^2+5x-6}{x-1} &= 0 \\ \text{set } x^2+5x-6 &= 0 \\ (x+6)(x-1) &= 0 \\ x = 1 &\text{ or } x = -6 \\ \text{plug into denominator: } 1-1=0 &\text{ or } -6-1 \neq 0 \end{aligned}$$

Since $x = 1$ casues the left side to be undefined, the only solution is $x = -6$.

Q19

$$\begin{aligned} \frac{3x^2-5}{2e^x-7} &= 0 \\ \text{set } 3x^2-5 &= 0 \\ 3x^2 &= 5 \\ x^2 &= \frac{5}{3} \\ x &= \pm\sqrt{\frac{5}{3}} \end{aligned}$$

The only value that makes the denominator 0 is $x = \ln\frac{7}{2}$. Both our solutions are valid. $x = \pm\sqrt{\frac{5}{3}}$.

Q21

$$\begin{aligned} \frac{\ln x-4}{3-x} &= 0 \\ \text{set } \ln x-4 &= 0 \\ \ln x &= 4 \\ x &= e^4 \\ \text{plug in } 3-e^4 &\neq 0 \end{aligned}$$

Our solution is valid. $x = e^4$.

Q23

$$\begin{aligned}
\frac{5}{(u+1)^2} &= \frac{u}{u+1}\\
\frac{5}{(u+1)^2} - \frac{u}{u+1} &= 0\\
\frac{5}{(u+1)^2} - \frac{u(u+1)}{(u+1)^2} &= 0\\
\frac{5-u-u^2}{(u+1)^2} &= 0\\
\text{set } u^2+u-5 &= 0\\
x &= \frac{-1 \pm \sqrt{1-4(1)(-5)}}{2}\\
x &= \frac{-1 \pm \sqrt{21}}{2}\\
\text{plug in } \left(\frac{-1 \pm \sqrt{21}}{2} + 1\right)^2 &\neq 0
\end{aligned}$$

Our solutions are valid. $x = \frac{-1 \pm \sqrt{21}}{2}$.

Q1

a $\lim_{x\to-3^-} f(x) = 1$

b $\lim_{x\to-3^+} f(x) = 2$

c $\lim_{x\to-2} f(x) = 1$

d $\lim_{x\to0} f(x)$ D.N.E.

e $\lim_{x\to4^-} f(x) = \infty$

f $\lim_{x\to4^+} f(x) = 1$

Q3

This function is continuous on its domain because it is the quotient of an exponential function and a polynomial. Its domain is all real numbers, because x^2+3 is never 0.

Q5

We will use the one-sided limits to compute the two-sided limit of $f(x)$.

$$\begin{aligned} &\lim_{x\to0^-} f(x) \\ &= \lim_{x\to0^-} \sin(2x) \\ &= 0 \end{aligned} \qquad \begin{aligned} &\lim_{x\to0^+} f(x) \\ &= \lim_{x\to0^+} -x^2 \\ &= 0 \end{aligned}$$

So $\lim_{x\to0} f(x) = 0$. On the other hand $f(0) = 4$. Since $\lim_{x\to0} f(x) \neq f(0)$, f is not continuous at $x = 0$.

Q7

We will use the one-sided limits to compute the two-sided limit of $f(x)$.

$$\begin{aligned} &\lim_{x\to1^-} f(x) \\ &= \lim_{x\to1^-} x+5 \\ &= 6 \end{aligned} \qquad \begin{aligned} &\lim_{x\to1^+} f(x) \\ &= \lim_{x\to1^+} x^2+4x+1 \\ &= 6 \end{aligned}$$

So $\lim_{x\to1} f(x) = 6$. Furthermore, $f(1) = 6$. Since $\lim_{x\to1} f(x) = f(1)$, f is continuous at $x = 1$.

Q9

$$
\begin{aligned}
\lim_{x\to 3}\frac{x-3}{x^2-9} &= \lim_{x\to 3}\frac{x-3}{(x-3)(x+3)} \\
&= \lim_{x\to 3}\frac{1}{x+3} && \text{since the functions agree except at } x=3 \\
&= \frac{1}{3+3} && \text{since } \frac{1}{x+3} is continuous at x=3 \\
&= \frac{1}{6}
\end{aligned}
$$

Q11

$$
\begin{aligned}
\lim_{x\to 9}\frac{2x-18}{\sqrt{x}-3} &= \lim_{x\to 9}\frac{2(\sqrt{x}-3)(\sqrt{x}+3)}{\sqrt{x}-3} \\
&= \lim_{x\to 3} 2\sqrt{x}+3 && \text{since the functions agree except at } x=9 \\
&= 2\sqrt{9}+3 && \text{since } 2\sqrt{x}+3 is continuous at x=9 \\
&= 9
\end{aligned}
$$

Q13

Let $g(x)=\sin x-2x+1$. g is continuous since it is the sum of a linear function and a trig function. $g(0)=1>0$. $g(1)=\sin(1)-1<0$, since $\sin(x)$ ranges from -1 to 1. Thus by the Intermediate Value Theorem there is a number c in $[0,1]$ such that $g(c)=0$. Since $g(c)=0$, $\sin c=2c-1$.

Q15

Nothing. $f(x)$ is not continuous on $[-1,1]$ because 0 is not in its domain. The Intermediate Value Theorem does not apply to this situation.

Q17

$$
\begin{aligned}
\lim_{x\to\infty}\frac{x^2+2x-9}{3x-6} &= \lim_{x\to\infty}\frac{x^2\left(1+\frac{2}{x}-\frac{9}{x^2}\right)}{3x\left(1-\frac{2}{x}\right)} \\
&= \lim_{x\to\infty}\frac{x^2}{x} \\
&= \lim_{x\to\infty} x \\
&= \infty
\end{aligned}
$$

Q19

This is a composition of functions. Let $v = \frac{1}{x}$. Let $w = e^v$.

$$\begin{aligned}
\lim_{x\to\infty} v &= \lim_{x\to\infty} \frac{1}{x} \\
&= 0 \\
\lim_{v\to 0} w &= \lim_{v\to 0} e^v \\
&= e^0 \\
&= 1 \\
\lim_{x\to\infty} \sqrt{e^{1/x}} &= \lim_{w\to 1} \sqrt{w} \\
&= \sqrt{1} \\
&= 1
\end{aligned}$$

Q21

This is a composition of functions. Let $v = e^x$.

$$\begin{aligned}
\lim_{x\to -\infty} v &= \lim_{x\to -\infty} e^x \\
&= 0 \\
\lim_{x\to -\infty} e^{e^x} &= \lim_{v\to 0} e^v \\
&= e^0 \\
&= 1
\end{aligned}$$

Q23

a Average rate of change is computed

$$\frac{\text{rise}}{\text{run}} = \frac{f(5)-f(2)}{5-2} = \frac{125-8}{3} = 39$$

b The line has slope 39 and passes through $(2,8)$. It's point-slope equation is

$$y-8=39(x-2)$$

c

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\
&= \lim_{h\to 0} \frac{(2+h)^3-2^3}{h} \\
&= \lim_{h\to 0} \frac{8+12h+6h^2+h^3-8}{h} \\
&= \lim_{h\to 0} \frac{12h+6h^2+h^3}{h} \\
&= \lim_{h\to 0} 12+6h+h^2 \\
&= 12
\end{aligned}$$

Q25

$$\begin{aligned}
f'(6) &= \lim_{h\to 0} \frac{f(6+h)-f(6)}{h} \\
&= \lim_{h\to 0} \frac{3(6+h)^2-7-3(6)^2+7}{h} \\
&= \lim_{h\to 0} \frac{108+36h+3h^2-7-108+7}{h} \\
&= \lim_{h\to 0} \frac{36h+3h^2}{h} \\
&= \lim_{h\to 0} 36+3h \\
&= 36
\end{aligned}$$

Q27

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h} \\
&= \lim_{h\to 0} \frac{x^2-x^2-2xh-h^2}{x^2(x+h)^2h} \\
&= \lim_{h\to 0} \frac{-2xh-h^2}{x^2(x+h)^2h} \\
&= \lim_{h\to 0} \frac{-2x-h}{x^2(x+h)^2} \\
&= \frac{-2x}{x^2x^2} \\
&= -\frac{2}{x^3}
\end{aligned}
$$

Q29

a $\frac{d}{dx}(5x^7 - 3x^2 + \frac{5}{x^2}) = \frac{d}{dx}(5x^7 - 3x^2 + 5x^{-2}) = 35x^6 - 6x - 10x^{-3}$

b $\frac{d}{dx}\frac{4x^5 - 2x^2 + 3x + 4}{x} = \frac{d}{dx}(4x^4 - 2x + 3 - 4x^{-1}) = 16x^3 - 2 + 4x^{-2}$

c $\frac{d}{dx}(x^2 + 2x)\sin x = (2x + 2x)\sin x + (x^2 + 2x)\cos x$

d $\frac{d}{dx}\frac{e^x}{x^2} = \frac{d}{dx}(e^x x^{-2}) = e^x x^{-2} - 2x^{-3}e^x$

e $\frac{d}{dx}\sqrt{x - 5} = \frac{1}{2\sqrt{x-5}}$

f $\frac{d}{dx}\cos(4x) = -4\sin(4x)$

g $\frac{d}{dx}\sin(e^x) = \cos(e^x)e^x$

h $\frac{d}{dx}(x^2 + 5x + 4)^{60} = 60(x^2 + 5x + 4)^{59}(2x + 5)$

i $\frac{d}{dx}e^{x^2 \sin x} = e^{x^2 \sin x}(2x\sin x + x^2\cos x)$

j $\frac{d}{dx}\frac{\ln(x^2 + 2)}{x^2 + 3x} = \frac{\frac{1}{x^2+2}(2x)(x^2 + 3x) - (2x + 3)\ln(x^2 + 2)}{(x^2 + 3x)^2}$

Q31

$$f'(x) = 3\cos(3x)$$
$$f''(x) = -9\sin(3x)$$
$$f'''(x) = -27\cos(3x)$$

Q33

$f(x)$ is increasing where $f'(x) > 0$.

$$\begin{aligned} f'(x) &= 3x^2 - 2x \\ \text{set } 0 &< 3x^2 - 2x \\ 0 &< x(3x-2) \end{aligned}$$

We perform a sign analysis on $f'(x) = x(3x-2)$

x	$-$	$+$	$+$
$(3x-2)$	$-$	$-$	$+$
$f'(x)$	$+$	$-$	$+$
	0	$\frac{2}{3}$	

So f is increasing on $(-\infty, 0) \cup \left(\frac{2}{3}, \infty\right)$.

Q35

$f(x)$ is increasing where $f'(x) > 0$. Notice f has domain $[0, \infty)$.

$$\begin{aligned} f'(x) &= 512x^{-1/2} - 4x^3 \\ \text{set } 0 &< 512x^{-1/2} - 4x^3 \\ 0 &< 4x^{-1/2}(128 - x^{7/2}) \\ 0 &< 128 - x^{7/2} && \text{since } x > 0 \text{ on the domain} \\ x^{7/2} &< 128 \\ x &< 128^{2/7} \\ x &< 4 \end{aligned}$$

Thus $f(x)$ is increasing on $[0, 4)$.

Q1

The equation is $y - f(4) = f'(4)(x-4)$. We need to find $f(4)$ and $f'(4)$. Since we are given the point of tangency, we know $f(4) = 2$.

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{x}} \\ f'(4) &= \frac{1}{2\sqrt{4}} \\ &= \frac{1}{4} \end{aligned}$$

The equation of the tangent line is $y - 2 = \frac{1}{4}(x-4)$.

Q3

a The equation is $L(x) = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x - \frac{\pi}{3}\right)$. We need to solve for $f\left(\frac{\pi}{3}\right)$ and $f'\left(\frac{\pi}{3}\right)$.

$$\begin{aligned} f\left(\frac{\pi}{3}\right) &= \frac{\sqrt{3}}{2} \\ f'(x) &= \cos x \\ f'\left(\frac{\pi}{3}\right) &= \frac{1}{2} \end{aligned}$$

The linearization is $L(x) = \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}\left(x - \dfrac{\pi}{3}\right)$.

b We would need approximations of $\dfrac{\sqrt{3}}{2}$ and $\dfrac{\pi}{3}$. It would also be okay to have approximations of $\sqrt{3}$ and π. We would obtain the needed values by long division.

c Using $\dfrac{\sqrt{3}}{2} = 0.866$ and $\dfrac{\pi}{3} = 1.047$ we get

$$\begin{aligned} \sin(1) &\approx L(1) \\ &\approx 0.866 + \frac{1}{2}(1 - 1.047) \\ &\approx 0.866 + \frac{1}{2}(-0.047) \\ &\approx 0.866 - 0.024 \\ &\approx 0.842 \end{aligned}$$

Q5

a

$$\begin{aligned} L(t) &= f(5) + f'(5)(t-5) \\ &= 3 + 0.2(t-5) \end{aligned}$$

b $L(8) = 3 + 0.2(8-5) = 3.6$.

c It is an underestimate. If $m''(t) > 0$ then the derivative is increasing. The linearization assumes the derivative is constant. The actual function has a larger derivative and thus larger values for $t > 5$.

Q7

$f'(x) = 8x^{-1/3} - 1$. This is undefined at $x = 0$ since we cannot devide by 0. To find the other critical points we solve:

$$\begin{aligned} 8x^{-1/3} - 1 &= 0 \\ 8x^{-1/3} &= 1 \\ x^{-1/3} &= \frac{1}{8} \\ x^{1/3} &= 8 \\ x &= 512 \end{aligned}$$

$x = 0$ and $x = 512$ are the critical points of $f(x)$.

Q9

First we use the derivative to find critical points.

$$\begin{aligned} f'(x) &= 3x^2 - 75 \\ \text{set } 0 &= 3x^2 - 75 \\ 0 &= 3(x-5)(x+5) \\ x &= \pm 5 \end{aligned}$$

We apply the second derivative test using $f''(x) = 6x$.

$$f'(5) = 6(5) > 0 \qquad\qquad f'(-5) = 6(-5) < 0$$

The second derivative test tells us that $x = 5$ is a local minimum, while $x = -5$ is a local maximum.

Q11

$f(x)$ is continuous and $[-8, 1]$ is a closed interval, so the EVT guarantees a maximum and a minimum. We find the critical points. $f'(x) = \frac{2}{3x^{1/3}}$. This is never 0, but is undefined at $x = 0$. Thus $x = 0$ is the only critical point. We evaluate f at the critical point and at the endpoints.

$$f(-8) = 4 \qquad \text{maximum} f(1) = 1$$
$$f(0) = 0 \qquad \text{minimum}$$

Q13

$$\lim_{x \to 0^+} \frac{x \cos(x - \pi)}{e^x - 1} \qquad \frac{0}{0} \text{ form, apply l'Hôpital's}$$
$$= \lim_{x \to 0^+} \lim_{x \to 0^+} \frac{\cos(x - \pi) - x \sin(x - \pi)}{e^x} \qquad \text{continuous, plug in } x = 0$$
$$= \frac{-1}{1} \qquad = -1$$

Q15

$$\lim_{x \to \infty} \frac{x \ln x}{x^{5/2} + 3} \qquad \frac{\infty}{\infty} \text{ form, apply l'Hôpital's}$$
$$= \lim_{x \to \infty} \frac{\ln x + 1}{\frac{5}{2} x^{3/2}} \qquad \frac{\infty}{\infty} \text{ form, apply l'Hôpital's}$$
$$= \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{15}{4} x^{1/2}}$$
$$= \lim_{x \to \infty} \frac{4}{15 x^{3/2}}$$
$$= 0$$

Q1

There are infinitely many. Two examples are $\frac{1}{2}x^2 + 5x$ and $\frac{1}{2}x^2 + 5x + 1$.

Q3

$4 \sin x + 2x^3 + c$.

Q5

$F(x) - G(x) = c$ for some constant c, so $3F(x) - bG(x) = (3-b)F(x) - bc$. This is an antiderivative $f(x)$ when $b = 2$.

Q7

$(3)(2) - 2 + (3)(3) - 2 + (3)(4) - 2 + (3)(5) - 2 = 34$.

Q9

$\sum_{k=a}^{b} c = c+c+c+c+\cdots+c$ where there are $b-a+1$ occurances of c. The sum simplifies to $c(b-a+1)$.

Q11

This notation indicates that we add up the values of the function for k between 1 and k, which is nonsense. The index variable cannot also be a bound on itself.

Q13

a. This sequence is obtained by repeatedly adding 4. We can represent it with multiplication: $\sum_{k=0}^{4} 3 + 4k$.

b. This sequence is obtained by repeatedly multipying by 2. We can represent it with an exponentail: $\sum_{k=0}^{5} 6(2^k)$.

c. This sequence is obtained by repeatedly adding 1 in the numerator and denominator. We can model them seperately: $\sum_{k=0}^{5} \frac{3+k}{4+k}$.

Q15

No. On the interval $\left(\frac{1}{2}, 1\right)$, $y \ln x$ is below the $x-$axis. $\int_{1/2}^{1} \ln x \; dx$ computes the signed area, which is the negative of the area below the x-axis annd above $y = \ln x$.

Q17

We will omit the graph. To divide $[1, 11]$ into 5 subintervals means each has length $\frac{11-1}{5} = 2$. Let left endpoints are 1, 3, 5, 7, and 9. The sum of the areas of the rectangles is

$$\begin{aligned}&\sqrt{1}(2) + \sqrt{3}(2) + sqrt5(2) + \sqrt{7}(2) + \sqrt{9}(2)\\&= 2(4 + \sqrt{3} + \sqrt{5} + \sqrt{7})\end{aligned}$$

Q19

The graph is a horizontal line of height 7. The area under this graph over $[3, 8]$ is a rectangle of length 5 and height 7. This area and the integral have a value of $(7)(5) = 35$.

Q21

By the Fundamental Theorem of Calculus, $g(x)$ is an antiderivative of $f(x)$. This means $g'(x) = f(x)$, or more specifically $g'(8) = f(8)$.

Q23

If f is increasing, then $f'(x)$ is positive. That means the integral of $f'(x)$ over $[22, 31]$ is positive. We can argue this geometrically, noting that $y = f'(x)$ is above the x-axis and hence has a positive signed area beneath it. Instead we could note that the integral is a limit of rectangle approximations, which are all positive.

Q25

$\displaystyle\int f(x)\; dx$ is an indefinite integral. It is the general antiderivative of $f(x)$ and is thus a family of functions. $\displaystyle\int_a^b f(x)\; dx$ is a definite integral. It represents the signed area under $y = f(x)$ from a to b. It is a number.

Q27

$$\begin{aligned}\int_1^8 x - \frac{3}{x}\ dx &= \frac{x^2}{2} - 3\ln x\Big|_1^8 \\ &= \frac{64}{2} - 3\ln 8 - \frac{1}{2} + 0 \\ &= \frac{63}{2} - 3\ln 8\end{aligned}$$

Q29

$\displaystyle\int e^x - 6x^2\ dx = e^x - 2x^3 + c$

Q31

$\displaystyle\int \sqrt{t}\ dt = \frac{2}{3}x^{3/2} + c.$

Q33

$\displaystyle\int \frac{3}{5}\sin y\ dy = -\frac{3}{5}\cos y + c.$

Q35

$$\begin{aligned}\int_{\pi/6}^{3\pi/4} 2\cos v\ dv &= 2\sin v\Big|_{\pi/6}^{3\pi/4} \\ &= 2\sin(3\pi/4) - 2\sin(\pi/6) \\ &= 2\left(\frac{\sqrt{2}}{2}\right) - 2\left(\frac{1}{2}\right) \\ &= \sqrt{2} - 1\end{aligned}$$

Q37

We can use the chain rule for these.

a $\displaystyle\int f(x+a)\ dx = F(x+a) + c.$

b $\displaystyle\int f(ax)\ dx = \frac{1}{a}F(ax) + c.$

Q39

Perform a u-substitution. Let $u = 7x$.

$$\begin{aligned} 7\ dx &= du \\ dx &= \frac{1}{7}\ du \\ \int e^{7x}\ dx &= \int \frac{1}{7}e^{u}\ du \\ &= \frac{1}{7}e^{u} + c \\ &= \frac{1}{7}e^{7x} + c \end{aligned}$$

Q41

Perform a u-substitution. Let $u = \frac{\theta}{3}$.

$$\begin{aligned} \frac{1}{3}\ d\theta &= du \\ dx &= 3\ du \\ \int \cos\left(\frac{\theta}{3}\right)\ d\theta &= \int 3\cos(u)\ du \\ &= 3\sin(u) + c \\ &= 3\sin\left(\frac{\theta}{3}\right) + c \end{aligned}$$

Q43

Perform a u-substitution. Let $u = \pi t$.

$$\begin{aligned} \pi\ dx &= du & x = 0 &\Rightarrow u = 0 \\ dx &= \frac{1}{\pi}\ du & x = \frac{1}{4} &\Rightarrow u = \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} \int_0^{1/4} \sin(\pi t)\ dt &= \int_0^{\pi/4} \frac{1}{\pi}\sin u\ du \\ &= -\frac{1}{\pi}\cos u\Big|_0^{\pi/4} \\ &= -\frac{1}{\pi}\frac{\sqrt{2}}{2} + \frac{1}{\pi}(1) \\ &= \frac{2-\sqrt{2}}{2\pi} \end{aligned}$$

Q45

Perform a u-substitution. Let $u = x^5 - 2x$.

$$\begin{aligned} 5x^4 - 2\ dx &= du \\ \int (x^5 - 2x)(5x^4 - 2)\ dx &= \int u\ du \\ &= \frac{u^2}{2} + c \\ &= \frac{(x^5 - 2x)^2}{2} + c \end{aligned}$$

Q1

$f(x) - g(x)$ is the height of the line segment at x above $g(x)$ and below $f(x)$.

Q3

We can compute the inverses $x = g^{-1}(y)$ and $x = f^{-1}(y)$. If, for instance, $x = g^{-1}(y)$ is the left hand boundary of the region and $x = f^{-1}(y)$ is the right hand boundary, the area is $\int_a^b f^{-1}(y) - g^{-1}(y)\, dy$.

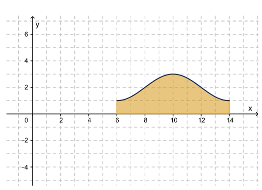
Q5

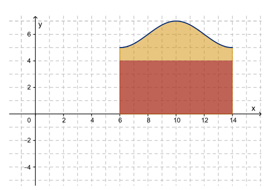
We need to solve for their intersections and test between each pair of intersections. We integrate the larger function minus the smaller function over that interval. Since we are integrating a positive function, we get a positive result for area.

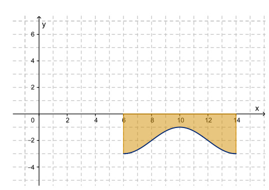
Q7

a The area would shift up and include an extra k by $(b - a)$ rectangle, increasing the integral by $k(b - a)$.

b Not necessarily. If $y = f(x)$ is less than k units above the x-axis, then some of the area would shift below the axis. If we counted this as geometric area, we may end up with more area than we stared with.

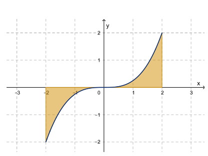






Q9

In the first case, positive and negative signed area between $y = f(x)$ and $y = 0$ would be made positive and added together. In the second, they would be added (and perhaps cancel) before being made positive.



Q11

If $y = g(x)$ is above $y = f(x)$ then $g(x) - f(x)$ is positive and $\int_a^b g(x) - f(x)\,dx$ computes the geometric area between them. This is the same test and the same integral that we use when the graphs are above the x-axis. Thus the signs of $f(x)$ and $g(x)$ don't matter. The formula works either way.

Q13

$$\begin{aligned} x^3 &= 4x \\ x^3 - 4x &= 0 \\ x(x-2)(x+2) &= 0 \\ x = 0 \text{ or } x = 2 &\text{ or } x = -2 \end{aligned}$$

These graphs don't cross between $x = 3$ and $x = 5$. We can use a test at $x = 3$, $3^3 > 4(3)$. The area is

$$\begin{aligned} \text{Area } &= \int_3^5 x^3 - 4x\ dx \\ &= \frac{x^4}{4} - 2x^2 \Big|_3^5 \\ &= \frac{5^4}{4} - 2(25) - \frac{3^4}{4} + 2(9) \\ &= 104 \end{aligned}$$

Q15

Solve for intersections

$$\begin{aligned} x^2 &= \sqrt{x} \\ x^2 - \sqrt{x} &= 0 \\ \sqrt{x}(x^{3/2} - 1) &= 0 \\ x = 1 &\text{ or } x = 0 \end{aligned}$$

We can use the test point $\frac{1}{2}$. $\sqrt{\frac{1}{2}} > \frac{1}{4}$ so we compute the area:

$$\begin{aligned} \text{Area } &= \int_0^1 \sqrt{x} - x^2\ dx \\ &= \frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \Big|_0^1 \\ &= \frac{2}{3} - \frac{1}{3} - 0 + 0 \\ &= \frac{1}{3} \end{aligned}$$

Q17

We solve for the intersections

$$\begin{aligned} x^2 &= 2x - 1 \\ x^2 - 2x + 1 &= 0 \\ (x-1)^2 &= 0 \\ x &= 1 \end{aligned}$$

So our x bounds are $x = 1$ and $x = -3$ (given). We can use $x = 0$ to test for top and bottom.

$$\begin{aligned} y &= x^2 & \qquad y &= 2x - 1 \\ y &= 0 & \qquad y &= 0 - 1 \end{aligned}$$

So $y = x^2$ is on top. We can now set up the integral and compute the area.

$$\begin{aligned} \text{Area } &= \int_{-3}^{1} x^2 - (2x-1)\ dx \\ &= \frac{x^3}{3} - x^2 + x \Big|_{-3}^{1} \\ &= \frac{1}{3} - 1 + 1 + \frac{27}{3} + 9 + 3 \\ &= \frac{64}{3} \end{aligned}$$

Q19

The intersection points are where $\sin x = \cos x$. According to our unit circle this occurs at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. We can use test points of $x = 0$, $x = \pi$ and $x = 2\pi$, or we can draw the graphs to see which is the top. The resulting area is

$$\begin{aligned} \text{Area } &= \int_{0}^{\pi/4} \cos x - \sin x\ dx + \int_{\pi/4}^{5\pi/4} \sin x - \cos x\ dx + \int_{5\pi/4}^{2\pi} \cos x - \sin x\ dx \\ &= (\sin x + \cos x)\Big|_{0}^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{5\pi/4} + (\sin x + \cos x)\Big|_{5\pi/4}^{2\pi} \\ &= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 0 - 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + 0 + 1 + \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} \\ &= 4\sqrt{2} \end{aligned}$$

Q21

Intersections:

$$
\begin{aligned}
xe^{x^2} &= ex \\
xe^{x^2} - ex &= 0 \\
x(e^{x^2} - e) &= 0 \\
x = 0 \text{ or } x = -1 &\text{ or } x = 1
\end{aligned}
$$

We can use $x = \frac{1}{2}$ and $x = -\frac{1}{2}$ as test points.

$$\frac{1}{2}e^{\left(\frac{1}{2}\right)^2} < frac12e \qquad -\frac{1}{2}e^{\left(-\frac{1}{2}\right)^2} > -frac12e$$

Now we can compute the area. We need a u-subtitution for half of the integrand, so we should divide each intergral into a sum of two. The substituion is $u = x^2$.

$$
\begin{aligned}
\text{Area } &= \int_{-1}^{0} ex - xe^{x^2}\, dx + \int_{0}^{1} xe^{x^2} - ex\, dx \\
&= \int_{-1}^{0} ex\, dx - \int_{-1}^{0} xe^{x^2}\, dx + \int_{0}^{1} xe^{x^2}\, dx - \int_{0}^{1} ex\, dx \\
&= \int_{-1}^{0} ex\, dx - \int_{1}^{0} \frac{1}{2}e^{u}\, du + \int_{0}^{1} \frac{1}{2}e^{u}\, du - \int_{0}^{1} ex\, dx \\
&= \left.\frac{ex^2}{2}\right|_{-1}^{0} + \left.\frac{1}{2}e^{u}\right|_{0}^{1} + \left.\frac{1}{2}e^{u}\right|_{0}^{1} - \left.\frac{ex^2}{2}\right|_{0}^{1} \\
&= -\frac{e}{2} + \frac{e}{2} - \frac{1}{2} + \frac{e}{2} - \frac{1}{2} - \frac{e}{2} \\
&= 1
\end{aligned}
$$

Q23

If $f(x) - g(x)$ has a double root at $x = a$, then the sign will not change at a. The previous problem had a double root at $x = 0$, since x^2 was a factor of both functions.

Q25

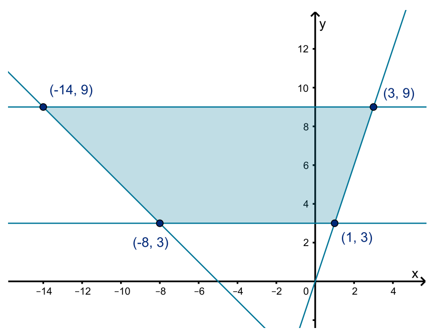
Since we have three curves of the form $y =$, we know there cannot be one top and one bottom curve. $y = \sqrt{x}$ and $y = -2x$ meet at $(0, 0)$. We invert both of them

$$y = \sqrt{x} \qquad\qquad y = -2x$$
$$y^2 = x \qquad\qquad -\frac{y}{2} = x$$

Between $y = 0$ and $y = 6$, $y^2 > -\frac{y}{2}$ so our integral is

$$\begin{aligned}\text{Area } &= \int_0^6 y^2 + \frac{y}{2}\, dy \\ &= \frac{y^3}{3} + \frac{y^2}{4}\Big|_0^6 \\ &= \frac{216}{3} + \frac{36}{4} - 0 - 0 \\ &= 81\end{aligned}$$

Q27



1 We can write the area as a sum or difference of dx integrals. Here is one way to do it:

$$\text{Area} = \int_{-14}^{-8} 9 - (-5 - x)\, dx + \int_{-8}^{1} 9 - 3\, dx + \int_{1}^{3} 9 - 3x\, dx$$

2 We can write the area as a single dy integral.

$$\text{Area} = \int_3^9 \frac{y}{3} - (-5 - y)\, dy$$

3 This is a trapezoid. You might remember the formula from geometry.

$$\text{Area} = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}(17 + 9)6$$

Q29

The intersections are

$$\begin{aligned}\sqrt{x} &= x^2\\ 0 &= x^2 - \sqrt{x}\\ 0 &= \sqrt{x}(x^{3/2} - 1)\\ x &= 0 \text{ or } x = 1\end{aligned}$$

Plugging these into $y = x^2$ gives $(0,0)$ and $(1,1)$ as intersection points.

a We can look at the graphs or use a test point like $x = \frac{1}{2}$ to see that $y = \sqrt{x}$ is on top.

$$\text{Area} = \int_0^1 \sqrt{x} - x^2 \, dx$$

b We invert each function.

$$\begin{aligned}y &= \sqrt{x} \qquad\qquad & y &= x^2\\ y^2 &= x & \pm\sqrt{x} &= x^2\end{aligned}$$

Since we are between $y = 0$ and $y = 1$ we can use a test point like $y = \frac{1}{2}$ to see that $x = \sqrt{y}$ is to the right of $x = y^2$.

$$\text{Area} = \int_0^1 \sqrt{y} - y^2 \, dy$$

Q31

If $f'(a) > g'(a)$ then f grows faster than g to the right of a. Since we know that g doesn't catch up until the next intersection, we can say $f(x) > g(x)$ until that intersection. By the same reasoning f will be below g to the left of a, until their next intersection. We can make the opposite conclusions if $g'(a) > f'(a)$.

Q1

A plane is intersected with the solid. The part of the plane that is inside the solid is the cross-section.

Q3

Prisms, because we know a cvolukme formula for them already.

Q5

A ball, cube, and cone have volume.

Q7

The volume is the same either way, but the numerical value will be higher with cubic centimeters. Cubic centimeters are smaller, so we can fit more of them into the solid than cubic inches.

Q9

The cross sections are circles. The grow from radius 0 at $x = -5$ to radius 5 at $x = 0$. Then they shrink back to radius 0 at $x = 5$.

Q11

S is a prism, though it might be slanted or twisted if the cross sections are located or rotated differently from each other.

Q13

a The sum of the prisms approximates the volume of S. $V \approx 5.1 + 6 + 7.2 + 9.6$

b The prisms have height 3. We can divide their volumes by 3 to get the volumes of the cross-sections we used as their bases: $1.7, 2, 2.4, 3.2$.

Q15

We'll take three cross-sections, one cross-section at the base, one cross-section a third of the way up, and the last cross-section two thirds of the way up. The height of each prism is 3.

- At the base, the cross-section is a square of side length 6. Its area is 36. The volume of the prism is 108
- The next cross-section is one third smaller. It's side length is 4. Its area is 16. The volume of the prism is 48.
- The last cross-section is two-thirds smaller than the base. Its side length is 2, its area is 4 and the volume of the prism is 12.

In total, we approximate the volume to be $V \approx 108 + 48 + 12 = 168$.

Q17

The cross-sections have area $A(x) = (e^x)^2 = e^{2x}$. We compute the volume:

$$\begin{aligned}\text{Volume} &= \int_0^3 e^{2x}\,dx \\ &= \frac{1}{2}e^{2x}\Big|_0^3 \\ &= \frac{1}{2}\left(e^6 - e^0\right) \\ &= \frac{e^6 - 1}{2}\end{aligned}$$

Q19

The bases are the veritcal distance between $y = 3x$ and $y = x^2$. $x^2 < 3x$ when $0 < x < 3$, so the base at x is $3x - x^2$.

$$\begin{aligned}A(x) &= \frac{1}{2}bh \\ &= \frac{1}{2}(3x - x^2)(3x - x^2) \\ &= \frac{9}{2}x^2 - 3x^3 + \frac{1}{2}x^4 \\ \text{Volume} &= \int_0^3 \frac{9}{2}x^2 - 3x^3 + \frac{1}{2}x^4\,dx \\ &= \frac{3}{2}x^3 - \frac{3}{4}x^4 + \frac{1}{10}x^5\Big|_0^3 \\ &= \frac{81}{2} - \frac{243}{4} + \frac{243}{10} \\ &= \frac{81}{20}\end{aligned}$$

Q21

$$
\begin{aligned}
A(x) &= \pi(\sqrt{x})^2 \\
&= \pi x \\
\text{Volume } &= \int_0^9 \pi x \; dx \\
&= \frac{pi}{2} x^2 \Big|_0^9 \\
&= \frac{81\pi}{2}
\end{aligned}
$$

Q23

$$
\begin{aligned}
A(x) &= \pi(4 - x^2)^2 \\
&= \pi(16 - 8x^2 + x^4) \\
\text{Volume} &= \int_{-2}^{2} \pi(16 - 8x^2 + x^4) \; dx \\
&= \pi \left(16x - \frac{8x^3}{3} + \frac{x^5}{5} \right) \Big|_{-2}^{2} \\
&= \pi \left(32 - \frac{64}{3} + \frac{32}{5} + 32 - \frac{64}{3} + \frac{32}{5} \right) \\
&= \frac{512\pi}{15}
\end{aligned}
$$

Q25

$y = -\frac{1}{2}x + 3$ meets the x-axis where $0 = -\frac{1}{2}x + 3$, or $x = 6$. The bases of the cross sections have length $-\frac{1}{2}x + 3$.

$$
\begin{aligned}
A(x) &= \frac{1}{2} bh \\
&= \frac{1}{2}(-\frac{1}{2}x + 3)(8) \\
&= -2x + 12 \\
\text{Volume } &= \int_0^6 -2x + 12 \; dx \\
&= -x^2 + 12x \Big|_0^6 \\
&= -36 + 72 \\
&= 36
\end{aligned}
$$

Q27

We can place the triangle on the xy-plane, with a leg on the x-axis and vertices $(0,0)$, $(4,0)$, and $(4,3)$. The diameters are the heights from the axis to the hypotenuse: $y = \frac{3x}{4}$. The radii are half that.

$$\begin{aligned}
A(x) &= \frac{1}{2}\pi r^2 \\
&= \frac{1}{2}\pi\left(\frac{3x}{8}\right)^2 \\
&= \frac{9\pi x^2}{128} \\
\text{Volume } &= \int_0^4 \frac{9\pi x^2}{128}\,dx \\
&= \left.\frac{3\pi x^3}{128}\right|_0^4 \\
&= \frac{192\pi}{128} \\
&= \frac{3\pi}{2}
\end{aligned}$$

Q29

a Solve for the intersections of $y = x^2 - 6x$ and $y = 0$.

$$\begin{aligned}
x^2 - 6x &= 0 \\
x(x-6) &- 0 \\
x = 0 &\text{ or } x = 6
\end{aligned}$$

Our x bounds are from 0 to 6. To see which graph is on top, use a test point. I used $x = 1$.

$$1^2 - 6(1) = -5 < 0$$

So $y = 0$ is on top.

b $A(x)$ is the area of a semicircle with diameter $6x - x^2$ This semicircle has radius $\frac{6x-x^2}{2}$ and area $\frac{1}{2}\pi\left(\frac{6x-x^2}{2}\right)^2$

Q31

There are several options. Since the information we have is not evenly-spaced, the prisms can have different heights. We compute our estimate by taking the area of each base times the height and summing the volumes.

$$\text{Volume } = (10)(1) + (12)(4) + (11)(2) + (7)(3) + (2)(2) = 105$$

Q1

Integrands that are products of two functions are good candidates, though there are some other functions we can integrate by parts.

Q3

Each factor in the product is sorted into its category: I, L, A, T or E. The one in the category farther to the left should be u, the one to the right should be dv.

Q5

This looks like something that could be the result of the product rule. $f(x)$ could be $\sin x$ and $g(x)$ could be $\tan^{-1} x$. The antiderivative is thus $\sin x \tan^{-1} x + c$.

Q7

Choose $u = \ln x$ and $dv = \frac{1}{x^3}\,dx$. We compute $du = \frac{1}{x}\,dx$ and $v = -\frac{1}{2x^2}$.

$$\begin{aligned}
\int \frac{\ln x}{x^3}\,dx &= -\frac{1}{2x^2}\ln x - \int -\frac{1}{2x^2}\frac{1}{x}\,dx \\
&= -\frac{\ln x}{2x^2} + \int \frac{1}{2x^3}\,dx \\
&= -\frac{\ln x}{2x^2} - \frac{1}{4x^2} + c \\
\text{or} \quad &= -\frac{2\ln x + 1}{4x^2} + c
\end{aligned}$$

Q9

Choose $u = \tan^{-1} x$ and $dv = dx$.

$$\begin{aligned}
\int \tan^{-1} x\,dx &= x\tan^{-1} x - \int x\frac{1}{1+x^2}\,dx \\
&= x\tan^{-1} x - \int \frac{1}{2u}\,du && \text{substitute } u = x^2 + 1 \\
&= x\tan^{-1} x \frac{1}{2}\ln|u| + c \\
&= x\tan^{-1} x \frac{1}{2}\ln(x^2+1) + c
\end{aligned}$$

Q11

Choose $u = \sin^{-1} x$ and $dv = dx$. $du = \frac{1}{\sqrt{1-x^2}}$

$$\begin{aligned}
\int \sin^{-1} x \ dx &= x \sin^{-1} x - \int x \frac{1}{\sqrt{1-x^2}} \ dx \\
&= x \sin^{-1} x + \int \frac{1}{2} u^{-1/2} \ du && \text{substitute } u = 1 - x^2 \\
&= x \sin^{-1} x + u^{1/2} + c \\
&= x \sin^{-1} x + \sqrt{1-x^2} + c
\end{aligned}$$

Q13

$$\begin{aligned}
\int x^2 \cos(x+2) \ dx &= x^2 \sin(x+2) - \int 2x \sin(x+2) \ dx && u = x^2 \quad dv = \cos(x+2) \ dx \\
&= x^2 \sin(x+2) + 2x \cos(x+2) - \int 2 \cos(x+2) \ dx && u = 2x \quad dv = \sin(x+2) \ dx \\
&= x^2 \sin(x+2) + 2x \cos(x+2) - 2 \sin(x+2) + c
\end{aligned}$$

Q15

We will want $dv = x^{-3} \sin(x^{-2}) \ dx$, so that $v = \frac{1}{2} \cos(x^{-2})$.

$$\begin{aligned}
&\int x^{-7} \sin(x^{-2}) \ dx \\
&= \frac{1}{2} x^{-4} \cos(x^{-2}) + \int 2x^{-5} \cos(x^{-2}) \ dx && u = x^{-4} \quad dv = x^{-3} \sin(x^{-2}) \ dx \\
&= \frac{1}{2} x^{-4} \cos(x^{-2}) - x^{-2} \sin(x^{-2}) - \int 2x^{-3} \sin(x^{-2}) \ dx && u = 2x^{-2} \quad dv = x^{-3} \cos(x^{-2}) \ dx \\
&= \frac{1}{2} x^{-4} \cos(x^{-2}) - x^{-2} \sin(x^{-2}) - \cos(x^{-2}) + c
\end{aligned}$$

Q17

$$\begin{aligned}
\int e^{3x}\sin x\,dx &= \frac{1}{3}e^{3x}\sin x - \int \frac{1}{3}e^{3x}\cos x\,dx && u=\sin x \qquad dv=e^{3x}\,dx\\
&= \frac{1}{3}e^{3x}\sin x - \int \frac{1}{3}e^{3x}\cos x\,dx && u=\frac{1}{3}\cos x \qquad dv=e^{3x}\,dx\\
&= \frac{1}{3}e^{3x}\sin x - \frac{1}{9}e^{3x}\cos x - \int \frac{1}{9}e^{3x}\sin x\,dx\\
\frac{10}{9}\int e^{3x}\sin x\,dx &= \frac{1}{3}e^{3x}\sin x - \frac{1}{9}e^{3x}\cos x\\
\int e^{3x}\sin x\,dx &= \frac{3}{10}e^{3x}\sin x - \frac{1}{10}e^{3x}\cos x + c
\end{aligned}$$

Q19

Choose $u = x^2$ and $dv = xe^{x^2}\,dx$

$$\begin{aligned}
\int x^3e^{x^2}\,dx &= \frac{1}{2}x^2e^{x^2} - \int xe^{x^2}\,dx\\
&= \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + c
\end{aligned}$$

Q21

First we find the intersection points

$$\begin{aligned}
xe^x &= ex\\
x(e^x - e) &= 0\\
x = 0 \text{ or } x &= 1
\end{aligned}$$

For a test point $x = \frac{1}{2}$ we have $\frac{1}{2}e^{1/2} < \frac{1}{2}e$.

$$\begin{aligned}
\text{Area} &= \int_0^1 ex - xe^x\,dx\\
&= \int_0^1 ex\,dx - \int_0^1 xe^x\,dx\\
&= \frac{ex^2}{2}\bigg|_0^1 - xe^x\Big|_0^1 + \int e^x\,dx\\
&= \frac{ex^2}{2} - xe^x + e^x\bigg|_0^1\\
&= \frac{e}{2} - e + e - 1\\
&= \frac{e}{2} - 1
\end{aligned}$$

Q23

We will need the result that $\int \ln x \; dx = x \ln x - x + c$.

$$\begin{aligned}
A(x) &= \pi r^2 \\
&= \pi (\ln x)^2 \\
\text{Volume} &= \int_1^5 \pi (\ln x)^2 \; dx \\
&= \pi x (\ln x)^2 \Big|_1^5 - \int_1^5 \pi x \frac{2 \ln x}{x} \; dx \qquad\qquad u = \pi(\ln x)^2 \qquad\qquad dv = 1 \; dx \\
&= \pi x (\ln x)^2 - 2\pi (x \ln x - x) \Big|_1^5 \\
&= \pi \left(5(\ln 5)^2 - 10 \ln 5 + 10 - 2 \right) \\
&= \pi \left(5(\ln 5)^2 - 10 \ln 5 + 8 \right)
\end{aligned}$$

Q1

The approximated value minus the actual value

Q3

Midpoint converges most quickly, trapezoid's error bound is always twice a large, left and right coverge much more slowly.

Q5

The largest approximation is $2((-4)^2+(-2)^2+2^2+4^2) = 80$. The smallest is $2((-2)^2+0^2+0^2+2^2) = 16$. The largest difference they can obtain is $80 - 16 = 64$.

Q7

$\Delta x = 15/5 = 3$. The left endpoints are $1, 4, 7, 10,$ and 13

$$\begin{aligned} L_5 &= \Delta x(f(1) + f(4) + f(7) + f(10) + f(13)) \\ &= 3(1^{3/2} + 4^{3/2} + 7^{3/2} + 10^{3/2} + 13^{3/2}) \\ &= 3(9 + 7^{3/2} + 10^{3/2} + 13^{3/2}) \end{aligned}$$

Q9

$\Delta x = 2/4 = 0.5$. The left endpoints are $0, 0.5, 1$ and 1.5

$$\begin{aligned} L_4 &= \Delta x(f(0) + f(0.5) + f(1) + f(1.5)) \\ &= 0.5\left(0 + \frac{e^{1/2}}{8} + e + \frac{27e^{3/2}}{8}\right) \end{aligned}$$

Q11

$f'(x) = \frac{1}{3}x^{-2/3}$. This is a decreasing positive function on $[1, 8]$ so the largest value occurs at $x = 1$. We can use

$$\begin{aligned} S &= f'(1) \\ &= \frac{1}{3} \\ |E_L| &\leq \frac{S(b-a)^2}{2n} \\ &\leq \frac{\frac{1}{3}(8-1)^2}{(2)(14)} \\ &\leq \frac{7}{12} \end{aligned}$$

Q13

$f'(x) = \frac{1}{x \ln 2}$, which is a positive decreasing function on $[2, 8]$. The largest value occurs at $x = 2$.

$$\begin{aligned} S &= f'(2) \\ &= \frac{1}{2\ln 2} \\ |E_L| &\leq \frac{S(b-a)^2}{2n} \\ &\leq \frac{\frac{1}{2\ln 2}(8-2)^2}{2n} \\ \text{Set } \frac{1}{10000} &\geq \frac{9}{n\ln 2} \\ n &\geq \frac{90000}{\ln 2} \\ n &\geq 129,842.553 \end{aligned}$$

We need at least $129,843$ rectangles to guarantee that the error is less that $\frac{1}{10000}$.

Q15

$L_4 < L_8 < M_4 = M_8 =$ actual value $< R_8 < R_4$

Q17

$\Delta x = \frac{16-1}{3}$, the endpoints of the subintervals are 1, 6, 11, and 16.

$$\begin{aligned} T_3 &= \frac{1}{2}\Delta x(f(1) + f(6) + f(6) + f(11) + f(11) + f(16)) \\ &= \frac{1}{2}(5)(0 + 30 + 30 + 110 + 110 + 240) \\ &= 1300 \end{aligned}$$

Q19

$\Delta x = \frac{9-1}{4} = 2$, the midpoints are 2, 4, 6, and 8.

$$\begin{aligned} M_4 &= \Delta x(f(2) + f(4) + f(6) + f(8)) \\ &= 2\left(\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{4\pi}{3}\right) + \cos(3\pi) + \cos\left(\frac{16\pi}{3}\right)\right) \\ &= 2\left(\frac{1}{2} - \frac{1}{2} - 1 - \frac{1}{2}\right) \\ &= -3 \end{aligned}$$

Q21

a $\Delta x = \frac{15-3}{2} = 6$. The midpoints are 6 and 12.

$$\begin{aligned} M_2 &= \Delta x(f(6) + f(12)) \\ &= (6)(11 + 13) \\ &= 144 \end{aligned}$$

b $\Delta x = \frac{18-0}{3} = 6$. The endpoints are 0, 6, 12 and 18.

$$\begin{aligned} T_3 &= \frac{1}{2}\Delta x(f(0) + f(6) + f(6) + f(12) + f(12) + f(18)) \\ &= \frac{1}{2}(6)(10 + 11 + 11 + 13 + 13 + 9) \\ &= 201 \end{aligned}$$

Q23

The error bound on R_n requires a bound K on the second derivative.

$$\begin{aligned} f'(x) &= -3x^{-4} \\ f''(x) &= 12x^{-5} \end{aligned}$$

This is a decreasing function so it is largest at $x = 3$. We can use $K = \frac{12}{3^5}$. We can set the bound in the midpoint remainder theorem to be less than $\frac{1}{10000}$

$$\begin{aligned} \frac{1}{10000} &\geq \frac{K(b-a)^3}{24n^2} \\ \frac{1}{10000} &\geq \frac{12(2)^3}{(3^5)(24)n^2} \\ n^2 &\geq \frac{(10000)(12)(8)}{(3^5)(24)} \\ n &\geq \sqrt{\frac{(10000)(12)(8)}{(3^5)(24)}} \end{aligned}$$

We can simplify this, but the problem does not require us to. The resulting number is between 164 and 165 so in practice we could use any $n \geq 165$.

Q25

a $\Delta x = \frac{12-0}{3} = 4$. The midpoints are 2, 6, and 10.

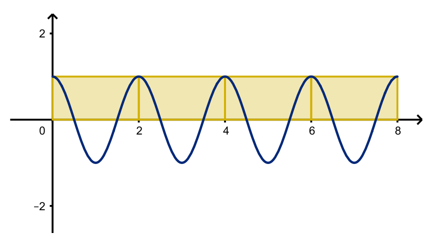
$$\begin{aligned} M_3 &= \Delta x(f(2) + f(6) + f(10)) \\ &= 4(5 + 9 + 4) \\ &= 72 \end{aligned}$$

b

$$\begin{aligned} |E_M| &\leq \frac{K(b-a)^3}{24n^2} \\ &\leq \frac{\frac{1}{4}(12-0)^3}{24(3)^2} \\ &\leq 2 \end{aligned}$$

Q27

We can pick a function where the endpoints have higher values than the points in between them. A trig function would work well. Here is $f(x) = \cos(\pi x)$ for $0 \leq x \leq 8$.



Q29

a $\Delta x = \frac{8-2}{3} = 2$. The endpoints are 2, 4, 6 and 8.

$$
\begin{aligned}
T_3 &= \frac{1}{2}\Delta x(f(2) + f(4) + f(4) + f(6) + f(6) + f(8)) \\
&= \frac{1}{2}(2)(8 + 64 + 64 + 216 + 216 + 512) \\
&= 1080
\end{aligned}
$$

b $f''(x) = 6x$, which is an increasing, positive function. Its greatest value is at $x = 8$.

$$
\begin{aligned}
K &= 6(8) \\
&= 48 \\
|E_M| &= \frac{48(8-2)^3}{24(3)^2} \\
&= 48
\end{aligned}
$$

c The function is concave up on $[2, 8]$. Thus the trapezoids will be above the graph. T_3 overestimates the actual value, meaning our error is positive.

Q31

First, notice $L_n + R_n = 2T_n$, so $U_n = \dfrac{M_n + 3T_n}{4}$. However, we know that M_n and T_n give the exact value of the integral or a linear function. Denote that value as V. We can compute

$$
U_n = \frac{M_n + 3T_n}{4} = \frac{V + 3V}{4} = V
$$

U_n also gives the exact value of the integral.

Q1

An integral of an unbounded region, either because the x-values are unbounded or the function is unbounded.

Q3

Improper integrals are limits. If the limit exists, the integral converges. If it does not exist, the integral diverges.

Q5

1, 3, 4

Q7

a $\lim_{x\to\infty} \frac{x^2+3x+5}{e^x} = 0$

b $\lim_{x\to-\infty} \frac{x^2+3x+5}{e^x} = \infty$

Q9

$\frac{x^2}{x} = x$ except at $x = 0$. Thus they have equal integrals.

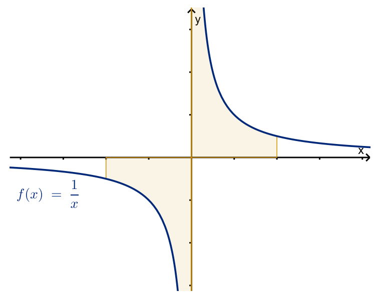
$$\begin{aligned}\int_0^3 \frac{x^2}{x}\,dx &= \int_0^3 x\,dx\\ &= \left.\frac{x^2}{2}\right|_0^3\\ &= \frac{9}{2}\end{aligned}$$

Q11

$$\begin{aligned}
\int_1^8 g(x)\ dx &= \int_1^4 g(x)\ dx + \int_4^6 g(x)\ dx + \int_6^8 g(x)\ dx \\
&= \int_1^4 \sqrt{x}\ dx + \int_4^6 3\ dx + \int_6^8 \frac{1}{x^2}\ dx \\
&= \frac{2x^{3/2}}{3}\bigg|_1^4 + 3x\bigg|_4^6 + -\frac{1}{x}\bigg|_6^8 \\
&= \frac{16}{3} - \frac{2}{3} + 18 - 12 - \frac{1}{8} + \frac{1}{6} \\
&= \frac{257}{24}
\end{aligned}$$

Q13

a



b $\frac{1}{x}$ has a vertical asymptote at $x = 0$. We'll compute the limits from each side. We will use different variables for each limit to avoid confusion, but this is not strictly necessary.

$$\lim_{s \to 0^-} \int_{-2}^{s} \frac{1}{x}\, dx + \lim_{t \to 0^+} \int_{t}^{2} \frac{1}{x}\, dx$$

c We will examine the first limit.

$$\begin{aligned} \lim_{s \to 0^-} \int_{-2}^{s} \frac{1}{x}\, dx &= \lim_{s \to 0^-} \ln|x| \Big|_{-2}^{s} \\ &= \lim_{s \to 0^-} \ln|-s| - \ln 2 \text{ because } s < 0, \text{ we know } |s| = -s \end{aligned}$$

As $x \to 0$, $\ln x$ goes to $-\infty$, so this limit does not exist. We do not need to examine the second limit. We already know this integral diverges.

Q15

This function has vertical asymptotes at $x = 0$ and $x = 4$. We can break up the integral at $x = 1$ (which has a nice square root) and write each half as a limit.

$$\begin{aligned}
\int_0^1 \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{4-x}}\,dx &= \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{4-x}}\,dx \\
&= \lim_{t\to 0^+} 2\sqrt{x} - 2\sqrt{4-x}\Big|_t^1 \\
&= \lim_{t\to 0^+} 2\sqrt{1} - 2\sqrt{3} - 2\sqrt{t} + 2\sqrt{4-t} \\
&= 2 - 2\sqrt{3} + 4 \\
\int_1^4 \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{4-x}}\,dx &= \lim_{t\to 4^-} \int_1^t \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{4-x}}\,dx \\
&= \lim_{t\to 4^-} 2\sqrt{x} - 2\sqrt{4-x}\Big|_1^t \\
&= \lim_{t\to 4^-} 2\sqrt{t} - 2\sqrt{4-t} - 2\sqrt{1} + 2\sqrt{3} \\
&= 4 - 2 + 2\sqrt{3} \\
\int_0^4 \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{4-x}}\,dx &= \int_0^1 \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{4-x}}\,dx + \int_1^4 \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{4-x}}\,dx \\
&= 2 - 2\sqrt{3} + 4 + 4 - 2 + 2\sqrt{3} \\
&= 4
\end{aligned}$$

Q17

a $\Delta x = \frac{16-4}{3} = 4$

b $\Delta x = \frac{b-a}{n}$

c It doesn't matter how big n is. The length of the interval $[a, \infty)$ is infinite. No set of n finite rectangles can cover it.

Q19

$$\begin{aligned}\lim_{t\to-\infty}\int_t^0 e^x\ dx &= \lim_{t\to-\infty} e^x\Big|_t^0\\ &= \lim_{t\to-\infty} e^0 - e^t\\ &= 1\end{aligned}$$

This limit exists, so $\displaystyle\int_{-\infty}^0 e^x\ dx = 1$.

Q21

We used integration by parts to show the antiderivative of $\ln x$ is $x\ln x - x + c$.

$$\begin{aligned}\int_0^1 \ln x\ dx &= \lim_{t\to 0^+}\int_t^1 \ln x\ dx\\ &= \lim_{t\to 0^+} x\ln x - x\Big|_t^1\\ &= \lim_{t\to 0^+} 0 - 1 - t\ln t + t\\ &= -1 - \lim_{t\to 0^+} t\ln t\\ &= -1 - \lim_{t\to 0^+}\frac{\ln t}{\frac{1}{t}} && \frac{\infty}{\infty}\text{ form}\\ &= -1 - \lim_{t\to 0^+}\frac{\frac{1}{t}}{-\frac{1}{t^2}} && \text{l'Hôpital's rule}\\ &= -1 - \lim_{t\to 0^+} -t\\ &= -1\end{aligned}$$

Q23

We break this integral into two limits. For both limits, we'll need an antiderivative of xe^{-x^2}.

$$\int xe^{-x^2}\,dx$$
$$= -\int \frac{1}{2}e^u\,du$$
$$= -\frac{1}{2}e^u + c$$
$$= -\frac{1}{2}e^{-x^2} + c$$

u-substitution

$$u = -x^2$$
$$du = -2x\,dx$$

We choose $a = 0$ as the break point. Both limits must converge for the integral to converge.

$$\lim_{s\to-\infty}\int_s^0 xe^{-x^2}\,dx$$
$$= \lim_{s\to-\infty} -\frac{1}{2}e^{-x^2}\Big|_s^0$$
$$= \lim_{s\to-\infty} -\frac{1}{2} + \frac{1}{2}e^{-s^2}$$
$$= -\frac{1}{2}$$

$$\lim_{t\to\infty}\int_0^t xe^{-x^2}\,dx$$
$$= \lim_{t\to\infty} -\frac{1}{2}e^{-x^2}\Big|_0^t$$
$$= \lim_{t\to\infty} -\frac{1}{2}e^{-t^2} + \frac{1}{2}$$
$$= \frac{1}{2}$$

Both limits converge. This means that $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$ converges. Its value is their sum: $-\frac{1}{2} + \frac{1}{2} = 0$.

Q25

We choose $x = -2$ as the break point and compute the limits.

$$\lim_{s\to-\infty}\int_s^{-2} f(x)\,dx$$
$$= \lim_{s\to-\infty}\int_s^{-2} \frac{1}{x^3}\,dx$$
$$= \lim_{s\to-\infty}\left(-\frac{1}{2x^2}\right)\Big|_s^{-2}$$
$$= \lim_{s\to-\infty} -\frac{1}{8} + \frac{1}{2s^2}$$
$$= -\frac{1}{8}$$

$$\lim_{t\to\infty}\int_{-2}^t f(x)\,dx$$
$$= \lim_{t\to\infty}\int_{-2}^t \frac{1}{(x+4)^2}\,dx$$
$$= \lim_{t\to\infty}\left(-\frac{1}{x+4}\right)\Big|_{-2}^t$$
$$= \lim_{t\to\infty} -\frac{1}{t+4} + \frac{1}{2}$$
$$= \frac{1}{2}$$

So $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}$

Q27

a Each branch of the function is continuous. To check for continuity at $x = 8$ we will compute the limit.

$$\begin{aligned}\lim_{x\to 8^-} f(x) &= \lim_{x\to 8^-} \sqrt[3]{x} \\ &= \sqrt[3]{8} \\ &= 2\end{aligned} \qquad \begin{aligned}\lim_{x\to 8^+} f(x) &= \lim_{x\to 8^+} 10 - x \\ &= 10 - 8 \\ &= 2\end{aligned}$$

So $\lim_{x\to 8} f(x) = 2$. This is equal to $f(8) = 10 - 8 = 2$. We conclude that $f(x)$ is continuous at $x = 8$. Since we already knew it was continuous everywhere else, it is continuous on all real numbers.

b First we should solve for where each branch is above and below the x-axis.

$$\begin{aligned}\sqrt[3]{x} &\geq 0 \\ x &\geq 0\end{aligned} \qquad \begin{aligned}10 - x &\geq 0 \\ 10 &\geq x\end{aligned}$$

The enclosed region is between $x = 0$ and $x = 10$. $f(x) > 0$ on this region.

$$\begin{aligned}\text{Area } &= \int_0^{10} f(x)\ dx \\ &= \int_0^8 f(x)\ dx + \int_8^{10} f(x)\ dx \\ &= \int_0^8 \sqrt[3]{x}\ dx + \int_8^{10} 10 - x\ dx \\ &= \frac{3x^{4/3}}{4}\bigg|_0^8 + 10x - \frac{x^2}{2}\bigg|_8^{10} \\ &= \frac{3(16)}{4} - 0 + 100 - \frac{100}{2} - 80 + \frac{64}{2} \\ &= 14\end{aligned}$$

Q29

a

$$\begin{aligned}
\text{Area} &= \int_1^\infty \frac{1}{x}\,dx \\
&= \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx \\
&= \lim_{t\to\infty} \ln x\Big|_1^t \\
&= \lim_{t\to\infty} \ln t \\
&= \infty
\end{aligned}$$

This interval diverges.

b

$$\begin{aligned}
A(x) &= \pi r^2 \\
&= \frac{\pi}{x^2} \\
\text{Volume} &= \int_1^\infty A(x)\,dx \\
&= \lim_{t\to\infty} \int_1^t \frac{\pi}{x^2}\,dx \\
&= \lim_{t\to\infty} -\frac{\pi}{x}\Big|_1^t \\
&= \lim_{t\to\infty} -\frac{\pi}{t} + \pi \\
&= \pi
\end{aligned}$$

c I find it pretty annoying. Your milage may vary.

Q1

A continuous random variable can take on values on an interval, or union of intervals. A discrete random variable has a finite number of possible outcomes (actually they can have infinitely many, but they have to be spaced out).

Q3

It must be non-negative and it must integrate to 1.

Q5

Infinitely many

Q7

$P(X = 13) = 0$. No outcome of a continuous random variable has positive probability.

Q9

The probability that T is less than or equal to 5.

Q11

$4 \leq U^2 \leq 9$ solves to $-3 \leq U \leq -2$ and $2 \leq U \leq 3$.

$$P(4 \leq U^2 \leq 9) = \int_{-3}^{-2} f_U(u)\ du + \int_{2}^{3} f_U(u)\ du$$

Q13

$$\begin{aligned} P(2 \leq W \leq 9) &= \int_2^9 f_W(w)\ dw \\ &= \int_2^6 f_W(w)\ dw + \int_6^9 f_W(w)\ dw \\ &= \int_2^6 \frac{36 - w^2}{144}\ dw + \int_6^9 0\ dw \\ &= \int_2^6 \frac{1}{4} - \frac{w^2}{144}\ dw \\ &= \left(\frac{w}{4} - \frac{w^3}{432}\right)\Bigg|_2^6 \\ &= \frac{6}{4} - \frac{216}{432} - \frac{2}{4} + \frac{8}{432} \\ &= \frac{14}{27} \end{aligned}$$

Q15

The density function $f_U(u)$ is $\frac{1}{7.5-4} = \frac{2}{7}$ over $[4, 7.5]$ and 0 elsewhere.

$$\begin{aligned} P(U \leq 5.5) &= \int_{-\infty}^{5.5} f_U(u)\ du \\ &= \int_{4}^{5.5} \frac{2}{7}\ du \\ &= \left.\frac{2u}{7}\right|_{4}^{5.5} \\ &= \frac{2(5.5-4)}{7} \\ &= \frac{3}{7} \end{aligned}$$

Q17

We will set the probaility equal to $\frac{2}{7}$ and solve for λ

$$\begin{aligned} P(W \geq 1) =& \frac{2}{7} \\ \int_{1}^{\infty} \lambda e^{-\lambda w}\ dw =& \frac{2}{7} \\ \lim_{t\to\infty} \int_{1}^{t} \lambda e^{-\lambda w}\ dw =& \frac{2}{7} \\ \lim_{t\to\infty} \left. -e^{-\lambda w}\right|_{1}^{t} =& \frac{2}{7} \\ \lim_{t\to\infty} -e^{-\lambda t} + e^{-\lambda} =& \frac{2}{7} \\ e^{-\lambda} =& \frac{2}{7} \\ -\lambda =& \ln \frac{2}{7} \\ \lambda =& \ln \frac{7}{2} \end{aligned}$$

Q19

a In order for $f(x)$ to be positive, we need $b > 0$. To find the exact value we set

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)\,dx &= 1\\
\int_{2}^{\infty} bx^{-3}\,dx &= 1\\
\lim_{t\to\infty}\int_{2}^{t} bx^{-3}\,dx &= 1\\
\lim_{t\to\infty} -\frac{bx^{-2}}{2}\bigg|_{2}^{t} &= 1\\
\lim_{t\to\infty} -\frac{b}{2t^2}+\frac{b}{8} &= 1\\
\frac{b}{8} &= 1\\
b &= 8
\end{aligned}$$

b

$$\begin{aligned}
E[Z] &= \int_{-\infty}^{\infty} xf(x)\,dx\\
&= \int_{2}^{\infty} x8x^{-3}\,dx\\
&= \lim_{t\to\infty}\int_{2}^{t} 8x^{-2}\,dx\\
&= \lim_{t\to\infty} -8x^{-1}\Big|_{2}^{t}\\
&= \lim_{t\to\infty} -\frac{8}{t}+\frac{8}{2}\\
&= 4
\end{aligned}$$

Q21

a You would need to check that $f_X(x) \geq 0$ for all x, and you would need to check that $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$

b

$$\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x)\;dx \\
&= \int_0^4 x\frac{3\sqrt{x}}{16}\;dx && \text{because } f_X(x) = 0 \text{ outside } [0,4] \\
&= \int_0^4 \frac{3}{16}x^{3/2}\;dx \\
&= \frac{3}{40}x^{5/2}\Big|_0^4 \\
&= \frac{3}{40}4^{5/2} - 0 \\
&= \frac{12}{5}
\end{aligned}$$

Q23

The expected value of a uniform random variable is the midpoint of the interval. $\dfrac{5.2+9.4}{2} = 7.3.$

Q25

We can't divide by 0, but we also can't define a uniform random variable on an interval of length 0. There is no function that we can integrate over a single point and get 1.

Q27

$E[X] = \frac{1}{p}$, while $E[Y] = \frac{1}{2p}$, so $E[Y]$ is half as large as $E[X]$.

Q29

a $\lambda = 3$ so

$$f_X(x) = \begin{cases} 3e^{-3x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

b $E[X] = \dfrac{1}{\lambda} = \dfrac{1}{3}$

c

$$\begin{aligned} P(X > 1) &= \int_1^\infty 3e^{-3x}\ dx \\ &= \lim_{t\to\infty} \int_1^t 3e^{-3x}\ dx \\ &= \lim_{t\to\infty} -e^{-3x}\Big|_1^t \\ &= \lim_{t\to\infty} -e^{3t} + e^{-3} \\ &= e^{-3} \end{aligned}$$

Q31

Let m be the median. m is the solution to $P(X \geq m) = 0.5$

$$\begin{aligned} P(X \geq m) &= 0.5 \\ \int_m^\infty f_X(x)\ dx &= 0.5 \\ \int_m^b \frac{1}{b-a}\ dx + \int_b^\infty 0\ dx &= 0.5 \\ \int_m^b \frac{1}{b-a}\ dx &= 0.5 \\ \frac{1}{b-a}\ x\Big|_m^b &= 0.5 \\ \frac{b}{b-a} - \frac{m}{b-a} &= 0.5 \\ b - m &= 0.5(b-a) \\ \frac{a+b}{2} &= m \end{aligned}$$

Q33

Let m be the median. m is the solution to $P(T \geq m) = 0.5$

$$
\begin{aligned}
P(T \geq m) &= 0.5 \\
\int_m^\infty f_T(t)\ dt &= 0.5 \\
\int_m^1 \frac{2\sqrt{t}}{3}\ dt + \int_1^\infty 0\ dx &= 0.5 \\
\int_m^1 \frac{2\sqrt{t}}{3}\ dt &= 0.5 \\
t^{3/2}\Big|_m^1 &= 0.5 \\
1 - m^{3/2} &= 0.5 \\
\frac{1}{2} &= m^{3/2} \\
\frac{1}{2^{2/3}} &= m
\end{aligned}
$$

Q35

$[3,4]$ should be half of the interval $[a,b]$. There are a few ways to accomplish this.

- If $[3,4]$ covers the right half of $[a,b]$, then 3 is the midpoint of $[a,b]$ so $3 < b \leq 4$ and $a = 6 - b$.
- If $[3,4]$ covers the left half of $[a,b]$, then 4 is the midpoint of $[a,b]$ so $3 \leq a < 4$ and $b = 8 - a$.
- If $[3,4]$ covers the middle of $[a,b]$, then $[a,b]$ has length 2. $2 \leq a \leq 3$ and $b = a + 2$.

Q37

a

$$
\begin{aligned}
\int_{-\infty}^\infty g(x)\ dx &= \int_{-\infty}^0 g(x)\ dx + \int_0^\infty g(x)\ dx \\
1 &= \int_{-\infty}^0 g(x)\ dx + 0 \\
1 &= \int_{-\infty}^0 g(x)\ dx
\end{aligned}
$$

In words, since X cannot be greater than 0, $P(X \leq 0) = 1$.

b $E[X] = \displaystyle\int_{-\infty}^\infty xg(x)\ dx$. Since $g(x) \geq 0$ when $x \leq 0$ and $g(x) = 0$ when $x > 0$, the integrand of this is either 0 or negative for all x. Thus $E[X] < 0$.

Q1

A function of a random variable is itself a random variable.

Q3

We assume that the random variable is uniform over the specified interval.

Q5

We could let $Y = 60X$.

Q7

Dominic made $\$P$ where $P = 200V - 12000$.

Q9

$E[Y] = \int_{-\infty}^{\infty} cx f_X(x)\, dx$. By the constant multiple rule, we can factor out the c and get c times the formula for $E[X]$.

Q11

Y ranges from 20 to 50. For a and b in this range, $P(a < Y < b) = P\left(\frac{a}{10} < x < \frac{b}{10}\right) = \frac{\frac{1}{10}(b-a)}{3} = \frac{b-a}{30}$. This indicates that Y is a uniform random variable. Its density function is

$$f_Y(y) = \begin{cases} \frac{1}{30} & \text{if } 20 \leq y \leq 50 \\ 0 & \text{otherwise} \end{cases}$$

Q13

$$\begin{aligned}
E\left[\frac{1}{W}\right] &= \int_{-\infty}^{\infty} \frac{1}{w} f_W(w)\ dw \\
&= \int_{-\infty}^{0} \frac{1}{w} 0\ dw + \int_{0}^{6} \frac{1}{w}\frac{36-w^2}{144}\ dw + \int_{6}^{\infty} \frac{1}{w} 0\ dw \\
&= \int_{0}^{6} \frac{1}{4w} - \frac{w}{144}\ dw \\
&= \lim_{t\to 0^+} \int_{t}^{6} \frac{1}{4w} - \frac{w}{144}\ dw && \text{vertical asymptote at } x = 0 \\
&= \lim_{t\to 0^+} \frac{\ln w}{4} - \frac{w^2}{288}\bigg|_{t}^{6} \\
&= \lim_{t\to 0^+} \frac{\ln t}{4} - \frac{t^2}{288} - \frac{\ln 6}{4} + \frac{36}{288} \\
&= -\infty
\end{aligned}$$

This integral diverges, so the expected value of $1/W$ is undefined.

Q15

$$\begin{aligned}
E[X^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)\ dx \\
&= \int_{-\infty}^{0} 0\ dx + \int_{0}^{\infty} x^2 \lambda e^{-\lambda x}\ dx \\
&= \lim_{t\to\infty} \int_{0}^{t} x^2 \lambda e^{-\lambda x}\ dx \\
&= \lim_{t\to\infty} -x^2 e^{-\lambda x}\bigg|_0^t + \int_0^t 2x e^{-\lambda x}\ dx && u = x^2 \qquad dv = \lambda e^{-\lambda x}\ dx \\
&= \lim_{t\to\infty} -x^2 e^{-\lambda x}\bigg|_0^t - \frac{2x e^{-\lambda x}}{\lambda}\bigg|_0^t + \int_0^t \frac{2}{\lambda} e^{-\lambda x}\ dx && u = 2x \qquad dv = e^{-\lambda x}\ dx \\
&= \lim_{t\to\infty} -x^2 e^{-\lambda x} - \frac{2x e^{-\lambda x}}{\lambda} - \frac{2e^{-\lambda x}}{\lambda^2}\bigg|_0^t \\
&= \lim_{t\to\infty} -\frac{t^2}{e^{\lambda t}} - \frac{2t}{\lambda e^{\lambda t}} - \frac{2}{\lambda^2 e^{\lambda t}} + 0 + 0 + \frac{2}{\lambda^2}
\end{aligned}$$

The ratios with t are of indeterminate form. We can use l'Hôpital's rule to show that their limits are 0. Thus $E[X^2] = \dfrac{2}{\lambda^2}$.

Q17

a This means that $\int_a^b f(x)\,dx = 0$. Thus the signed area under $y = f(x)$ from a to b is 0. In other words, the area above the x-axis is equal to the area below the x-axis.

b In this case, there is no area below the x-axis. Thus there is also no area above the x-axis. The function has value 0 from $x = a$ to $x = b$.

Q19

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{3-0}\int_0^3 x^2\,dx \\ &= \left.\frac{x^3}{9}\right|_0^3 \\ &= \frac{27}{9} - 0 \\ &= 3 \end{aligned}$$

Q21

$$\begin{aligned} f_{\text{ave}} &= \frac{1}{2-0}\int_0^2 x^2e^{3x}\,dx \\ &= \frac{1}{2}\left(\left.\frac{x^2e^{3x}}{3}\right|_0^2 - \int_0^2 \frac{2xe^{3x}}{3}\,dx\right) && \text{by parts } u = x^2, dv = e^{3x}dx \\ &= \frac{1}{2}\left(\left.\frac{x^2e^{3x}}{3}\right|_0^2 - \left.\frac{2xe^{3x}}{9}\right|_0^2 + \int_0^2 \frac{2e^{3x}}{9}\,dx\right) && \text{by parts } u = 2x, dv = e^{3x}/3dx \\ &= \frac{1}{2}\left.\left(\frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2e^{3x}}{27}\right)\right|_0^2 \\ &= \frac{1}{2}\left(\frac{4e^6}{3} - \frac{4e^6}{9} + \frac{2e^6}{27} - 0 + 0 - \frac{2e^0}{27}\right) \\ &= \frac{13e^6}{27} - \frac{1}{27} \end{aligned}$$

Q23

We already know $E[X] = \frac{1}{\lambda}$ from an earlier example and $E[X^2] = \frac{2}{\lambda^2}$ from a previous exercise. We'll look for opportunities to apply this as we evaluate the variance formula.

$$\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty} \left(x - \frac{1}{\lambda}\right)^2 f_X(x)\ dx \\
&= \int_0^{\infty} \left(x - \frac{1}{\lambda}\right)^2 \lambda e^{-\lambda x}\ dx \\
&= \int_0^{\infty} \left(x^2 - \frac{2x}{\lambda} + \frac{1}{\lambda^2}\right) \lambda e^{-\lambda x}\ dx \\
&= \int_0^{\infty} x^2 \lambda e^{-\lambda x}\ dx - \int_0^{\infty} \frac{2x}{\lambda} \lambda e^{-\lambda x}\ dx + \int_0^{\infty} \frac{1}{\lambda^2} \lambda e^{-\lambda x}\ dx \\
&= \int_0^{\infty} x^2 \lambda e^{-\lambda x}\ dx - \frac{2}{\lambda} \int_0^{\infty} x \lambda e^{-\lambda x}\ dx + \frac{1}{\lambda^2} \int_0^{\infty} \lambda e^{-\lambda x}\ dx \\
&= E[X^2] - \frac{2}{\lambda} E[X] + \frac{1}{\lambda^2}(1) \\
&= \frac{2}{\lambda^2} - \frac{2}{\lambda}\frac{1}{\lambda} + \frac{1}{\lambda^2} \\
&= \frac{1}{\lambda^2}
\end{aligned}$$

Q25

First we compute $E[W]$

$$\begin{aligned} E[W] &= \int_{-\infty}^{\infty} w f_W(w)\ dw \\ &= \int_{-\infty}^{0} w f_W(w)\ dw + \int_{0}^{6} w f_W(w)\ dw + \int_{6}^{\infty} w f_W(w)\ dw \\ &= \int_{0}^{6} w \frac{36 - w^2}{144}\ dw \\ &= \int_{0}^{6} \frac{w}{4} - \frac{w^3}{144}\ dw \\ &= \frac{w^2}{8} - \frac{w^4}{576}\bigg|_0^6 \\ &= \frac{36}{8} - \frac{1296}{576} \\ &= \frac{9}{4} \end{aligned}$$

Next we can compute the variance.

$$\begin{aligned} \sigma^2 &= \int_{-\infty}^{\infty} \left(w - \frac{9}{4}\right)^2 f_W(w)\ dw \\ &= \int_{-\infty}^{0} \left(w - \frac{9}{4}\right)^2 f_W(w)\ dw + \int_{0}^{6} \left(w - \frac{9}{4}\right)^2 f_W(w)\ dw + \int_{6}^{\infty} \left(w - \frac{9}{4}\right)^2 f_W(w)\ dw \\ &= \int_{0}^{6} \left(w - \frac{9}{4}\right) \frac{36 - w^2}{144}\ dw \\ &= \int_{0}^{6} \left(w^2 - \frac{9w}{2} + \frac{81}{16}\right)^2 \frac{36 - w^2}{144}\ dw \\ &= \int_{0}^{6} -\frac{w^4}{144} + \frac{w^3}{32} + \frac{55w^2}{256} - \frac{9w}{8} + \frac{81}{64}\ dw \\ &= -\frac{w^5}{720} + \frac{w^4}{128} + \frac{552}{768} - \frac{9x^2}{16} + \frac{81w}{64}\bigg|_0^6 \\ &= \frac{171}{80} \end{aligned}$$

Q27

If $X > a$, then $Y > ca$. These events have the same probability. We can convert that equivalence into a statement about integrals.

$$P\left(a \leq X \leq b\right) = P(ca \leq Y \leq cb)$$
$$\int_a^b f_X(x)\ dx = \int_{ca}^{cb} f_Y(y)\ dy$$

The change in bounds could be achieved by a u-substituion. If we apply this substitution to the integrand, we will know that the two integrals area equal.

$$ucx$$
$$du = c\ dx$$
$$\frac{1}{c}\ du = dx$$
$$P(a \leq x \leq b) = int_a^b f_X(x)\ dx = \int_{ca}^{cb} \frac{1}{c} f_X(u)\ du$$

So $\frac{1}{c} f_X(u) = \frac{1}{c} f_X(cx)$ is a function we can integrate from ca to cb to obtain the probability that $a \leq X \leq b$ and $ca \leq Y \leq cb$. Thus if we change the name of the variable, we get a density function for Y:

$$f_Y(y) = \frac{1}{c} f_X(cy)$$

Q29

We'll begin with the formula we have and derive $E[X^2] - E[X]^2$

$$\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x)\ dx \\
&= \int_{-\infty}^{\infty} (x^2 - 2xE[X] + (E[X])^2) f_X(x)\ dx \\
&= \int_{-\infty}^{\infty} x^2 f_X(x)\ dx - \int_{-\infty}^{\infty} 2xE[X] f_X(x)\ dx + \int_{-\infty}^{\infty} (E[X])^2 f_X(x)\ dx && \text{sum rule} \\
&= \int_{-\infty}^{\infty} x^2 f_X(x)\ dx - 2E[X] \int_{-\infty}^{\infty} x f_X(x)\ dx + (E[X])^2 \int_{-\infty}^{\infty} f_X(x)\ dx && \text{constant multiple rule} \\
&= E[X^2] - 2E[X]E[X] + (E[X])^2(1) \\
&= E[X^2] - 2(E[X])^2 + (E[X])^2 \\
&= E[X^2] - (E[X])^2
\end{aligned}$$

Q1

They approximate the values of the original function near the center, but they are easier to evaluate.

Q3

We would need to be able to evaluate $\ln x$ and its derivatives $\frac{1}{x}$, $-\frac{1}{x^2}$ etc at the center. The only place where all of these are rational numbers is at $x = 1$.

Q5

a $\sqrt[3]{6} = f(6)$.

b I'd expect $L(6)$ to overestimate $f(6)$. $f(x)$ is concave down over positive values of x, so it curves below its tangent lines. This means the values (heights) of the tangent line are above the values of f.

Q7

Yes, $T_1(x) = f(a) + f'(a)(x - a) = L(x)$.

Q9

The coefficient of the $(x - a)$ term is the first derivative. It should be negative.

Q11

a $T_8(x)$ requires 8 derivatives. Fortunately $f(x) = e^x$ has repetitive derivatives.

$$\begin{aligned} f(x) &= e^x & f(0) &= 1 \\ f'(x) &= e^x & f'(0) &= 1 \\ f''(x) &= e^x & f''(0) &= 1 \\ &\vdots & &\vdots \end{aligned}$$

We can plug these into the summation formula:

$$\begin{aligned} T_8(x) &= \sum_{k=0}^{8} \frac{f^{(k)}(0)}{k!}(x-0)^k \\ &= \sum_{k=0}^{8} \frac{1}{k!}x^k \\ &= \frac{1}{0!}(1) + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \frac{1}{7!}x^7 + \frac{1}{8!}x^8 \end{aligned}$$

b To approximate e, we approximate $f(1) = e^1$ by evaluating $T_8(1)$.

$$\begin{aligned} T_8(1) &= \frac{1}{0!}(1) + \frac{1}{1!}1 + \frac{1}{2!}1^2 + \frac{1}{3!}1^3 + \frac{1}{4!}1^4 + + \frac{1}{5!}1^5 + \frac{1}{6!}1^6 + \frac{1}{7!}1^7 + \frac{1}{8!}1^8 \\ &= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} \end{aligned}$$

There is a bit more arithmetic to do here. It could be done by hand in a reasonable amount of time.

c Since $f^{(k)}(0) = 1$ for all k, we can write

$$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!}x^k$$

Q13

The derivatives follow the pattern $\cos x$, $-\sin x$, $-\cos x$, $\sin x \ldots$. At $x = \pi$ these are $-1, 0, 1, 0, -1, \ldots$. The Taylor polynomial is

$$T_{10}(x) = -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4 + \frac{1}{6!}(x-\pi)^6 - \frac{1}{8!}(x-\pi)^8 + \frac{1}{10!}(x-\pi)^{10}$$

Q15

a $\displaystyle\sum_{k=0}^{4} 15(3)^k$

b $\displaystyle\sum_{k=0}^{6} 24 - 5k$

c $\displaystyle\sum_{k=2}^{7} \frac{1}{2k^2}$

Q17

We compute the first few derivatives to see if we can find an expression for $f^{(k)}(x)$

$$\begin{aligned} f(x) &= \ln(x) & f(1) &= 0 \\ f'(x) &= x^{-1} & f'(1) &= 1 \\ f''(x) &= -x^{-2} & f''(1) &= -1 \\ f'''(x) &= 2x^{-3} & f'''(1) &= 2 \\ f^{(4)}(x) &= -6x^{-4} & f^{(4)}(1) &= -6 \\ f^{(5)}(x) &= 24x^{-5} & f^{(5)}(1) &= 24 \end{aligned}$$

These answers look like factorials, but they're shifted by 1. They're also alternating signs, which we can model with $(-1)^k$, except that the even powers are negative. The power of x is $-k$. One way to model this is $f^{(k)}(x) = (-1)^{k+1}(k-1)!x^{-k}$. Plugging in $x = 1$ gives $f^{(k)}(1) = (-1)^{k+1}(k-1)!$ except at $k = 0$. For that case we compute $\ln 1 = 0$. This means we can leave it out of the summation. The form for the remaining terms allows for some nice simplification.

$$\begin{aligned} T_{53}(x) &= \sum_{k=1}^{53} \frac{(-1)^{k+1}(k-1)!}{k!}(x-1)^k \\ &= \sum_{k=1}^{53} \frac{(-1)^{k+1}}{k}(x-1)^k \end{aligned}$$

Q19

We compute the first few derivatives to find a pattern.

$$\begin{aligned} f(x) &= \cos x & f(0) &= 1 \\ f'(x) &= -\sin x & f'(0) &= 0 \\ f''(x) &= -\cos x & f''(0) &= -1 \\ f'''(x) &= \sin x & f'''(0) &= 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1 \\ f^{(5)}(x) &= -\sin x & f^{(5)}(0) &= 0 \end{aligned}$$

The coefficient is only nonzero for even order derivatives, so we can use x^{2k} to produce only even terms. To create a degree 100 polynomial, k will go to 50. Since they alternate, we can use $(-1)^k$ as the derivative for each.

$$T_{100}(x) = \sum_{k=0}^{50} \frac{(-1)^k}{(2k)!} x^{2k}$$

Q21

In some cases the Taylor function is exactly equal to the function. For example a linear function is exactly equal to its Taylor polynomials of degree ≥ 1. In this case the rror is 0. More generally, any method with a minimum error is unlikely to be much use, as we want to be able to make our errors small, preferably arbitrarily small.

Q23

We'll use Taylor's inequality. $|R_4(5)| \leq \left|\frac{M}{5!}(5-1)^5\right|$. M is a bound on $f^{(5)}(x)$ on $[1,5]$.

$$f^{(5)}(x) = \frac{d}{dx} f^{(4)}(x) = 3x^2 e^{x^3}$$

This derivative is increasing (at least where x is positive), so its largest value occurs at $x = 5$.

$$M = f^{(5)}(5) = (3)(25)e^{125}$$

Putting this into Taylor's inequality gives us $|R_4(5)| \leq \left|\frac{75e^{125}}{5!}(4)^5\right|$. This tells us that the difference between $T_4(5)$ and $f(5)$ is no larger than $\frac{75e^{125}}{5!}(4)^5$.

Q25

a The bound on $R_3(x)$ produced by Taylor's Inequality is dependent on $f^{(4)}(x)$.

b $f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$. We cannot put a bound on this, because $\lim_{x\to 0^+} -\frac{15}{16}x^{-7/2} = -\infty$.

c $f^{(4)}(x)$ decreases in absolute value as x increases. Thus on the interval $[4, 5]$, it has largest absolute value at $x = 4$. $f^{(4)}(4) = -\frac{15}{16}4^{-7/2} = -\frac{15}{2048}$. We can write the bound

$$|f^{(4)}(x)| \leq \left|\frac{15}{2048}\right|$$

d

$$\begin{aligned} |R_3(5)| &\leq \left|\frac{\frac{15}{2048}}{4!}(5-4)^4\right| \\ &\leq \frac{5}{16384} \end{aligned}$$

Q27

a All the derivatives of e^x are e^x and have value 1 at $x = 0$.

$$T_5(x) = \sum_{k=0}^{5} \frac{1}{k!}x^k$$

b

$$\frac{1}{\sqrt{e}} = e^{-1/2} \approx T_5\left(-\frac{1}{2}\right) = \sum_{k=0}^{5} \frac{1}{2^k k!}$$

c Taylor's Inequality requires a bound M of $|f^{(6)}(x)|$ over $\left[-\frac{1}{2}, 0\right]$. $f^{(6)}(x) = e^x$, which is positive and increasing. Its largest value over $\left[-\frac{1}{2}, 0\right]$ is at $x = 0$. We can use $M = |e^0| = 1$.

$$\begin{aligned} \left|R_5\left(-\frac{1}{2}\right)\right| &\leq \left|\frac{M}{6!}\left(-\frac{1}{2} - 0\right)^6\right| \\ &\leq \frac{1}{2^6 6!} \end{aligned}$$

Q29

a We compute the first 4 derivatives of f at 0:

$$\begin{aligned} f(x) &= \cos 3x & f(0) &= 1 \\ f'(x) &= -3\sin 3x & f'(0) &= 0 \\ f''(x) &= -9\cos 3x & f''(0) &= -9 \\ f'''(x) &= 27\sin 3x & f'''(0) &= 0 \\ f^{(4)}(x) &= 81\cos 3x & f^{(4)}(0) &= 81 \end{aligned}$$

$$\begin{aligned} T_4(x) &= \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} x^k \\ &= 1 - \frac{9}{2}x^2 + \frac{81}{24}x^4 \end{aligned}$$

b

$$\cos\frac{3\pi}{4} = f\left(\frac{\pi}{4}\right) \approx T_4\left(\frac{\pi}{4}\right) = 1 - \frac{9}{2}\left(\frac{\pi}{4}\right)^2 + \frac{81}{24}\left(\frac{\pi}{4}\right)^4$$

c Taylor's inequality requires a bound on $\left|f^{(5)}(x)\right|$ over $\left[0, \frac{\pi}{4}\right]$. $f^{(5)}(x) = -243\sin 3x$, which is not stritcly increasing or decreasing on $\left[0, \frac{\pi}{4}\right]$. Instead we can use the fact that $-0 \leq \sin x \leq 1$ and use $M = |(-243)(1)| = 243$.

$$\begin{aligned} \left|R_5\left(\frac{\pi}{4}\right)\right| &\leq \left|\frac{M}{5!}\left(\frac{\pi}{4} - 0\right)^5\right| \\ &\leq \frac{243}{120}\left(\frac{\pi}{4}\right)^5 \\ &\leq \frac{81\pi^5}{40960} \end{aligned}$$

Q31

a We compute the first 3 derivatives of f at $x = 2$:

$$\begin{aligned} f(x) &= x^3 - 3x + 5 & f(2) &= 7 \\ f'(x) &= 3x^2 - 3 & f'(2) &= 9 \\ f''(x) &= 6x & f''(2) &= 12 \\ f'''(x) &= 6 & f'''(2) &= 6 \end{aligned}$$

$$\begin{aligned} T_3(x) &= \sum_{k=0}^{3} \frac{f^{(k)}(2)}{k!}(x-2)^k \\ &= 7 + 9(x-2) + 6(x-2)^2 + (x-2)^3 \end{aligned}$$

b Taylor's inequality bounds $|R_3(x)|$ using a bound on the fourth derivative of f. The fourth derivative of f is 0 at all x. Thus $|R_3(x)| \leq 0$, meaning $R_3(x) = 0$.

c Since the error is always 0 for all x, this suggests that $f(x) = T_3(x)$ for all x.

d

$$\begin{aligned} T_3(x) &= 7 + 9(x-2) + 6(x-2)^2 + (x-2)^3 \\ &= 7 + 9x - 18 + 6x^2 - 24x + 24 + x^3 - 6x^2 + 12x - 8 \\ &= x^3 - 3x + 5 \\ &= f(x) \end{aligned}$$

e If $f(x)$ is a degree n polynomial, then the nth (or higher) Taylor polynomial of $f(x)$ is equal to $f(x)$. We can verify this because $|R_n(x)| \leq \left|\frac{M}{(n+1)!}(x-a)^{n+1}\right|$, but M is a bound on $f^{(n+1)}(x) = 0$. This tells us that $R_n(x) = 0$ for all x.

Q1

The index for a sqeuence only takes integer values, we typically use x for real numbers.

Q3

When $a_n = f(n)$ for all integers n (perhaps ignoring some number of initial terms), and $\lim_{x\to\infty} f(x)$ exists.

Q5

There are multiple ways to express these, especially since the problem does not specifiy what value of n to start with. The following solutions all begin with $n = 1$.

a $a_n = n^2 + 1$ or $\{n^2+1\}_{n=1}^{\infty}$

b $a_n = \frac{-3}{(-2)^n}$ or $\left\{\frac{-3}{(-2)^n}\right\}_{n=1}^{\infty}$

c $a_n = \frac{1}{n(n+1)}$ or $\left\{\frac{1}{n(n+1)}\right\}_{n=1}^{\infty}$

Q7

The distance between $\frac{\sin n}{n^2}$ and 0 is less than $\frac{1}{n^2}$ which we can make as small as we want by choosing a large n.

Q9

No. It could be increasing, but slower and slower so that it never exceeds a certain number. For instance $\frac{n-1}{n}$ is increasing, but all the terms are less than 1.

Q11

a $f(x) = 2^x$.

b No, it goes to ∞.

c No. The theorem requires that $\lim_{x\to\infty} f(x)$ converges to L.

d Yes. Since each term of 2^n is twice the term before it, 2^n eventually grows much larger than any finite number, so its limit cannot exist.

Q13

n dominates $\log n$, so this limits to 0. We could use l'Hôpital's rule instead.

Q15

This is a ratio of polynomials. They have the same degree, so the limit is the ratio of their leading coefficients. $\lim_{n\to\infty} \frac{n^3+3}{4n^3-9} = \frac{1}{4}$.

Q17

e^n dominates $\sqrt{n}$ so this limit is ∞. We could use l'Hôpital's rule instead.

Q19

$n!$ dominates 5^n, so $\lim_{n\to\infty} \frac{n!}{5^n} = \infty$.

Q21

Each term in $a_n = n!$ is n times as large as the last. Each term in $b_n = n^n$ is $\frac{n^n}{(n-1)^{n-1}} = \left(\frac{n}{n-1}\right)^{n-1} n$ times as large. This suggests n^n grows faster than $n!$.

Q23

Only the limit of $g(x)$ is relevent. When computing a limit we can ignore any finite number of terms. In this case specifically, ignoring the firt 342 terms shows that $f(x)$ has no bearing on the limit of the sequence.

Q1

A sequence is an ordered list of nmumbers. A series is a sum of a list of numbers.

Q3

A geometric series has a constant ratio between consecutive terms. If this ratio r has absolute value less than 1, the sum is $\frac{a}{1-r}$.

Q5

a k^p becomes a factor of $\frac{(k+1)^p}{k^p}$, which limits to 1.

b c^k becomes a factor of $\frac{c^{k+1}}{c^k} = c$. The ratio test takes an absolute value so this factor contributes c.

c $k!$ becomes a factor of $\frac{(k+1)!}{k!} = k+1$. This limits to ∞ (unless something cancels it out).

Q7

a e

b $\frac{2}{3}$

Q9

a $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}$

b $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$

c $\lim\limits_{n\to\infty} \frac{n}{n+1} = 1$

d Notice that $\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$. Thus we have

$$\begin{aligned} s_n &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots + \frac{1}{n(n+1)} \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \\ &= \frac{n}{n+1} \end{aligned}$$

This is called a **telescoping series** because when we write each term as a difference, the components cancel and the expression collapses into part of the first term minus part of the last term.

Q11

The sums are

$$\begin{aligned} &-1 = -1 \\ &-1 + 1 = 0 \\ &-1 + 1 - 1 = -1 \\ &-1 + 1 - 1 + 1 = 0 \end{aligned}$$

The sequence of partial sums looks like it will continue to oscillate between -1 and 0, meaning the limit does not exist. This series diverges.

Q13

By our argument $s_{2^n} > 1 + \frac{n}{2}$ so if we set $20 = 1 + \frac{n}{2}$ and solve, we get $n = 38$. That means $s_{2^{38}} > 20$.

Q15

A geometric series has a common ratio between terms. The ratios between these terms are $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$, which are not the same. This is not a geometric series.

Q17

The common ratio $r = \dfrac{7.5}{5} = 1.5$. The next term is $7.5 * 1.5 = 11.25$.

Q19

$a = 5$ and $r = 0.3$. $|r| < 1$, so $\displaystyle\sum_{k=0}^{\infty} 5(0.3)^k = \frac{5}{1-0.3} = \frac{50}{7}$.

Q21

$a = \dfrac{15}{125} = \dfrac{3}{25}$. $r = \frac{1}{5}$. Since $|r| < 1$,

$$\begin{aligned}\sum_{j=3}^{\infty} \frac{15}{5^j} &= \frac{\frac{3}{25}}{1-\frac{1}{5}} \\ &= \frac{\frac{3}{25}}{\frac{4}{5}} \\ &= \frac{3}{20}\end{aligned}$$

Q23

Our first term (when $k = 4$) is $a = \dfrac{81}{(16)(18)} = \dfrac{9}{64}$. $r = \frac{3}{2}$. Since $|r| > 1$, this series diverges.

Q25

The common ratio is $\frac{3}{z}$. For convergence we need

$$\begin{aligned}\left|\frac{3}{z}\right| &< 1 \\ -1 < \frac{3}{z} &< 1 \\ z < -3 \text{ or } z &> 3\end{aligned}$$

Q27

$$\begin{aligned} s_n &= a_1 + a_2 + a_3 + \cdots + a_k \\ &\geq \frac{1}{100} + \frac{1}{100} + \frac{1}{100} + \cdots + \frac{1}{100} \\ &\geq \frac{n}{100} \end{aligned}$$

Thus $s_n > \frac{n}{100}$.

Q29

$\lim\limits_{k\to\infty} \dfrac{1}{k^3} = 0$, so the divergence test doesn't tell us whether this diverges or converges.

Q31

$\lim\limits_{k\to\infty} \ln k = \infty$, so the divergence test tells us that this series diverges.

Q33

Yes. If $L > 1$ then the terms of the series are growing in magnitude. They either diverge or go to $\pm\infty$. Assuming we can compute $\lim\limits_{n\to\infty} a_n$ at all, its value will not be 0.

Q35

First we compute the ratio, and then we take a limit.

$$\begin{aligned} \left|\frac{a_{k+1}}{a_k}\right| &= \left|\frac{\frac{(k+1)!}{4^{k+1}}}{\frac{k!}{4^k}}\right| \\ &= \left|\frac{(k+1)!4^k}{k!4^{k+1}}\right| \\ &= \left|\frac{k+1}{4}\right| && \text{(cancel the matching factors)} \\ &= \frac{k+1}{4} && \text{(since } k+1>0\text{)} \\ \lim_{k\to\infty} \frac{k+1}{4} &= \infty \end{aligned}$$

The limit of the ratios is infinite, so by the ratio test, the series diverges.

Q37

$$\begin{aligned}
\left|\frac{a_{k+1}}{a_k}\right| &= \left|\frac{\frac{(-1)^k}{(k+1)^2}}{\frac{(-1)^{k-1}}{k^2}}\right| \\
&= \left|\frac{(-1)^k(k^2+2k+1)}{(-1)^{k-1}k^2}\right| \\
&= \left|\frac{-(k^2+2k+1)}{k^2}\right| && \text{(cancel the factors of } -1) \\
&= \frac{(k^2+2k+1)}{k^2} && \text{(apply the absolute value)}
\end{aligned}$$

$$\lim_{k\to\infty}\frac{(k^2+2k+1)}{k^2} = 1$$

The limit of the ratios is 1. The ratio test is indeterminate.

Q39

$$\begin{aligned}
\left|\frac{a_{k+1}}{a_k}\right| &= \left|\frac{\frac{(k+1)^2}{4^{k+1}}}{\frac{k^2}{4^k}}\right| \\
&= \left|\frac{(k+1)^2 4^k}{k^2 4^{k+1}}\right| \\
&= \left|\frac{k^2+2k+1}{4k^2}\right| && \text{(cancel the factors of } 4) \\
&= \frac{k^2+2k+1}{4k^2} && \text{(since the fraction is positive)}
\end{aligned}$$

$$\lim_{k\to\infty}\frac{k^2+2k+1}{4k^2} = \frac{1}{4}$$

The limit $\frac{1}{4}$ is less than 1. By the ratio test, this series converges.

Q41

$$\begin{aligned}
\left|\frac{a_{k+1}}{a_k}\right| &= \left|\frac{\frac{\sqrt{k+2}}{(k+1)^2}}{\frac{\sqrt{k+1}}{k^2}}\right| \\
&= \left|\frac{k^2\sqrt{k+2}}{(k+1)^2\sqrt{k+1}}\right| \\
&= \frac{k^2}{k^2+2k+1}\sqrt{\frac{k+2}{k+1}}
\end{aligned}$$

$$\lim_{k\to\infty}\frac{k^2}{k^2+2k+1}\sqrt{\frac{k+2}{k+1}} = 1\sqrt{1} = 1$$

Since the limit is 1, the ratio test is inconclusive.

Q43

Apply the divergence test. $\lim_{k\to\infty} \dfrac{k+1}{k} = 1$, which is not 0. Thus the series diverges.

Q45

We apply the divergence test first. By dominance, $\lim_{k\to\infty} \frac{ke^k}{4^{k+1}} = \infty$. Since this is not 0, we know the series diverges.

Q47

In both cases, we cannot directly compute the value using normal methods. However, we can compute any finite part of it. We evaluate both of these by taking a limit. In the integral case, we let the upper bound go to ∞. In the series case, we let the length of the partial sums go to ∞.

Q49

a The left endpoints are $0, 1, 2, 3$ and 4. The sum of the areas is $1(e^0 + e^{-1} + e^{-2} + e^{-3} + e^{-4})$.

b We would need infinitely many.

c We could represent the sum of all the areas by $\displaystyle\sum_{k=0}^{\infty} e^{-k}$.

d This is a geometric series with initial term 1 and ratio $\frac{1}{e}$. Its sum is $\dfrac{1}{1-\frac{1}{e}} = \dfrac{e}{e-1}$.

e $f(x) = e^{-x}$ is decreasing, which means any L_n will overestimate $\int_0^n f(x)\,dx$. This suggests that the limit of the L_n will be at least as large as $\int_0^\infty f(x)\,dx$.

Q51

a All the values of f_X are non-negative. We need to check that they sum to 1. Since $f_X(x)$ is only positive for positive integers, the sum of the values is the series $\sum_{k=1}^{\infty} \frac{1}{2^k}$. This is a geometric series with initial term $\frac{1}{2}$ and common ratio $\frac{1}{2}$. It converges to $\frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$. Thus the values of $f_X(x)$ sum to 1. We conclude that $f_X(x)$ is a valid probability distribution function.

b We add up the probabilities for all outcomes greater than 4. We obtain a geometric series.

$$\begin{aligned}
P(X > 4) &= f_X(5) + f_X(6) + f_X(7) + \cdots \\
&= \sum_{k=5}^{\infty} \frac{1}{2^k} && a = \frac{1}{32} \qquad r = \frac{1}{2} \\
&= \frac{\frac{1}{32}}{1-\frac{1}{2}} \\
&= \frac{1}{16}
\end{aligned}$$

c The expected value formula breaks down into a sum of infintely many geometric series.

$$\begin{aligned}
E[X] &= \sum_{k=1}^{\infty} \frac{k}{2^k} \\
&= \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \cdots \\
&= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \\
&\quad + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \\
&\quad + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \\
&\quad + \frac{1}{16} + \frac{1}{32} + \cdots \\
&\quad + \frac{1}{32} + \cdots \\
&\quad \vdots
\end{aligned}$$

Applying the geometric series formula to each of these sums gives

$$\begin{aligned}
E[X] &= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \\
&= \frac{1}{1-\frac{1}{2}} && \text{by the geometric series formula} \\
&= 2
\end{aligned}$$

Q1

A power series is a limit of partial sums which are polynomials. It is denoted like a polynomial of infinite degree, while a polynomial has finite many terms.

Q3

We can integrate a power series term-by-term.

Q5

a It would be convenient to use the exponents as an index variable. The coefficients are counting up by 5 from an initial value of 10.

$$10 + 15x + 20x^2 + 25x^3 + 30x^4 + \cdots = \sum_{k=0}^{\infty} (10 + 5k)x^k$$

b In this case the exponents are only even numbers, so we can use twice the index variable as our exponents. The coefficients form a geometric sequence with initial term $\frac{1}{2}$ and common ratio $-\frac{1}{2}$

$$\frac{1}{2} - \frac{1}{4}x^2 + \frac{1}{8}x^4 - \frac{1}{16}x^6 + \frac{1}{32}x^8 - \cdots = \sum_{k=0}^{\infty} \frac{1}{2}\left(-\frac{1}{2}\right)^k x^{2k}$$

Q7

a $r = 4x^2$

b The series has intial term 1 and common ratio $r = 4x^2$.

$$\begin{aligned} p(x) &= 1 + 4x^2 + 16x^4 + 64x^6 + \cdots \\ &= \sum_{k=0}^{\infty} 4^k x^{2k} \end{aligned}$$

c The doimain of $p(x)$ is the set of x for which it converges. According to the sum of a geometric series, $p(x)$ converges when

$$\begin{aligned} |r| &< 1 \\ 4x^2 &< 1 \\ x^2 &< \frac{1}{4} \\ -\frac{1}{2} < x &< \frac{1}{2} \end{aligned}$$

Q9

No. The ratio between terms is $\dfrac{a_{k+1}}{a_k} = \dfrac{4(k+1)^3(x+7)}{k^3}$, which is different for different values of k. This is not a geometric series, so the sum of a geometric series formula does not apply.

Q11

We know the ratio test will be an invalid at $x = 3$. The power series is 0 at $x = 3$. For all other x, we compute the ratios.

$$\begin{aligned}\frac{a_{k+1}}{a_k} &= \frac{2^{k+1}(x-3)^{k+1}}{2^k(x-3)^k} \\ &= 2(x-3)\end{aligned}$$

There is no k in the expression so this series has a constant ratio. It is a geometric series, so it only converges when $|r| < 1$. We solve

$$\begin{aligned}|2(x-3)| &< 1 \\ |x-3| &< \frac{1}{2} \\ -\frac{1}{2} < x - 3 &< \frac{1}{2} \\ \frac{5}{2} < x &< \frac{7}{2}\end{aligned}$$

Beacuse this series is geometric, there is no uncertainly about the endpoints. Geometric series diverge when $r = \pm 1$. Thus the domain of the power series is $\left(\frac{5}{2}, \frac{7}{2}\right)$

Q13

The ratio between terms will be undefinted at the center $x = 6$, but we know the series converges there. Everywhere else we can compute

$$\begin{aligned}\frac{a_{k+1}}{a_k} &= \frac{\left(\frac{1}{4}\right)^{k+1}(x-6)^{k+1}}{\left(\frac{1}{4}\right)^{k}(x-6)^{k}} \\ &= \frac{x-6}{3}\end{aligned}$$

This is a geometric series, so it only converges when $|r| < 1$. We solve

$$\begin{aligned}\left|\frac{x-6}{4}\right| &< 1 \\ |x-6| &< 4 \\ -4 < x - 6 &< 4 \\ 2 < x &< 10\end{aligned}$$

The domain is $(2, 10)$.

Q15

The ratio between terms will be undefinted at the center $x = -3$, but we know the series converges there. Everywhere else we can compute

$$\begin{aligned} \frac{a_{k+1}}{a_k} &= \frac{(k+1)(x+3)^{k+1}}{k(x+3)^k} \\ &= \frac{(k+1)(x+3)}{k} \end{aligned}$$

This is not constant. Since this is not a geometric series, we apply the ratio test.

$$\begin{aligned} \lim_{k\to\infty}\left|\frac{(k+1)(x+3)}{k}\right| &< 1 \\ |x+3| &< 1 \qquad\qquad \text{radius } = 1 \\ -1 < x+3 &< 1 \\ 2 < x &< 4 \end{aligned}$$

The series converges on $(2, 4)$. This test does not tell us whether it converges at the endpoints.

Q17

The ratio between terms will be undefinted at the center $x = 5$, but we know the series converges there. Everywhere else we can compute

$$\begin{aligned} \frac{a_{k+1}}{a_k} &= \frac{3^k 4(k+1)(x-5)^{k+1}}{3^{k+1}4k(x-5)^k} \\ &= \frac{(k+1)(x-5)}{3k} \end{aligned}$$

This is not constant. Since this is not a geometric series, we apply the ratio test.

$$\begin{aligned} \lim_{k\to\infty}\left|\frac{(k+1)(x-5)}{3k}\right| &< 1 \\ \left|\frac{x-5}{3}\right| &< 1 \\ |x-5| &< 3 \qquad\qquad \text{radius } = 3 \\ -3 < x-5 &< 3 \\ 2 < x &< 8 \end{aligned}$$

The series converges on $(2, 8)$. This test does not tell us whether it converges at the endpoints.

Q19

$$\int \sum_{k=0}^{\infty} 2^k (x-3)^k \;\; dx = \sum_{k=0}^{\infty} \frac{2^k}{k+1}(x-3)^{k+1} + c$$

Q21

$\frac{d}{dx}\sum_{k=0}^{\infty}\frac{1}{4^k}(x-6)^k = \sum_{k=1}^{\infty}\frac{k}{4^k}(x-6)^{k-1}$. It has the same radius of convergence we computed in an earlier exercise. It converges on $(2, 10)$, though we don't know the whether the endpoints are included.

Q23

The fifth derivative has the same radius of convergence as the original series. In an earlier exercise we computed this radius to be 1. Thus the fifth derivative converges on $(2, 4)$, though we don't know the whether the endpoints are included.

Q25

a We'll apply the ratio test.

$$\begin{aligned}
\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| &= \lim_{k\to\infty}\left|\frac{\frac{(k+1)^2+(k+1)}{5^{k+1}}(x+3)^{k+1}}{\frac{k^2+k}{5^k}(x+3)^k}\right| \\
&= \lim_{k\to\infty}\left|\frac{((k+1)^2+(k+1))(x+3)}{5(k^2+k)}\right| \\
&= \lim_{k\to\infty}\left|\frac{(k^2+3k+2)(x+3)}{5(k^2+k)}\right| \\
&= \left|\frac{x+3}{5}\right|
\end{aligned}$$

The ratio test says the series converges when this limit is less than 1 so

$$\begin{aligned}
\left|\frac{x+3}{5}\right| &< 1 \\
-1 < \frac{x+3}{5} &< 1 \\
-5 < x+3 &< 5 \\
-8 < x &< 2
\end{aligned}$$

The domain is the interval $(-8, 2)$. The endpoints may be part of the domain or they may not.

b We apply the theorem that allows us to integrate a power series term by term.

$$\begin{aligned}
\int p(x)\ dx &= C + \sum_{k=0}^{\infty}\frac{k^2+k}{5^k(k+1)}(x+3)^{k+1} \\
&= C + \sum_{k=0}^{\infty}\frac{k}{5^k}(x+3)^{k+1}
\end{aligned}$$

Q27

$f'(x) = \frac{1}{1+x^2}$. This is a geometric series of initial term 1 and common ratio $-x^2$. We express it as

$$f'(x) = \sum_{k=0}^{\infty}(-x^2)^k = \sum_{k=0}^{\infty}(-1)^k x^{2k}$$

$$\int f'(x)\ dx = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1} + c$$

The domain is where $|-x^2| < 1$, which is the interval $(-1, 1)$. On this interval $\tan^{-1} x$ is one of the antiderivatives of $f'(x)$. To get the correct one we solve for c. We plug in a value ($x = 0$ is easiest) and solve.

$$\begin{aligned} \tan^{-1} 0 = 0 &= \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}0^{2k+1} + c \\ 0 &= c \end{aligned}$$

Thus $\tan^{-1} x = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1}$ on $(-1, 1)$.

Q1

We can show that the error bound given by Taylor's inequality converges to 0 as the degree goes to ∞.

Q3

As long as the center is 0, we can mutliply the Taylor series for $f(x)$ by x^n or compose it with $g(x) = x^n$, and the result will still be a Taylor series.

Q5

We can't compute the sum of an infinite series by hand. We can only approximate $T(1.25)$ by its partial sums, $T_n(1.25)$. These are the Taylor polynomials. The series adds no information to the computation that we could not obtain with the Taylor polynomials.

Q7

The coefiicients would be multiplied by e and the x^k would be replaced with $(x-1)^k$.

Q9

We will compute the first few derivatives and look for a pattern.

$$\begin{aligned} f(x) &= \frac{1}{x} \\ f'(x) &= -\frac{1}{x^2} \\ f''(x) &= \frac{2}{x^3} \\ f'''(x) &= -\frac{6}{x^4} \\ f^{(4)}(x) &= -\frac{24}{x^5} \end{aligned}$$

From this we infer $f^{(k)}(x) = (-1)^k \dfrac{k!}{x^{k+1}}$. We can now write the Taylor series

$$\begin{aligned} T(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(4)}{k!}(x-4)^k \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \frac{k!}{4^{k+1}}}{k!}(x-4)^k \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{4^{k+1}}(x-4)^k \end{aligned}$$

Q11

We will compute the first eight derivatives of $f(x)$.

$$\begin{aligned} f(x) &= \cos x & f(0) &= 1 \\ f'(x) &= -\sin x & f'(0) &= 0 \\ f''(x) &= -\cos x & f''(0) &= -1 \\ f'''(x) &= \sin x & f'''(0) &= 0 \\ f^{(4)}(x) &= \cos x & f^{(4)}(0) &= 1 \\ f^{(5)}(x) &= -\sin x & f^{(5)}(0) &= 0 \\ f^{(6)}(x) &= -\cos x & f^{(6)}(0) &= -1 \\ f^{(7)}(x) &= \sin x & f^{(7)}(0) &= 0 \\ f^{(8)}(x) &= \cos x & f^{(8)}(0) &= 1 \end{aligned}$$

Since we only need the even terms, we can use $2k$ as the exponents of x. The derivatives alternate between 1 and -1 so $(-1)^k$ will model these.

$$T(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$$

Q13

We begin by applying Taylor's inequality to obtain a bound on $|R_n(x)|$. Every derivative of $f(x)$ is e^x, a positive, increasing function. We conclude

- If $x > 0$, then $f^{(n+1)}$ is bounded by e^x on $[0, x]$
- If $x < 0$, then $f^{(n+1)}$ is bounded by $e^0 = 1$ on $[x, 0]$

Taylor's inequality states

$$|R_n(x)| \leq \left| \frac{e^x}{(n+1)!} x^{n+1} \right| \qquad \text{or} \qquad |R_n(x)| \leq \left| \frac{1}{(n+1)!} x^{n+1} \right|$$

Both of these limit to 0 as $n \to \infty$. Thus $T(x) = f(x)$ for all x.

Q15

We will apply Taylor's inequality to obtain a bound on $|R_n(x)|$. Begin with the computation, shown in a previous exercise, that $f^{(k)}(x) = (-1)^k \dfrac{k!}{x^{k+1}}$. $|f^{(k)}(x)|$ is a decreasing function on $(2, 6)$.

- If we are approximating at $x < 4$ we can use $M = \dfrac{(n+1)!}{x^{n+2}}$ as a bound on $|f^{(n+1)}(x)|$
- If we are approximating at $x \geq 4$ we can use $M = \dfrac{(n+1)!}{4^{n+2}}$ as a bound on $|f^{(n+1)}(x)|$

$$|R_n(x)| \leq \left| \frac{\frac{(n+1)!}{x^{n+2}}}{(n+1)!}(x-4)^{n+1} \right| \quad \text{or} \quad |R_n(x)| \leq \left| \frac{\frac{(n+1)!}{4^{n+2}}}{(n+1)!}(x-4)^{n+1} \right|$$

$$\leq \left| \frac{1}{x^{n+2}}(x-4)^{n+1} \right| \quad \text{or} \quad \leq \left| \frac{1}{4^{n+2}}(x-4)^{n+1} \right|$$

$$\leq \frac{1}{x}\left(\frac{4-x}{x}\right)^{n+1} \quad \text{or} \quad \leq \frac{1}{4}\left(\frac{x-4}{4}\right)^{n+1}$$

Since $\frac{4-x}{x} < 1$ for x between 2 and 4 and $\frac{x-4}{4} < 1$ for x between 4 and 6 (actually 4 and 8), we conclude that both of these limit to 0 as $n \to \infty$. Thus $\lim\limits_{n\to\infty} |R_n(x)| = 0$, meaning $T(x) = f(x)$ on $(2, 6)$.

Q17

In our computation of Taylor's inequality for $|R_n(x)|$, we noted that for $x > 1$, we can use $M = n!$ for our bound on $f^{(n+1)}$ over $[1, x]$. We apply Taylor's iequality to $x = 2$.

$$\begin{aligned} |R_n(2)| &\leq \left| \frac{n!}{(n+1)!}(2-1)^{n+1} \right| \\ &\leq \frac{1}{n+1} \end{aligned}$$

Thus $\lim\limits_{\vec{n}\to\infty} |R_n(2)| = 0$. The error in the Taylor polynomials at $x = 2$ goes to 0 as the degree goes to infinity. In other words, $T(2) = \ln 2$, meaning $T(2)$ converges.

Q19

By previous work we have a Taylor series for $\cos x$. We multiply this by x^5.

$$\begin{aligned} T(x) &= x^5 \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k+5} \end{aligned}$$

Q21

Not easily. Distributing the x^2 across the Taylor series would give us a series of the form

$$T(x) = \sum_{k=0}^{\infty} c_k x^2 (x-1)^k$$

This does not fit the definition of a power series, as $c_k x^2$ is not a constant.

Q23

Let $f(x) = e^{x^3}$. We obtain the following series for $f(x)$ by substituing x^3 into our Taylor series $T(x)$ for e^x.

$$e^{x^3} = T(x^3) = \sum_{k=0}^{\infty} \frac{1}{k!} x^{3k}$$

The antiderivative is

$$\begin{aligned} g(x) &= \int \sum_{k=0}^{\infty} \frac{1}{k!} x^{3k}\ dx \\ &= \sum_{k=0}^{\infty} \frac{1}{(3k+1)k!} x^{3k+1} + c \end{aligned}$$

Q25

We can differentiate term by term

$$\begin{aligned} T'(x) &= \sum_{k=0}^{\infty} \frac{2k(-1)^k}{(2k)!} x^{2k-1} \\ T''(x) &= \sum_{k=1}^{\infty} \frac{2k(2k-1)(-1)^k}{(2k)!} x^{2k-2} && \text{(the } k = 0 termdifferentiatesto0\text{)} \\ T''(x) &= \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k-2)!} x^{2k-2} && \text{(cancel from the factorial)} \end{aligned}$$

Why does this make sense? Both $2k$s became $(2k-2)$s, so a substitution could simplify things. Set $j = k - 1$ and rewrite the summation:

$$T''(x) = \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(2j)!} x^{2j}$$

Notice that this is the same as our expression for $T(x)$ except that we've changed the name of the index variable (which doesn't change the value) and we've added an extra power of -1. We conclude that $T''(x) = -T(x)$, which makes sense because $\frac{d^2}{dx^2} \cos x = -\cos x$.

Q27

We had $\cos(2x) = \cos^2 x - \sin^2 x$. The pythagorean identity lets us substitue $1 - \cos^2 x$ for $\sin^2 x$. We combine like terms to obtain

$$\cos(2x) = 2\cos^2 x - 1$$

Q29

$e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1 + 0i = 1$

Q31

a First we compute the derivatives

$$\begin{aligned} f(x) &= \frac{1}{x^2} & f(4) &= \frac{1}{16} \\ f'(x) &= -\frac{2}{x^3} & f'(4) &= -\frac{1}{32} \\ f''(x) &= \frac{6}{x^4} & f'(4) &= \frac{3}{128} \\ f'''(x) &= -\frac{24}{x^5} & f'(4) &= -\frac{3}{128} \end{aligned}$$

From this we obtain

$$T_3(x) = \frac{1}{16} - \frac{1}{32}(x-4) + \frac{3}{256}(x-4)^2 - \frac{1}{256}(x-4)^3$$

b We would need a bound on $|f^{(4)}(x)| = |\frac{120}{x^6}|$. This positive and decreasing on $[2.5, 4]$ so it's largest value occurs at 2.5. We can use $M = \frac{120}{2.5^6}$. Taylor's inequality states

$$|R_3(2.5)| \leq \left| \frac{120}{2.5^6 4!}(2.5-4)^4 \right|$$

c The general derivative of f appears to be $f^{(5)}(x) = \frac{(-1)^k (k+1)!}{x^{k+2}}$. This means

$$\begin{aligned} T(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k (k+1)!}{4^{k+2} k!}(x-4)^k \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (k+1)}{4^{k+2}}(x-4)^k \end{aligned}$$

The ratio test gives us a ratio between terms of

$$\begin{aligned} \left| \frac{a_{n+1}}{a_n} \right| &= \left| \frac{\frac{(-1)^{n+1}(n+2)}{4^{n+3}}(x-4)^{n+1}}{\frac{(-1)^n (n+1)}{4^{n+2}}(x-4)^n} \right| \\ &= \left| \frac{(n+2)(x-4)}{4(n+1)} \right| \end{aligned}$$

We apply the ratio test

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &< 1\\
\lim_{n\to\infty}\left|\frac{(n+2)(x-4)}{4(n+1)}\right| &< 1\\
\left|\frac{x-4}{4}\right| &< 1\\
|x-4| &< 4\\
-4 < x-4 &< 4\\
0 < x &< 8
\end{aligned}$$

The series converges on $(0, 8)$. We do not know about the endpoints from the ratio test, though the divergence test will show that $T(x)$ diverges at both endpoints.

Q33

a Yes. A Taylor series is a type of power series, and the domain of a power series centered at 10 is an interval centered at 10. If 5 is in the domain, then so is every point between 5 and 10 (in fact, 5 and 15).

b If the error approaches 0, this means the Taylor polynomials approach $f(x)$, but the Taylor polynomials are the partial sums of the Taylor series. Thus, the Taylor series converges to $f(x)$. On the other hand, just because the Taylor series converges, does not mean that it converges to $f(x)$.

c We should just use the Taylor polynomials. The Taylor series is of no use to compute $f(7)$, because we do not know how to evaluate it. We could approximate it using partial sums, but the partial sums are the Taylor polynomials.

Q1

a in the x-direction, b in the y-direction and c in the z-direction

Q3

The y-axis is the points that can be reached from the origin by a displacement in the y-direction. A general point is $(0, b, 0)$.

Q5

We can sketch the graph in the plane without that variable, then we can extend that graph in the direction of the free variable.

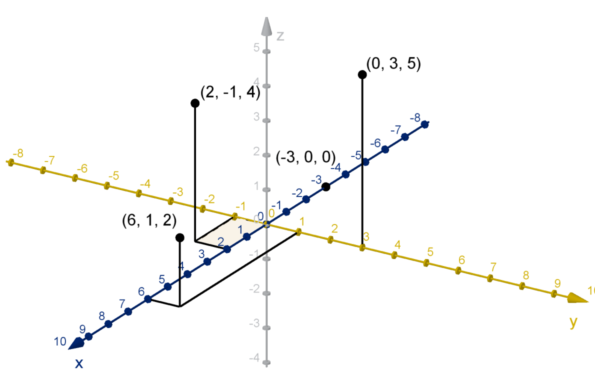
Q7

Points on the y axis, could not be expressed this way, since the line through P and the origin would be vertical. Also, coordinates would not uniquely identify a point. Given any distance d and any slope m, there are two points that lie on both $x^2 + y^2 = d^2$ and $y = mx$. We would not be able to tell which one the coordinates were referring to.

Q9

a, b are the usual orientation. The others are not.

Q11



Q13

$$\begin{aligned} D &= \sqrt{(7-3)^2 + (3-6)^2 + (-10-2)^2} \\ &= \sqrt{16 + 9 + 144} \\ &= \sqrt{169} \\ &= 13 \end{aligned}$$

Q15

$$\begin{aligned} D &= \sqrt{(11-10)^2 + (9-12)^2 + (105-109)^2} \\ &= \sqrt{1 + 9 + 16} \\ &= \sqrt{26} \end{aligned}$$

Q17

No. If we plug the coordinates into the equation we get $8 = 4^2 - 2$. This is false, so $(4, 3, 8)$ does not lie on the graph of $z = x^2 - 2$.

Q19

This is an empty graph. It contains no points. Since the square of any real numer is non-negative, there are no coordinates y and z that will make $y^2 + z^2 = -1$ a true equation.

Q21

Any point that has an x-coordinate of 2 or a y-coordinate of 3 lies on this graph. Thus the graph consists of the vertical line $x = 2$ and the horizontal line $y = 3$.

Q23

No. The graphs are parallel planes, extending in the x and y directions. They do not intersect. Also, if they did intersect then they would contain a point (x, y, z), but z would have to be both 4 and 6 to lie on both graphs.

Q25

a $x=-4$ is a plane parallel to the yz plane through $(0,0,-4)$.

b $x^2+y^2=9$ is a circle of radius 3 in the xy plane, projected in the z direction to make a cylinder.

c We will complete the square to handle the $4x$ and $-2z$ terms.

$$\begin{aligned}
x^2+4x+y^2+z^2-2z&=4\\
x^2+4x+4+y^2+z^2-2z+1&=9\\
(x+2)^2+y^2+(z-1)^2&=9
\end{aligned}$$

This is a sphere of radius 3, centered at $(-2,0,1)$.

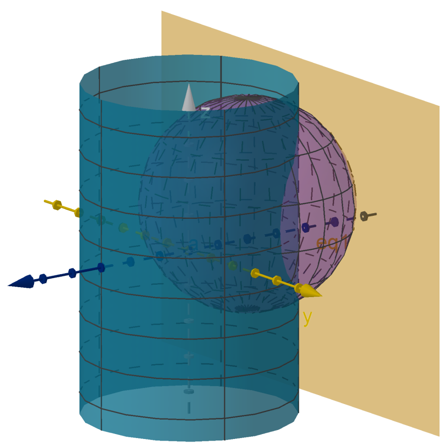
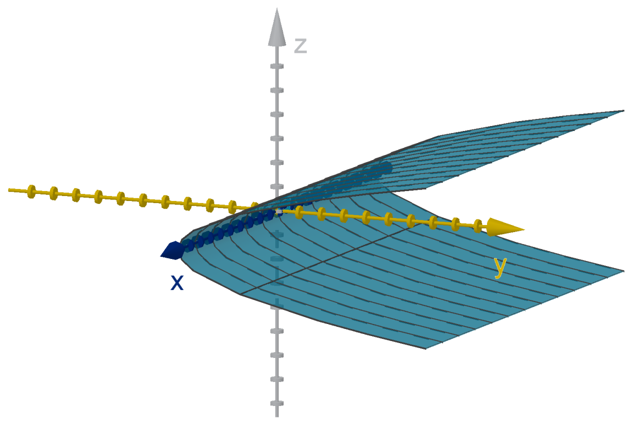


Figure: A plane, a cylinder, and a sphere

Q27



Q29

In general, an equation seems to reduce dimension by 1 from the ambient space. We expect this to be a 5-dimensional graph.

Q31

Maybe not. $y = x^2$ is a curve in the xy-plane but a surface in three-space. Without more information, we don't know which graph is "correct".

Q33

If y does not appear in the equation of the plane, then m_y must be 0.

Q35

It is easiest to compute m_x, if we have two points with the same y-coordinate. Given the points we have, a y-coordinate of 5 or 2 would be best.

Q37

a $z+1=3(x-2)-(y-5)$

b Set $x=y=0$.

$$\begin{aligned}z+1&=3(0-2)-(0-5)\\z+1&=-6+5\\z&=10\end{aligned}$$

Q39

We use $(3,0,0)$ and $(0,0,-1)$ to get $m_x=\frac{-1-0}{0-3}=\frac{1}{3}$. We use $(0,7,0)$ and $(0,0,-1)$ to get $m_x=\frac{-1-0}{0-7}=\frac{1}{7}$. $(0,0,-1)$ is the z-intercept. The equation is

$$z=\frac{1}{3}x+\frac{1}{7}y-1$$

Q41

We use $(6,7,-2)$, and $(8,7,1)$ to get $m_x=\frac{1-(-2)}{8-6}=\frac{3}{2}$. We use $(6,4,1)$ and $(6,7,-2)$ to get $m_y=\frac{7-4}{-2-1}=-1$. We don't have an intercept, but we can call it b and solve for it. We will plug in $(6,4,1)$, but any point will work.

$$\begin{aligned}z&=\frac{3}{2}x-y+b\\1&=\frac{3}{2}6-4+b\\-4&=b\\z&=\frac{3}{2}x-y-4\end{aligned}$$

Q43

Between the points $(3,4,2)$ and $(7,4,6)$, the only change is z is attributable to the change in x, since y doesn't change. We can get $m_x = \frac{7-3}{6-2} = 1$. From $(3,4,2)$ o $(5,5,6)$, we can first increase x by 2. With a slope of 1, this gets us to $(5,4,4)$. With this we can compute $m_y = \frac{6-4}{5-4} = 2$. Finally, we can plug $(3,4,2)$ into the equation we have so far and solve for b.

$$\begin{aligned} z &= x + 2y + b \\ 2 &= 3 + (2)(4) + b \\ -9 &= b \\ z &= x + 2y - 9 \end{aligned}$$

Q45

We could draw a sphere of radius 5 in $x_1x_3x_4$-space. Since x_2 is a free variable, we could extend this sphere in the x_2-direction.

Q47

A point on the x_2x_4-plane is displaced in only the x_2- and x_4-directions. Its x_1- and x_3-coodinates are 0. The equations to describe this are $x_1 = x_3 = 0$.

Q49

S contains two points that are $D = \sqrt{0^2 + 3^2 + 4^2} = 5$ units apart. There is no upper bound to how large the sphere can be. Even very large spheres have points that are close together. However, if the sphere is too small, it will not have any points 5 units apart. The farthest apart two points can be is the endpoints of a diameter. If the diameter, is less than 5, there will not be two points 5 units apart. Thus the radius must be equal to or larger than $\frac{5}{2}$.

Q51

No. There are many ways to demonstrate this. For instance, if x and y made a 60 degree angle, then $(0,0)$, $(1,0)$ and $(0,1)$ would make an equilateral triangle, suggesting that the distance from $(0,1)$ to $(1,0)$ is 1. The distance formula disagrees. It computes a distance of $\sqrt{2}$.

Q1

The value of the function f at (x, y).

Q3

Level curves and level surfaces.

Q5

$f(2, -8) = 13(2) + \frac{-8}{2} = 26 - 4 = 22$

Q7

No. The plus or minus means there are two possible outputs for some inputs. For instance, if $x = 2$ and $y = -1$, then is $f(2, -1) = 3$ or -3?

Q9

The only obstacle to evaluating this function is that we cannot divide by 0. The domain is all of $\mathbb{R}^2$ except the line $y = -x$.

Q11

The only obstacle to evaluating this function is that we cannot take the square root of a negative. The domain is all of $\mathbb{R}^2$ where $y^2 \geq 25$. This solves to $y \geq 5$ or $y \leq -5$, so the domain is all of $\mathbb{R}^2$ above $y = 5$ or below $y = -5$.

Q13

There are two obstacles to evaluating this function. We cannot have a negative square root so $x \geq -3$. We cannot divide by zero so $y^2 \neq x$. The domain is the region of $\mathbb{R}^2$ to the right of $x = -3$ except for the parabola $x = y^2$.

Q15

$(-84.38, 35.75)$ is two unit in the y direction (north) from $(-84.38, 33.75)$. According to the map, the temperatures there around 50 degrees, which is less than 59.

Q17

Observe South Dakota on the map (after looking up which state is South Dakota if necessary). The colors range from dark purple (10 degrees) in the northeast corner to green (40 degrees) in the southwest corner.

Q19

The integer points in the rectangle defined by $0 \le x \le 687$ and $0 \le y \le 1024$.

Q21

No. If f is a function, then each ordered pair can have at most one output. If these points are on the graph, that implies that $f(1,3) = 5$ and $f(1,3) = 7$.

Q23

The z-axis is where $x = y = 0$. If the graph passes through $(0,0,c)$, then $f(0,0) = c$.

Q25

The level curves have equations of the form $(x-2)^2 + (y+1)^2 = c$. These are circles of various radii, centered at $(2,-1)$.

Q27

The level curves have the form $\frac{x^2}{y} = c$ or $y = \frac{1}{c}x^2$. These are parabolas openning upward or dowanward with a vertex at $(0,0)$.

Q29

The level curve has equation of the form $x^3 + y^3 = c$. Plug in $(4,2)$ to solve for c: $4^3 + 2^3 = c$. The equation is

$$x^3 + y^3 = 72$$

Q31

Zero or one. If more than one passed through $(3,7)$ we would have $f(3,7) = c_1$ and $f(3,7) = c_2$. This would violate the property that a function can have only a single output for a given input.

Q33

In Kansas, the temperature is increasing more quickly as you travel south than if you're travelling south through, for example, Indiana, Kentucky, Tenessee and Georgia.

Q35

The farm fields are relatively flat. The ground does not rise or fall significantly enough to cross a level curve anywhere on the fields.

Q37

If we set $y = 2$ we obtain the equation $z = x^3 + 8$. The graph of this equation in the xz-plane is the $y = 2$ cross-section.

Q39

It tells us that $f(x, y)$ does not depend on y, so y is a free variable.

Q41

Solving for z gives $z = \sqrt{y - x^2}$ and $z = -\sqrt{y - x^2}$.

Q43

This equation has a z^2, a z and terms with no z's at all, so it is a quadratic in z

$$z^2 + xyz + (x^2 + y^2 - 20) = 0$$

$$z = \frac{-xy \pm \sqrt{x^2y^2 - 4(1)(x^2 + y^2 - 20)}}{2}$$

Q45

a $\mathbb{R}^6$

b $\mathbb{R}^5$

Q47

a $\mathbb{R}^2$

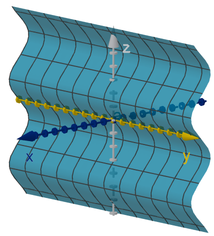
b In $\mathbb{R}$. A typical level set is a set of points (or numbers) on the real line.

Q49

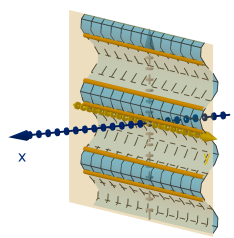
y is a free variable, so for each point on the graph, we can travel in the y direction and remain on the graph. On other words, the graph is the curve $z = x^2$ in the xz-plane, extended in the y direction.

Q51

a We can sketch this by drawing a sine wave veritically in the xz-plane, and then extending it in the y-direction.



b In the xz-plane $x = \frac{1}{2}$ meets $x = \sin z$ at $\left(\frac{1}{2}, \frac{\pi}{6}\right)$, $\left(\frac{1}{2}, \frac{5\pi}{6}\right)$, $\left(\frac{1}{2}, \frac{13\pi}{6}\right)$, etc. These intersections extend in the y direction to create an infinite set of parallel horizontal lines.



Q1

One variable is changing, the other(s) is treated as a constant.

Q3

Many examples are possible. Any situation in which changing one variable would cause other variables to change as well would be a valid example.

Q5

This line is tangent to the graph in the y-direction, since x is fixed at 2. Thus its slope is $f_y(2,0)$.

$$f_y(x,y) = 3x^2 e^{3xy}$$
$$f_y(2,0) = 12$$

In point-slope, the equation is

$$z - 4 = 12(y - 0)$$

.

Q7

$f_x(x,y) = 14x - 5y\sin x$. The y in $5y\cos x$ is treated as a constant multiple. The e^y term is treated as a constant (derivative 0).

Q9

As we travel in the x-direction, we pass from higher values to lower values of f. Thus $f_x(3,0) < 0$.

Q11

Since we're approximating f_x, we should choose a point with the same y-value. The point is approximately $(-2.25, 1.25)$. We can compute the rate of change in z with respect to x:

$$f_x(4,-1.25) \approx \frac{50-40}{-2.25-4} = -\frac{10}{1.75} = -\frac{40}{7}$$

Q13

a $f_x = 2x$

$f_y = -2y$

b We can rewrite this as $f(x, y) = y^{1/2}x^{-1/2}$. In each partial derivative, the other factor is a constant multiple.

$f_x = y^{1/2}\left(-\frac{1}{2}x^{-3/2}\right)$

$f_y = \frac{1}{2}y^{-1/2}x^{-1/2}$

c For f_x we use the chain rule.

- The outer function is ye^x. Its derivative is ye^x.
- The inner function is xy. Its derivative is y.
- By the chain rule

$$\frac{\partial}{\partial x}ye^x = ye^{xy}y = y^2e^{xy}$$

For f_y we cannot treat the initial y as a constant multiple. We need the product rule. We still use the chain rule on the second factor e^{xy}.

- The outer function is e^x. Its derivative is e^x.
- The inner function is xy. Its derivative is x.

We are now ready to apply the product rule.

$$\begin{aligned}\frac{\partial}{\partial y}ye^{xy} &= \left(\frac{\partial}{\partial y}y\right)e^{xy} + y\left(\frac{\partial}{\partial y}e^{xy}\right)\\ &= (1)e^{xy} + ye^{xy}x\\ &= (1+xy)e^{xy}\end{aligned}$$

Q15

$f_x(x, y, z) = \lim_{h\to\vec{0}} \dfrac{f(x+h, y, z) - f(x, y, z)}{h}$. The two points where we evaluate the function have the same y and z coordinates. Thus y and z are treated as constant in this rate of change.

Q17

$\frac{\partial g}{\partial v} = ue^{uv+w^2}$.

Q19

$P = \frac{nrT}{V}$ suggests that more volume leads to smaller P. A partial derivative, which suggests that V is constant will overstate the actual growth rate of P, in which V is increasing.

Q21

$$
\begin{aligned}
g &= f_t && = \frac{\partial}{\partial t} f \\
h &= (f_t)_s && = \frac{\partial}{\partial s}\frac{\partial}{\partial t} f \\
&= f_{ts} && = \frac{\partial^2 f}{\partial s \partial t}
\end{aligned}
$$

Q23

First we compute f_y. The chain rule gives us $f_y = \cos(3x + x^2 y)(x^2)$.

Computing $(f_y)_x$ will require the product rule and the chain rule.

$$
\begin{aligned}
\frac{\partial}{\partial x}\cos(3x + x^2 y)(x^2) &= \frac{\partial}{\partial x}\left(\cos(3x + x^2 y)\right)(x^2) + \cos(3x + x^2 y)\frac{\partial}{\partial x}\left(x^2\right) \\
&= -\sin(3x + x^2 y)(3 + 2xy)(x^2) + \cos(3x + x^2 y)(2x)
\end{aligned}
$$

Q25

a Using the product rule, we obtain $\frac{\partial g}{\partial y} = e^{xy^2} + 2xy^2 e^{xy^2}$.

b

$$
\begin{aligned}
\frac{\partial g}{\partial x} &= 6x^2 z + y^3 e^{xy^2} \\
\frac{\partial^2 g}{\partial x^2} &= 12xz + y^5 e^{xy^2}
\end{aligned}
$$

Q27

ii and iii

Q29

The roles of x and y are identical in this function. We can change each x into a y and vice versa, without changing the function. This means that $\frac{\partial f}{\partial y}$ can be obtained from $\frac{\partial f}{\partial x}$ by changing each x into y and vice versa.

Q31

The only way that $f_x(x,y)$ would not be a function would be if $f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$ can have more than one possible value. However a function cannot have more than one limit at a single point. It cannot be getting arbitratily close to two different values, eventually to converge to one, it must stay away from the other.

Q1

The partial derviatives of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$.

Q3

They are equivalent. To produce the linearization from the tangent plane, solve for z and then substitute $L(x, y)$ for z.

Q5

a $z = p(x, y)$ is a plane. 3 is the slope in the x-direction. 5 is the slope in the y-direction. -2 is the z-intercept.

b $p_x = 3$ and $p_y = 5$ so the tangent plane has equation

$$z - 21 = 3(x - 1) + 5(y - 4)$$

c If we distribute and combine terms of the tangent plane equation, we obtain $z = 3x + 5y - 2$, which is equivalent to $z = p(x, y)$. This makes sense, because the tangent plane of a plane should be the plane iteself.

Q7

No. It means that $f_y(x_0, y_0) = 0$. Having a zero derivative at one point does not mean the function never changes as y changes.

Q9

Writing the formula requires us to fill in 5 values.

1. $x_0 = 2$ is given.
2. $y_0 = 2$ is given.
3. $z_0 = 4$ is given.
4. $f_x(x_0, y_0)$ requires the chain rule

$$f_x(x,y) = \frac{1}{2\sqrt{36 - 4x^2 - y^2}}(-8x)$$
$$f_x(4,0) = \frac{1}{2\sqrt{36 - (4)(2)^2 - 2^2}}(-8)(2) = -2$$

5. $f_y(x_0, y_0)$ also requires the chain rule

$$f_y(x,y) = \frac{1}{2\sqrt{36 - 4x^2 - y^2}}(-2y)$$
$$f_y(4,0) = \frac{1}{2\sqrt{36 - (4)(2)^2 - 2^2}}(-2)(2) = -\frac{1}{2}$$

We plug these values into the tangent plane formula.

$$z - 4 = -2(x-2) - \frac{1}{2}(y-2)$$

Q11

We can write $f(x,y) = y^{1/2}x^{-1/2}$ for easier differentiation.

$$f_x(x,y) = y^{1/2}\left(-\frac{1}{2}x^{-3/2}\right) \qquad f_x(4,36) = -\frac{3}{8}$$
$$f_y(x,y) = \frac{1}{2}y^{-1/2}x^{-1/2} \qquad f_y(4,36) = \frac{1}{24}$$

The equation of the plane is

$$z - 3 = -\frac{3}{8}(x-4) + \frac{1}{24}(y-36).$$

Q13

$$f(x,y) = ye^{xy} \qquad f(3,2) = 2e^6$$
$$f_x(x,y) = y^2e^{xy} \qquad f_x(3,2) = 4e^6$$
$$f_y(x,y) = (1+xy)e^{xy} \qquad f_y(3,2) = 7e^6$$

The equation is

$$L(x,y) = 2e^6 + 4e^6(x-3) + 7e^6(y-2)$$

Q15

The linearization at $(5,0)$ will have a constant term of $\sqrt{5}$ and a coefficient of $\frac{1}{2\sqrt{5}}$. Unless we already know this value, this will not be an effective strategy for evaluating by hand. Using the linearization at $(4,0)$ is a better choice.

Q17

$(4,1)$ is a neaby point where we can compute the value and derivatives by hand. Here are the values we need to compute.

$$f(x,y) = \frac{x^2}{y} \qquad f(4,1) = 16$$
$$f_x(x,y) = \frac{2x}{y} \qquad f_x(4,1) = 8$$
$$f_y(x,y) = -\frac{x^2}{y^2} \qquad f_y(4,1) = -16$$

We can now write the linearization and evaluate it at $(3.97, 1.05)$ to approximate $\frac{3.97^2}{1.05}$.

$$\begin{aligned} L(x,y) &= f(4,1) + f_x(4,1)(x-4) + f_y(4,1)(y-1) \\ &= 16 + 8(x-4) - 16(y-1) \end{aligned}$$
$$\begin{aligned} \frac{3.97^2}{1.05} = f(3.97, 1.05) \approx L(3.97, 1.05) &= 16 + 8(3.97-4) - 16(1.05-1) \\ &= 16 - 0.24 - 0.8 \\ &= 14.96 \end{aligned}$$

We conclude that $\frac{3.97^2}{1.05} \approx 14.96$.

Q19

Compute the partial derivatives of f using the quotient rule.

$$f_x(x,y) = \frac{0-2xy}{(x^2+y^2)^2} \qquad f_y(x,y) = \frac{1(x^2+y^2)-2y^2}{(x^2+y^2)^2}$$

$$f_x(4,3) = -\frac{24}{625} \qquad f_y(4,3) = \frac{7}{625}$$

The differential is

$$\begin{aligned} df &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \\ &= -\frac{24}{625}dx + \frac{7}{625}dy \end{aligned}$$

Q21

Our area function is $A(l,w) = lw$

$$\begin{aligned} dA &= \frac{\partial A}{\partial l}dl + \frac{\partial A}{\partial w}dw \\ &= wdl + ldw \\ &= 30dl + 50dw \max dA &= 30(0.2) + 50(0.2) \\ &= 16m^2 \end{aligned}$$

Q23

$dz = 2dx - 12dy$. Since dx and dy can be any numbers, there is no maximum value for dz.

Q25

If $f_{yy}(x,y) < 0$, then f is concave down in the y-direction. This means that $L(x,y)$ will be above $f(x,y)$ if we travel in the y-direction. We can conclude that $L(x,y) < f(x,y)$ as long as $x = 3$ and $y \neq 2$. We are not able to compare $f(x,y)$ and $L(x,y)$ at any point that requires travel in the x-direction from $(3,2)$, since we don't know how the function changes in the x-direction.

Q1

A vector and a point can both be represented in terms of coordinates. The formula for length of a vector is similar to the formula for distance between points. A vector and a number can both be added, subtracted or multiplied.

Q3

Two vectors point in the same direction, if they are scalar multiples of each other, and the scalar is positive. They point in opposite directions, if the scalar is negative.

Q5

iii. and v. are vectors. The others are not.

Q7

B and C must be the same point. We reach both of them by starting an A and travelling along the same displacment vector.

Q9

The x-displacement is $2-8=-6$. The y-displacement is $3-7=-4$. The z-displacement is $15-11=4$.

a $\overrightarrow{AB}=\langle -6,-4,4\rangle$

b $\overrightarrow{AB}=-6\vec{i}-4\vec{j}+4\vec{k}$

Q11

The slope is $\frac{10}{-4}=-\frac{5}{2}$.

Q13

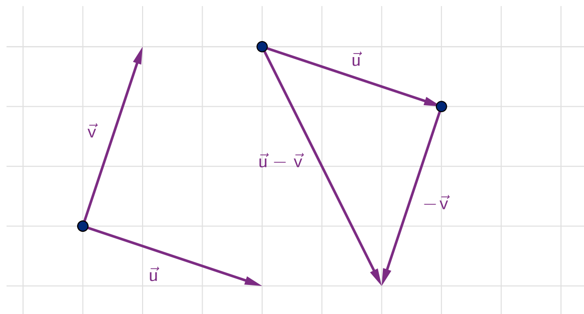
They are parallel.

Q15

They have the same direction, but $2\vec{u}$ has twice the magnitude of $\vec{u}$.

Q17

$-\vec{v}$ has the same magnitude as $\vec{v}$ but points in the opposite direction. We can place $\vec{u}$ and $-\vec{v}$ head to tail and connect the ends to create $\vec{u}-\vec{v}$.



Suppose instead that we place $\vec{u}$ and $\vec{v}$ to have the same initial point (tail to tail). Notice that $\vec{u}-\vec{v}$ is also the vector from the terminal point of $\vec{v}$ to the terminal point of $\vec{u}$. In terms of displacements, we can think of this as traveling backward along $\vec{v}$ and then forward along $\vec{u}$. This is useful if we are already thinking of $\vec{u}$ and $\vec{v}$ as position vectors.

Q19

D is the center of a parallelogram with vectices A, B and C. A simpler and more interesting conclusion is that D is the midpoint of the segment from B to C.

Q21

$\vec{u}+\vec{v}=9\vec{i}+\vec{j}$

Q23

The vector from Sam's house to lindsey's house is $\langle 3,-5\rangle$. The vector from Lindsey's house to Russel's house is $\langle 0,-2\rangle$. These vectors are head to tail, so to get from Sam's house to Russel's house, we add them. The result is $\langle 3,-7\rangle$, which means $3mi$ east and $7mi$ south.

Q25

a $\overrightarrow{EB} = \vec{v}$

b $\overrightarrow{CG} = \frac{1}{2}\vec{u}$

c $\overrightarrow{BC} = \vec{v} - \vec{u}$

d $\overrightarrow{AF} = \frac{1}{2}\vec{u} + \frac{1}{2}\vec{v}$

e $\overrightarrow{GB} = \frac{1}{2}\vec{u} - \vec{v}$

Q27

$5\vec{i} + 2\vec{j}$

Q29

$|\vec{u}| = \sqrt{5^2 + 12^2} = 13.$

Q31

The length of this vector is $\sqrt{3^2 + 1^2} = \sqrt{10}$. If we multiply the vector by $\frac{1}{\sqrt{10}}$ we will obtain a vector in the same direction whose length is 1. Our unit vector is

$$\vec{u} = \frac{3}{\sqrt{10}}\vec{i} - \frac{1}{\sqrt{10}}\vec{j}$$

Q33

If we place both $\vec{u}$ and $\vec{v}$ at the origin, they extend into the first quadrant. Their angle is largest when one is close to the positive x-axis and one is close to the positive y-axis. The angle between them can be made arbitrarily close to $\frac{\pi}{2}$.

Q35

At the crossing there are two different angles depending on which way we travel. They sum to π. One of them is equal to the angle between the vectors.

Q37

$$\begin{aligned} 3\vec{u} - 4\vec{v} &= 3\langle 2,0,3\rangle - 4\langle 5,6,0\rangle \\ &= \langle 6,0,9\rangle - \langle 20,24,0\rangle \\ &= \langle -14,-24,9\rangle \end{aligned}$$

Q39

$|\vec{v}| = \sqrt{2^2+7^2+6^2} = \sqrt{89}$.

Q41

a There are two unit vectors orthogonal to a given vector in $\mathbb{R}^2$, pointing in opposite directions.

b There are infinitely many unit vectors orthogonal to a given vector in $\mathbb{R}^3$. Their terminal points trace out a circle.

Q43

We will place $\vec{v}$ so that its initial point is in p, then check whether the terminal point is also in p. A convenient intitial point is the z-intercept, $(0,0,-7)$. The terminal point would be $(0+2,0+3,-7+8) = (2,3,1)$. We plug this into the equation for p

$$1 = 2 + 2(3) - 7$$

This is true, so $(2,3,1)$ lies in p. Thus $\vec{v}$ must be parallel to p.

Q45

$t\vec{u} + (1-t)\vec{v} = \vec{v} + t(\vec{u}-\vec{v})$. Since $\vec{v} = \overrightarrow{AC}$ and $\vec{u} - \vec{v} = \overrightarrow{CB}$, this means that when $\vec{v}$ and $t(\vec{u}-\vec{v})$ are placed head to tail, we end up somewhere on the line from C to B.

Q47

a This length would be infinite.

b This length would be $\sqrt{1+\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}+\cdots}$. The series under the sqaure root is geometric with intial term 1 and common ratio $\frac{1}{4}$. Its sum is

$$\frac{a}{1-r}=\frac{1}{1-\frac{1}{4}}=\frac{4}{3}$$

Thus the length of the vector is $\sqrt{\frac{4}{3}}$ or $\frac{2}{\sqrt{3}}$.

Q1

The dot product is commutative. It distributes across vector addition and is associative with scalar multiplication.

Q3

The formula $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$ relates the angle to the dot product.

Q5

$\vec{v} \cdot \vec{i}$ is equal to the x-component of $\vec{v}$. $\vec{v} \cdot \vec{j}$ is equal to the y-component of $\vec{v}$.

Q7

a $(4)(-1) + (5)(-2) = -4 - 10 = -14$

b $(5)(1) + (6)(-2) = 5 - 12 = -7$

c $(2)(0) + (4)(-1) + (-10)(-2) = -4 + 20 = -16$

Q9

a $\vec{u} \cdot \vec{u} = 13$ and $\vec{u} \cdot \vec{v} = 5$ and $\vec{u} \cdot \vec{v} = -4$.

b $\vec{v} \cdot \vec{u} = 5$ which is equal to $\vec{u}\dot{\vec{v}}$.

c $|\vec{u}| = \sqrt{13}$, $\vec{u} \cdot \vec{u} = (|\vec{u}|)^2$

d $(3\vec{u}) \cdot (3\vec{v}) = 45$. This is $\vec{u} \cdot \vec{v}$ times 3 times 3.

e $\vec{v} + \vec{w} = \langle -1, 1 \rangle$. $\vec{u} \cdot (\vec{v} + \vec{w}) = 1$. This is $\vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$

f The dot product follows the rules of a numerical product: it is commutative and associative and has a distributive property. If it were a sum, we'd expect $(3\vec{u}) \cdot (3\vec{v})$ to equal $3(\vec{u} \cdot \vec{v})$.

g Assign variables to the components of each vector, for instance

$$\vec{u} = \langle a, b \rangle \qquad \vec{v} = \langle c, d \rangle \qquad \vec{w} = \langle e, f \rangle$$

Then perform these computations and see that we get identical expressions in terms of the variables.

Q11

$(\vec{a} - 3\vec{b}) \cdot (5\vec{c} + 2\vec{d}) = 5\vec{a} \cdot \vec{c} - 15\vec{b} \cdot \vec{c} + 2\vec{a} \cdot \vec{d} - 6\vec{b} \cdot \vec{d}$.

Q13

a Since $\vec{u}$ and $\vec{v}$ are parallel, we have

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \pm|\vec{u}||\vec{v}| \\ -28 &= \pm|\vec{u}|(4) \pm 7 \qquad\qquad = |\vec{u}| \end{aligned}$$

Since the length of a vector is positive $|\vec{u}| = 7$.

b Since $\vec{u}$ and $\vec{v}$ are parallel, they either point in the same direction or opposite directions. Since the dot product is negative, they must point in opposite directions.

Q15

Substituing the given information into the cosine formula gives

$$\begin{aligned}\vec{u}\cdot\vec{v} &= |\vec{u}||\vec{v}|\cos\theta\\ 15 &= 5|\vec{v}|\cos\theta\\ \frac{3}{\cos\theta} &= |\vec{v}|\end{aligned}$$

Since $\cos\theta$ can be no larger than 1, $|\vec{v}|\geq 3$. There is no limit to how large $|\vec{v}|$ can be, since $\cos\theta$ can be arbitrarily close to 0.

Q17

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\theta \qquad\qquad = |\vec{v}|\cos\theta$$

Since $|\vec{v}| = \sqrt{7^2+2^2+1^2} = \sqrt{54} = 3\sqrt{6}$ is fixed, the only unknown that determines $\vec{u}\cdot\vec{v}$ is θ. $\cos\theta$ is maximized when $\theta = 0$. This means $\vec{u}$ and $\vec{v}$ have the same direction. To obtain a unit vector in the direction of v we use the formula:

$$\begin{aligned}\vec{u} &= \frac{1}{|\vec{v}|}\vec{v}\\ &= \frac{1}{3\sqrt{6}}(7\vec{i}-2\vec{j}+\vec{k})\\ &= \frac{7}{3\sqrt{6}}\vec{i}-\frac{2}{3\sqrt{6}}\vec{j}+\frac{1}{3\sqrt{6}}\vec{k}\end{aligned}$$

Q19

Apply the cosine formula

$$\begin{aligned}\vec{u}\cdot\vec{v} &= |\vec{u}||\vec{v}|\cos\theta\\ (6)(7)+(1)(0)+(4)(2) &= \sqrt{6^2+1^2+4^2}\sqrt{7^2+0^2+2^2}\cos\theta\\ 50 &= \sqrt{53}\sqrt{53}\cos\theta\\ \frac{50}{53} &= \cos\theta\\ \cos^{-1}\frac{50}{53} &= \theta\end{aligned}$$

Q21

For convenience, let $A = (0,0,0)$, $B = (1,0,0)$ and $C = (1,1,1)$. We can apply the cosine formula.

$$\begin{aligned}
\overline{AB} \cdot \overline{AC} &= |\overline{AB}||\overline{AC}| \cos\theta \\
(1)(1) + (0)(1) + (0)(1) &= \sqrt{1^2 + 0^2 + 0^2}\sqrt{1^2 + 1^2 + 1^2} \cos\theta \\
1 &= \sqrt{3}(1) \cos\theta \\
\frac{1}{\sqrt{3}} &= \cos\theta \\
\cos^{-1} \frac{1}{\sqrt{3}} &= \theta
\end{aligned}$$

Q23

We could use

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}.$$

If the expression on the right is ± 1 then the vectors are parallel. This is much more work than just checking whether $\vec{u}$ and $\vec{v}$ are scalar multiples of each other.

Q25

a No. For instance, if the dot product is 0, then $\vec{v}$ could be any vector orthogonal to the chosen vector.

b You would need dot products equal to the dimension of the vector. The simplest method would be to use the standard basis vectors. The dot product of a vector with a standard basis vector is just the corresponding coordinate. If we asked for the dot product with all standard basis vectors, we would know all the components of $\vec{v}$.

Q1

We need a point on the plane and a vector normal to the plane to write the normal equation.

Q3

The coefficients of the variables are the components of a normal vector.

Q5

$\vec{v} \cdot \langle 6, 6, 3 \rangle = 48 - 18 - 30 = 0$. That means that $\vec{v}$ is orthogonal to the normal vector of the plane, so $\vec{v}$ is parallel to the plane.

Q7

There are infinitely many normal vectors for each, but the easiest to choose are

a $\langle 3, -8, 10 \rangle$

b $\langle 4, -5, -1 \rangle$

Q9

A parallel plane has the same normal vectors, so we can use the same a, b and c. We plug in the origin and solve for d.

$$\begin{aligned} 7(0) - 11(0) + 8(0) + d &= 0 \\ d &= 0 \end{aligned}$$

So one equation of this plane is $7x - 11y + 8z = 0$.

Q11

All we know is that $(0, 0, 0)$ lies on the plane. We plug it in and see what we can solve for.

$$\begin{aligned} a(0) + b(0) + c(0) + d &= 0 \\ d &= 0 \end{aligned}$$

We know that $d = 0$. We don't know anything about a, b or c.

Q13

The normal vectors are $\langle 4, 6, 8 \rangle$ and $\langle 10, 15, 20 \rangle$, the second vector is $\frac{5}{2}$ times the first. This means the normal vectors are parallel, so we can conclude that the planes are parallel.

Q15

Every x y and z that satisfies the first equation must satisfy the second. We might notice that this can hold if the second equation is -2 times the first equation, which suggets that $k = (-2)(10) = -20$. Another approach would be to find a point that satisfies the first equation and plug it into the second equation. The y-intercept is an easy choice. It is $(0, 10, 0)$.

$$\begin{aligned} \text{solve for } y - intercept \qquad & 3(0) - y + 4(0) + 10 = 0 \\ & 10 = y \\ \text{plug in } (0, 10, 0) \qquad & -6(0) + 2(10) - 8(0) + k = 0 \\ & k = -20 \end{aligned}$$

Q17

We will plug thee points into the normal equation and solve for a, b, c and d.

$$\begin{aligned} a(10) + b(0) + c(0) + d = 0 \qquad & a = -\frac{d}{10} \\ a(0) + b(-5) + c(0) + d = 0 \qquad & b = \frac{d}{5} \\ a(0) + b(0) + c(2) + d = 0 \qquad & c = -\frac{d}{2} \end{aligned}$$

Any value of d will work. A natural choice is $d = 10$. Plugging this back in gives the equation

$$-x + 2y - 5z + 10 = 0$$

Q19

We'll plug in the points and solve the system of equations.

$$
\begin{aligned}
a(4)+b(3)+c(0)+d&=0 & &\\
a(5)+b(1)+c(1)+d&=0 &\Longleftarrow\qquad & (-10a-2b-2c-2d=0)\\
a(-2)+b(5)+c(2)+d&=0 &\Longleftarrow\qquad & +(-2a+5b+2c+d=0)\\
& & & -12a+3b-d=0\\
4a+3b+d&=0 &\Longleftarrow\qquad & +(-4a-3b-d=0)\\
& & & -16a-2d=0\\
& & & -\frac{d}{8}=a\\
-4\frac{d}{8}+3b+d&=0 & &\\
3b+\frac{d}{2}&=0 & &\\
b&=-\frac{d}{6} & &\\
5a+b+c+d&=0 & &\\
-5\frac{d}{8}-\frac{d}{6}+c+d&=0 & &\\
c&=-\frac{5d}{24} & &
\end{aligned}
$$

Any value of d will work. A natural choice is $d=24$. Plugging this back in gives the equation

$$-3x-4y-5z+24=0.$$

Q21

$(6,3)$ is 3 units away from one point on the line, but that may not be the closest point. Looking at the graph, we expect that there is a point on the line closer to $(6,3)$ somewhere in the first quadrant.

Q23

$$d=\frac{3(5)+2(2)-5(1)+10}{\sqrt{3^2+2^2+5^2}}=\frac{24}{\sqrt{38}}$$

Q25

We compute the signed distances. We know $|\vec{n}|$ is always positive. Because we only care about which side we are on, we can leave $|\vec{n}|$ uncomputed.

$$\frac{3(6) - 10(7) + 9(1) + 46}{|\vec{n}|} = \frac{3}{|\vec{n}|} > 0$$
$$\frac{3(5) - 10(-3) + 9(-4) + 46}{|\vec{n}|} = \frac{-5}{|\vec{n}|} < 0$$

Since one signed distance is positive and the other is negative, these points lie on opposite sides of the plane.

Q27

a We plug the coordinates of each point into $L(x_1, x_2, x_3, x_4) = 2x_1 + 5x_2 - 4x_3 + 10x_4 + k$

Type	Measurements	$L(x_1, x_2, x_3, x_4)$
Cat	$(5, 1, 3, 6)$	$63 + k$
Dog	$(7, 3, 7, 2)$	$21 + k$
Dog	$(7, 2, 6, 4)$	$40 + k$
Dog	$(9, 1, 8, 5)$	$41 + k$
Cat	$(6, 4, 5, 5)$	$62 + k$
Cat	$(9, 2, 7, 6)$	$60 + k$

The smallest Cat value is $60 + k$, the largest dog value is $41 + k$. If $-60 < k < -41$, then the cats will all have positive values and the dogs will all have negative values.

b We dont know excactly where other cats and dogs will fall. It makes sense to keep our dividing hyperplane as far as possible from our existing cats an dogs to allow as much variation as possible without crossing the hyperplane. The midpoint, $k = 50.5$ would be a natural choice. There may be other reasonable approaches as well.

Q29

The plane crosses halfway between A and B and perpendicular to the segment between them. THe midpoint is

$$M = \left(\frac{1+7}{2}, \frac{-2+0}{2}, \frac{7+5}{2}\right) = (4, -1, 6)$$

The vector $\overrightarrow{AB}$ is perpendicular to the plane, so we can use it as a normal vector

$$\vec{n} = \langle 7-1, 0-(-2), 5-7\rangle = \langle 6, 2, -2\rangle$$

The equation of the plane is

$$6(x-4) + 2(y+1) - 2(z-6) = 0$$

Another apporach is to notice that the plane consists of the points (x, y, z) such that the following distances are equal:

$$\sqrt{(x-1)^2 + (y+2)^2 + (z-7)^2} = \sqrt{(x-7)^2 + y^2 + (z-5)^2}$$

This will simplify to an equivalent equation to the one we found via the first method.

Q31

We will pick three convenient points and solve for the coefficients in the plane. An easy choice is $(0, 0, 0)$, $(1, 0, 0)$ and $(0, 0, 1)$.

$$\begin{aligned} a(0) + b(0) + c(0) + d &= 0 & \qquad d &= 0 \\ a(1) + b(0) + c(0) + d &= 0 & \qquad a &= -d \\ a(0) + b(0) + c(1) + d &= 0 & \qquad c &= -d \end{aligned}$$

Thus a, c and d are all 0. The plane has equation $by = 0$ for any b. If we choose $b = 1$, this gives us the equation for the xz-plane.

Q1

The direction in which the function increases most quickly.

Q3

$D_{\vec{u}}f = \nabla f \cdot \vec{u}$.

Q5

a $\sqrt{4^2+3^2} = 5$

b

$$\frac{\text{rise}}{\text{run}} = \frac{10-12}{5} = -\frac{2}{5}$$

Q7

$\nabla f(x,y) = \left\langle 2x\sin(xe^y) + x^2e^y\cos(xe^y), x^3e^y\cos(xe^y)\right\rangle$

Q9

The level curve of f through (x_0, y_0) and the level curve of g through (x_0, y_0) meet at a right angle.

Q11

a We used the fact that $\vec{u}$ is a unit vector to replace our denominator (run) with 1.

b $\nabla f \cdot \vec{u}$ would still be equal to the numerator, or the rise. It would represent how much the linearization rises as (x, y) are displaced by $\vec{u}$.

Q13

$D_{\vec{u}}f(x,y) = |\nabla f(x,y)|\cos\theta$. This is smallest when $\theta = \pi$, meaning $\vec{u}$ and $\nabla f(x,y)$ are opposite vectors. $\vec{u}$ is a unit vector in the direction of $-\nabla f(x,y)$, which can be computed:

$$\vec{u} = \frac{1}{|-\nabla f(x,y)|}(-\nabla f(x,y)) = -\frac{1}{|\nabla f(x,y)|}(\nabla f(x,y))$$

Q15

This tells us that both components of the gradient vector are equal. If they are positive, then the greatest increase happens in the direction of $\langle 1, 1\rangle$. If they are negative, then the greatest increase happens in the direction of $\langle -1, -1\rangle$. The correspoding unit vectors are

$$\vec{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Q17

The maximum is in the direction of the gradient vector.

$$\begin{aligned}
\nabla f(x, y, z) &= \langle 3y, 3x, 2z\rangle \\
\nabla f(2, 1, -4) &= \langle 3, 6, -8\rangle \\
|\nabla f(2, 1, -4)| &= \sqrt{109} \\
\vec{u} &= \frac{1}{\sqrt{109}} \langle 3, 6, -8\rangle \\
&= \left\langle \frac{3}{\sqrt{109}}, \frac{6}{\sqrt{109}}, -\frac{8}{sqrt109} \right\rangle D_{\vec{u}} f(2, 1, -4) \quad = \langle 3, 6, -8\rangle \cdot \left\langle \frac{3}{\sqrt{109}}, \frac{6}{\sqrt{109}}, -\frac{8}{\sqrt{109}} \right\rangle \\
&= \frac{9}{\sqrt{109}} + \frac{36}{\sqrt{109}} + \frac{64}{\sqrt{109}} \\
&= \sqrt{109}
\end{aligned}$$

Q19

$$\begin{aligned}
\nabla f(x, y, z) &= \langle e^{yz}, xze^{yz}, xye^{yz}\rangle \\
\nabla f(3, 0, 4) &= \langle e^0, 12e^0, 0e^0\rangle \\
&= \langle 1, 12, 0\rangle \\
D_{\vec{u}} f(3, 0, 4) &= \langle 1, 12, 0\rangle \cdot \left\langle \frac{2}{3}, -\frac{1}{3}, -\frac{2}{3} \right\rangle \\
&= \frac{2}{3} - 4 + 0 \\
&= -\frac{10}{3}
\end{aligned}$$

Q21

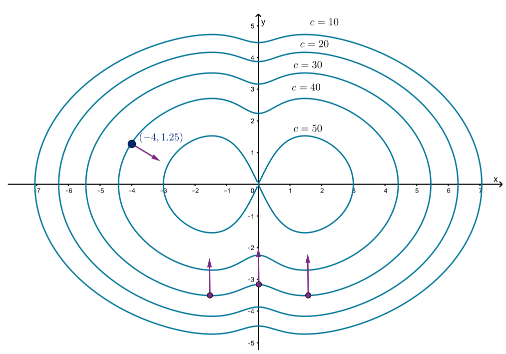
First we solve for $\vec{u}$. Since $|\langle 2,3\rangle| = \sqrt{13}$,

$$\vec{u} = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$$

Now we use the gradient vector to solve for the directional derivative.

$$\begin{aligned}
\nabla f(x,y) &= \langle 2x+3y, 3x\rangle \\
\nabla f(-1,4) &= \langle 10,-3\rangle \\
D_{\vec{u}} f(-1,4) &= \langle 10,-3\rangle \cdot \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle \\
&= \frac{20}{\sqrt{13}} - \frac{9}{\sqrt{13}} \\
&= \frac{11}{\sqrt{13}}
\end{aligned}$$

Q23



Q25

The gradient has a negative y-component, meaning B increases as we travel downward. If we travel above (x_0, y_0), we expect B to decrease, so the pixels will be dimmer.

Q27

Let $f(x,y) = x^3 + 8y^3 - 12xy$. $\nabla f(3, 1.5)$ is a normal vector to level curve.

$$\begin{aligned} \nabla f(x,y) &= \left\langle 3x^2 - 12y, 24y^2 - 12x \right\rangle \\ \nabla f(3, 1.5) &= \langle 9, 18\rangle \end{aligned}$$

A normal equation of the plane is

$$9(x-3) + 18(y-1.5) = 0.$$

Q29

We can use the gradient vector of $f(x,y,z) = z^3 - xz^2 - yx^2$ as a normal vector to the plane.

$$\begin{aligned} \nabla f(x,y,z) &= \left\langle -z^2 - 2xy, -x^2, 3z^2 - 2xz \right\rangle \\ \nabla f(4,-2,2) &= \langle 12, -16, -4\rangle \end{aligned}$$

A normal equation of the plane is

$$12(x-4) - 16(y+2) - 4(z-2) = 0$$

If you prefer, you could intead use a shorter normal vector like $\vec{n} = \langle 3, -4, -1\rangle$.

Q31

We can write the components of $\nabla f(5,-1)$ as variables and solve for them. Let $\nabla f(5,-1) = \langle a, b\rangle$. The dot products give us a system of two equations.

$$\begin{aligned} \langle a, b\rangle \cdot \langle -0.6, 0.8\rangle &= 4 \\ -0.6a + 0.8b &= 4 \end{aligned} \qquad \begin{aligned} \langle a, b\rangle \cdot \langle 0, -1\rangle &= -2 \\ -b &= -2 \\ b &= 2 \end{aligned}$$

$$\begin{aligned} -0.6a + 0.8(2) &= 4 \\ -0.6a + 1.6 &= 4 \\ -0.6a &= 2.4 \\ a &= -4 \end{aligned}$$

We conclude that $\nabla f(5,-1) = \langle -4, 2\rangle$

Q33

The usual equation is

$$L(x, y) = f(x_0, y_0 + f_x(x_0, y_0))(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The partial derivatives are the components of the gradient. We can rewrite this as a dot product.

$$L(x, y) = f(x_0, y_0 + \nabla f(x_0, y_0)) \cdot \langle x - x_0, y - y_0 \rangle$$

Q35

$\vec{u}$ must be orthogonal to $\nabla f(a, b, c)$. There are infintely many unit vectors orthogonal to a given nonzero vector in $\mathbb{R}^3$. Their terminal points lie in the plane normal to $\nabla f(a, b, c)$. If we place them with a common initial point, their terminal points make a circle.

Q37

a We compute the gradient of h:

$$\nabla h(x,y) = \langle 2x+2, 6\sqrt{y}\rangle$$
$$\nabla h(2,3) = \left\langle 6, 6\sqrt{3}\right\rangle$$

Let $\vec{u} = \langle a, b\rangle$. Since $\vec{u}$ is a unit vector, $b = \sqrt{1-a^2}$. We will set up the given directional derivative and solve.

$$\begin{aligned}
D_{\vec{u}}h(2,3) &= 6\\
\left\langle 6, 6\sqrt{3}\right\rangle \cdot \left\langle a, \sqrt{1-a^2}\right\rangle &= 6\\
6a + 6\sqrt{3}\sqrt{1-a^2} &= 6\\
\sqrt{3}\sqrt{1-a^2} &= 1-a\\
3-3a^2 &= a^2-2a+1\\
0 &= 4a^2-2a-2\\
0 &= 2(2a+1)(a-1)\\
a &= -\frac{1}{2} \text{ and } a = 1
\end{aligned}$$

If $a = -\frac{1}{2}$, then $b = \sqrt{1-\frac{1}{4}} = \pm\frac{\sqrt{3}}{2}$. If $a = 1$ then $b = 0$. However, since we squared both sides we need to check for extraneous solutions. We plug them back into the original equation.

$$\left\langle 6, 6\sqrt{3}\right\rangle \cdot \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle = 6 \qquad \text{(valid)}$$
$$\left\langle 6, 6\sqrt{3}\right\rangle \cdot \left\langle -\frac{1}{2}, -\frac{\sqrt{3}}{2}\right\rangle = -12 \qquad \text{(not valid)}$$
$$\left\langle 6, 6\sqrt{3}\right\rangle \cdot \langle 1, 0\rangle = 6 \qquad \text{(valid)}$$

So $\vec{u} = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2}\right\rangle$ or $\langle 1, 0\rangle$

b We know that $\nabla h(2,3)$ is normal to the tangent line. We can use the cosine formula to find the angle between $\vec{u}$ and $\nabla h(2,3)$.

$$\begin{aligned}
\nabla h(2,3)\cdot\vec{u} &= |\nabla h(2,3)||\vec{u}|\cos\theta\\
6 &= (\sqrt{36+108})(1)\cos\theta\\
\frac{1}{2} &= \cos\theta\\
\frac{\pi}{3} &= \theta
\end{aligned}$$

Since $\nabla h(2,3)$ is normal to the tangent line, the angle between $\vec{u}$ and the tangent line is

$$\frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$$

Q1

$x(t)$ and $y(t)$ give the coordinates of a point moving through the xy-plane at time t. The points trace out a curve. $z = f(x, y)$ is the height of the function above each point.

Q3

$\frac{\partial z}{\partial x}$ assumes that other variables are held constant. $\frac{dz}{dx}$ allows for them to change as x changes. We compute it using the chain rule.

Q5

The curve is a line through $(3, -2)$ with a slope of $\frac{4}{5}$. The slope is the ratio of the t coefficient in $y(t$ to the t coefficient in $x(t)$.

Q7

We solve for the tangent vector and the compute its length.

$$\begin{aligned}
\langle x'(t), y'(t)\rangle &= \langle -2\sin t, 3\cos t\rangle \\
\left\langle x'\left(\frac{\pi}{3}\right), y'\left(\frac{\pi}{3}\right)\right\rangle &= \left\langle -\sqrt{3}, \frac{3}{2}\right\rangle \\
\left|\left\langle x'\left(\frac{\pi}{3}\right), y'\left(\frac{\pi}{3}\right)\right\rangle\right| &= \sqrt{(\sqrt{3})^2 + \left(\frac{3}{2}\right)^2} \\
&= \frac{\sqrt{21}}{2}
\end{aligned}$$

Q9

No. Since $t^2 = (-t)^2$ there will be two points on this curve for each positive value of x. This graph will fail the vertical line test.

Q11

$\frac{df}{dt} = \nabla f \cdot \langle x'(t), y'(t)\rangle$. If $\langle x'(t), y'(t)\rangle$ is a unit vector, then this is the same as a directional derivative.

Q13

We are looking for the value of $\frac{d\omega}{dt}$. We will need the partial derivatives of ω

$$\frac{\partial \omega}{\partial v} = \frac{1}{r} \qquad \frac{\partial \omega}{\partial r} = -\frac{v}{r^2}$$
$$\frac{\partial \omega}{\partial v} = \frac{1}{8400000} \qquad \frac{\partial \omega}{\partial r} = -\frac{6900}{8400000^2}$$

Now we evaluate the chain rule

$$\begin{aligned}
\frac{d\omega}{dt} &= \frac{\partial \omega}{\partial v}\frac{dv}{dt} + \frac{\partial \omega}{\partial r}\frac{dr}{dt} \\
&= -\frac{1}{8400000}60 - \frac{6900}{8400000^2}100 \\
&= -\frac{16823}{2352000000}
\end{aligned}$$

Q15

$$\frac{\partial h}{\partial t} = \frac{\partial h}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial h}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial h}{\partial x_3}\frac{\partial x_3}{\partial t} + \frac{\partial h}{\partial x_4}\frac{\partial x_4}{\partial t}$$

Q17

We are looking for the value of $\frac{dL}{dt}$. First we solve for the partial derivatives of L

$$\frac{\partial L}{\partial r} = mv \qquad \frac{\partial L}{\partial m} = rv \qquad \frac{\partial L}{\partial v} = rm$$
$$\frac{\partial L}{\partial r} = (6000)(3100) \qquad \frac{\partial L}{\partial m} = (42000000)(3100) \qquad \frac{\partial L}{\partial v} = (42000000)(6000)$$
$$= 186 \times 10^5 \qquad = 1302 \times 10^8 \qquad \frac{\partial L}{\partial v} = 252 \times 10^9$$

Now we apply the chain rule.

$$\begin{aligned}
\frac{dL}{dt} &= \frac{\partial L}{\partial r}\frac{dr}{dt} + \frac{\partial L}{\partial m}\frac{dm}{dt} + \frac{\partial L}{\partial v}\frac{dv}{dt} \\
&= (186 \times 10^5)(8 \times 10^4) + (1302 \times 10^8)(0) + (252 \times 10^9)(20) \\
&= 6528 \times 10^9
\end{aligned}$$

Q19

We need the partial derivatvies of f.

$$\frac{\partial f}{\partial x} = 2xye^{(x^2y)} \qquad \frac{\partial f}{\partial y} = e^{(x^2y)} + yx^2e^{(x^2y)}$$

$$\frac{\partial f}{\partial x} = 2(3)(-5)e^{-45} = -30e^{-45} \qquad \frac{\partial f}{\partial y} = e^{-45} + (-5)(3)^2e^{-45} = -44e^{-45}$$

Now we apply the chain rule

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \\ &= (-30e^{-45})(2) + (-44e^{-45})(10) \\ &= -500e^{-45}\end{aligned}$$

Q21

We use implicit differentiation on $F(x, y) = y^3 - xy + x^2 - 4$.

$$F_x(x, y) = -y + 2x \qquad F_y(x, y) = 3y^2 - x$$

$$F_x(4, 2) = 6 \qquad F_y(4, 2) = 8$$

By the implicit differentiation formula we have

$$\frac{dy}{dx} = -\frac{F_x(4, 2)}{F_y(4, 2)} = -\frac{6}{8} = -\frac{3}{4}$$

Q23

We use implicit differentiation on $F(x, y) = x - y^2$.

$$F_x(x, y) = 1 \qquad F_y(x, y) = -2y$$

$$F_x(18, -3) = 1 \qquad F_y(18, -3) = 6$$

By the implicit differentiation formula we have

$$\frac{dy}{dx} = -\frac{F_x(18, -3)}{F_y(18, -3)} = -\frac{1}{6}$$

Q25

We'll need the partial derivatives of L

$$\frac{\partial L}{\partial r} = mv \qquad \frac{\partial L}{\partial m} = rv \qquad \frac{\partial L}{\partial v} = rm$$

Now we use the chain rule on the equation $L = c$. Since we want $\frac{dv}{dr}$, we differentiate witrh respect to r.

$$\begin{aligned}
L &= c \\
\frac{\partial L}{\partial r}\frac{dr}{dr} + \frac{\partial L}{\partial m}\frac{dm}{dr} + \frac{\partial L}{\partial v}\frac{dv}{dr} &= 0 \\
(mv)(1) + (rv)(0) + (rm)\left(\frac{dv}{dr}\right) &= 0 \\
(rm)\left(\frac{dv}{dr}\right) &= -mv \\
\frac{dv}{dr} &= -\frac{v}{r}
\end{aligned}$$

There is no m in the answer. The relationship between radius and linear speed is independent of mass.

Q27

a

$$\begin{aligned}
\frac{\partial f}{\partial t}(x,t) &= \frac{1}{2\sqrt{x+t}} \\
\frac{\partial f}{\partial t}(7,9) &= \frac{1}{2\sqrt{7+9}} \\
&= \frac{1}{8}
\end{aligned}$$

b By the chain rule

$$\begin{aligned}
\frac{df}{dt} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial t}\frac{dt}{dt} \\
&= \frac{1}{2\sqrt{x+t}}\frac{dx}{dt} + \frac{1}{2\sqrt{x+t}} \\
&= \left(\frac{1}{8}\right)(-3) + \frac{1}{8} \\
&= -\frac{1}{4}
\end{aligned}$$

c $\frac{\partial f}{\partial t}$ is the change in f as t increases but x is held constant. $\frac{df}{dt}$ is the change in f as t increases and x changes accounding to the function $x(t)$.

Q29

a Let $f(x,y) = x^2 + 2x - y^2$. Then

$$\begin{aligned}\nabla f(x,y) &= \langle 2x+2, -2y\rangle \\ \nabla f(5,-3) &= \langle 12, 6\rangle\end{aligned}$$

The slope of this vector is $\frac{6}{12} = \frac{1}{2}$. The tangent to the level curve is perpendicular to gradient, so its slope is the negative of the reciprocal, which is -2.

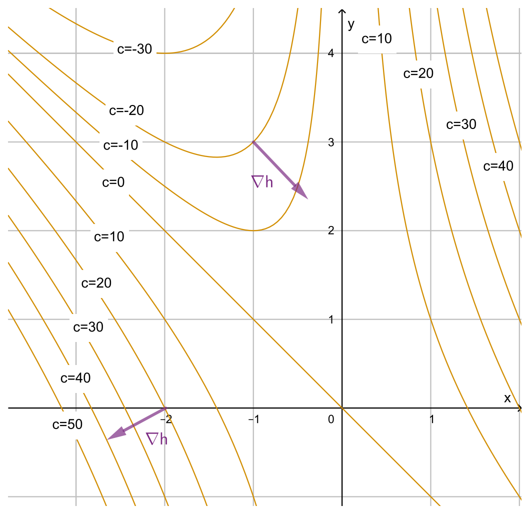
b

$$\begin{aligned}\frac{dy}{dx} &= -\frac{f_x}{f_y} \\ &= -\frac{2x+2}{-2y} \\ &= -\frac{12}{6} \\ &= -2\end{aligned}$$

Q31

a At $(2, 1)$ the positive y-direction points toward higehr-values level curves, so $h_y(2, 1) > 0$.

b Here is the diagram



c

$$\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt}$$
$$= -5\frac{\partial h}{\partial x} + 6\frac{\partial h}{\partial y}$$

From the direction of the gradient at $(2, 1)$ we can tell that $\frac{\partial h}{\partial x} > 0$ and $\frac{\partial h}{\partial y} \leq 0$. Thus $\frac{dh}{dt}$ is the sum of two negative numbers. It must be negative as well.

Q1

Local maximums occur only at critical points where the gradient is $\vec{0}$ or undefined. IF the gradient had any other value, then the function would have a positive directional dertivative in some direction.

Q3

The function must be continuous and the domain must be closed and bounded. It tells us that the function has a maximum and minimu on the domain.

Q5

$-x^2$ and $-y^2$ will always be negtaive, but $-10xy$ could be positive. Try a large positive value of x and a large negative value of y.

$$f(10, 10) = -100 - 100 + 1000 = 800$$

Since this is larger than $f(0, 0)$, we conclude $(0, 0)$ is not a maximum.

Q7

Larger values of x produce larger values of e^x. If (a, b) produces the smallest values of $f(x, y)$ in its neighborhood, then is also produces the smallest values of $e^{f(x,y)}$ in that neighborhood.

Q9

Any direction that makes an acute angle with $\langle -5, 11\rangle$ will work. Travelling the in positive y-direction is one simple way to guarantee a positive directional derivative.

Q11

No. Just because $|\nabla f(x, y)|$ is minimized does not mean it is 0. A function could have $|\nabla f(x, y)| > 0$ for all (x, y) and thus have no critical points at all. An example is $f(x, y) = x + y^3$.

Q13

f is a polynomial, so the gradient exists at all (x, y). We will set both partial derivatives equal to 0 and solve.

$$\begin{aligned} 4x^3 + 4y &= 0 \\ y &= -x^3 \end{aligned}$$

$$\begin{aligned} 4x + 4y^3 &= 0 \\ 4x + 4(-x^3)^3 &= 0 \\ 4(x - x^9) &= 0 \\ 4x(1 - x)(1 + x)(1 + x^2)(1 + x^4) &= 0 \\ x = 0 \text{ or } x = -1 &\text{ or } x = 1 \\ y = 0 \text{ or } y = 1 &\text{ or } y = -1 \end{aligned}$$

The critical points are $(0, 0)$, $(1, -1)$ and $(-1, 1)$.

Q15

Yes. We must have $f_{xy}(x_0, y_0) = 0$, then $D = 0f_(yy)(x_0, y_0) - 0^2 = 0$ and the second derivatives test is inconclusive. (x_0, y_0) could be a maximum.

Q17

h is a polynomial, so the gradient exists at all (x, y). We will set both partial derivatives equal to 0 and solve.

$$\begin{aligned} 2xy - 2x &= 0 & x^2 - 4y &= 0 \\ 2x(y-1) &= 0 & & \\ \text{if } x &= 0 & 0^2 - 4y &= 0 \\ & & y &= 0 \\ \text{if } y &= 1 & x^2 - 4(1) &= 0 \\ & & (x-2)(x+2) &= 0 \\ & & x = 2 \text{ or } x &= -2 \end{aligned}$$

The critical points are $(0, 0)$, $(2, 1)$ and $(-2, 1)$. For the second derivatives test we compute

$$\begin{aligned} D &= h_{xx}h_{yy} - (h_{xy})^2 \\ &= (2y-2)(-4) - (2x)^2 \\ &= 8 - 8y - 4x^2 \end{aligned}$$

at $(0, 0)$	$D = 8 > 0$ and $h_{xx} = -2 < 0$	local max
at $(\pm 2, 1)$	$D = -16 < 0$	saddle point

Q19

f is a polynomial, so the gradient exists at all (x, y). We will set both partial derivatives equal to 0 and solve.

$$
\begin{aligned}
6x^2 - 12y &= 0 & -12x + 6y &= 0 \\
& & y &= 2x \\
6x^2 - 12(2x) &= 0 & & \\
6x(x-4) &= 0 & & \\
\text{if } x &= 0 & -12(0) + 6y &= 0 \\
& & y &= 0 \\
\text{if } x &= 4 & -12(4) + 6y &= 0 \\
& & y &= 8
\end{aligned}
$$

The critical points are $(0, 0)$, and $(4, 8)$. For the second derivatives test we compute

$$
\begin{aligned}
& & D &= f_{xx}f_{yy} - (f_{xy})^2 & \\
& & D &= (12x)(6) - (-12)^2 & \\
& & D &= 72x - 144 & \\
&\text{at } (0,0) & D &= -144 < 0 & \text{saddle point} \\
&\text{at } (4,8) & D &= 144 > 0 \text{ and } h_{xx} = 48 > 0 & \text{local min}
\end{aligned}
$$

Q21

f is a polynomial, so the gradient exists at all (x, y). We will set both partial derivatives equal to 0 and solve.

$$3x^2 - 30x - 9 + 12y = 0 \qquad\qquad 12x - 6y - 18 = 0$$
$$12x - 18 = 6y$$
$$2x - 3 = y$$

$$\begin{aligned} 3x^2 - 30x - 9 + 12(2x - 3) &= 0 \\ 3x^2 - 6x - 45 &= 0 \\ 3(x-5)(x+3) &= 0 \\ \text{if } x &= 5 & 2(5) - 3 &= y \\ & & 7 &= y \\ \text{if } x &= -3 & 2(-3) - 3 &= 0 \\ & & -9 &= y \end{aligned}$$

The critical points are $(5, 7)$, and $(-3, -9)$. For the second derivatives test we compute

$$\begin{aligned} & D = f_{xx} f_{yy} - (f_{xy})^2 & \\ & D = (6x - 30)(-6) - (12)^2 & \\ & D = 36 - 36x & \\ \text{at } (5,7) \quad & D = -144 < 0 & \text{saddle point} \\ \text{at } (-3,-9) \quad & D = 144 > 0 \text{ and } h_{xx} = -48 < 0 & \text{local max} \end{aligned}$$

Q23

Since larger values of z give larger values of e^z, it sufficies to find the local minimums and maximums of $f(x, y) = x^3 + y^2 - 12x + 10y$. f is a polynomial, so the gradient exists at all (x, y). We will set both partial derivatives equal to 0 and solve.

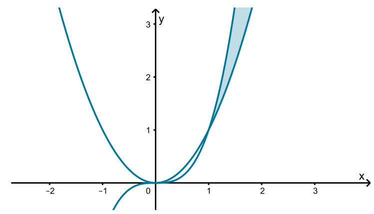
$$\begin{aligned} 3x^2 - 12 &= 0 & 2y + 10 &= 0 \\ 3(x-2)(x+2) &= 0 & 2(y+5) &= 0 \\ x &= \pm 2 & y &= -5 \end{aligned}$$

The critical points are $(\pm 2, -5)$. For the second derivatives test we compute

$$\begin{aligned} & D = f_{xx} f_{yy} - (f_{xy})^2 & \\ & D = (6x)(2) - 0^2 & \\ \text{at } (2,1) \quad & D = 12 > 0 \text{ and } h_{xx} = 6 > 0 & \text{local min} \\ \text{at } (-2,1) \quad & D = -12 < 0 & \text{saddle point} \end{aligned}$$

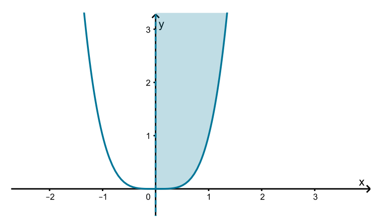
Q25

D contains its boundary points, so it is closed. D extends forever, so it is unbounded.



Q27

D does not contain the boundary line $x = 0$, so it is not closed. D extends forever, so it is unbounded.



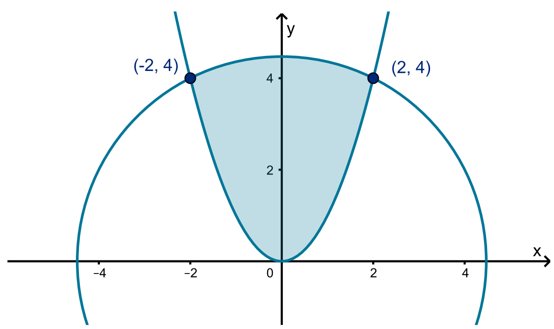
Q29

D is the region above the parabola $y = x^2$. This is unbounded. The Extreme Value Theorem does not apply.

Q31

We solve for where $y = x^2$ intersects $x^2 + y^2 = 20$

$$\begin{aligned} x^2 + y^2 &= 20 \\ x^2 + (x^2)^2 &= 20 \\ x^4 + x^2 - 20 &= 0 \\ (x^2+5)(x-2)(x+2) &= 0 \\ x = 2 \text{ or } x &= -2 \\ y = \pm 2^2 &= 4 \end{aligned}$$



The places we need to check are

- The interior of the region
- On the circle $x^2 + y^2 = 20$
- On the parabola
- At the points $(\pm 2, 4)$

Q33

f is continuous, and a triangle is closed and bounded, so the Extreme Value Theorem applies. We look for the maximum and minimum in the folowing places:

- In the interior. $\nabla f = \left\langle e^{x+3y}, 3e^{x+3y} \right\rangle$. This is never $\vec{0}$, so there are no interior critical points.
- On the segment from $(0,0)$ to $(6,0)$, $y = 0$.

$$\begin{aligned} f(x,0) &= e^{x+0} \\ &= e^x \\ f'(x) &= e^x \end{aligned}$$

  This is never 0. There are no critical points on this segment.
- On the segment from $(0,0)$ to $(0,3)$, $x = 0$.

$$\begin{aligned} f(0,y) &= e^{0+3y} \\ &= e^{3y} \\ f'(y) &= 3e^{3y} \end{aligned}$$

  This is never 0. There are no critical points on this segment.
- On the segment from $(6,0)$ to $(0,3)$, $y = 3 - \frac{1}{2}x$.

$$\begin{aligned} f\left(x, 3 - \frac{1}{2}x\right) &= e^{x+3\left(3-\frac{1}{2}x\right)} \\ f(x) &= e^{9-\frac{1}{2}x} \\ f'(x) &= -\frac{1}{2}e^{9-\frac{1}{2}x} \end{aligned}$$

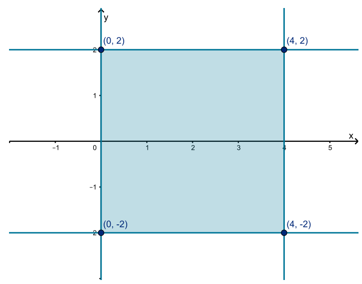
  This is never 0. There are no critical points on this segment.
- The endpoints $(0,0)$, $(6,0)$ and $(0,3)$ cannot be ruled out.

To find the maximum and minimum, we evaluate the function on the points we found.

$$\begin{aligned} f(0,0) &= 1 && \text{minimum} \\ f(6,0) &= e^6 \\ f(0,3) &= e^9 && \text{maximum} \end{aligned}$$

Q35

Here is the domain. It is closed and bounded, and f is continuous.



Now we look for critical points

- On interior

$$\nabla f(x,y) = \langle 3x^2 - 12, 3y^2 - 3\rangle$$

$$\begin{aligned} \text{Set } 3x^2 - 12 &= 0 & \text{and } 3y^2 - 3 &= 0 \\ 3(x-2)(x+2) &= 0 & 3(y-1)(y+1) &= 0 \\ x &= \pm 2 & y &= \pm 1 \end{aligned}$$

  We obtain four critical points $(\pm 2, \pm 1)$ but only two of them $(2, \pm 1)$ are in the domain.

- On $y = 2$

$$\begin{aligned} f(x,2) &= x^3 - 12x + 8 - 6 \\ f'(x) &= 3x^2 - 12 \\ \text{Set } 0 &= 3x^2 - 12 \\ 0 &= 3(x-2)(x+2) \\ \pm 2 &= x \end{aligned}$$

  We obtain two critical points $(\pm 2, 2)$ but only $(2, 2)$ is in the domain.

- On $y = -2$

$$\begin{aligned} f(x,2) &= x^3 - 12x - 8 + 6 \\ f'(x) &= 3x^2 - 12 \\ \text{Set } 0 &= 3x^2 - 12 \\ 0 &= 3(x-2)(x+2) \\ \pm 2 &= x \end{aligned}$$

  We obtain two critical points $(\pm 2, -2)$ but only $(2, -2)$ is in the domain.

- On $x = 0$

$$\begin{aligned} f(0,y) &= y^3 - 3y \\ f'(y) &= 3y^2 - 3 \\ \text{Set } 0 &= 3y^3 - 3 \\ 0 &= 3(y-1)(y+1) \\ \pm 1 &= y \end{aligned}$$

  We obtain two critical points $(0, \pm 1)$.

- On $x = 4$

$$
\begin{aligned}
f(0, y) &= 64 - 48 + y^3 - 3y \\
f'(y) &= 3y^2 - 3 \\
\text{Set } 0 &= 3y^3 - 3 \\
0 &= 3(y-1)(y+1) \\
\pm 1 &= y
\end{aligned}
$$

We obtain two critical points $(4, \pm 1)$.

- We cannot rule out the endpoints $(0, \pm 2)$ and $(4, \pm 2)$.

We plug in all of our potential points and evaluate.

$$
\begin{aligned}
f(0, 2) &= 2 \\
f(0, 1) &= -2 \\
f(0, -1) &= 2 \\
f(0, -2) &= -2 \\
f(2, 2) &= -14 \\
f(2, 1) &= -18 && \text{minimum} \\
f(2, -1) &= -14 \\
f(2, -2) &= -18 && \text{minimum} \\
f(4, 2) &= 14 && \text{maximum} \\
f(4, 1) &= 10 \\
f(4, -1) &= 14 && \text{maximum} \\
f(4, -2) &= 10
\end{aligned}
$$

Q37

a $\nabla f = \langle 2x - 4y, -4x + 8y \rangle$. We set this equal to $\vec{0}$ and solve.

$$
\begin{aligned}
2x - 4y &= 0 & -4x + 8y &= 0 \\
x &= 2y & & \\
& & -4(2y) + 8y &= 0 \\
& & 0 &= 0
\end{aligned}
$$

Thus every point on $x = 2y$ is a critical point.

b $D = (2)(8) - (-4)^2 = 0$, so the second derivatives test is inconclusive at every critical point.

c $f(x, y)$ can be factored into $(x - 2y)^2$. A square cannot be less than 0, and f is equal to 0 when $x = 2y$. This tells us that the critical points are all minimums.

Q1

A constraint is an equation that we require our inputs to satisfy in a maximization or minimization problem.

Q3

It is always closed. It may or may not be bounded.

Q5

$13x + 22y + 11z = 230$

Q7

If we let $g(x, y, z) = x^2 + y^2 + z^2$, the sphere is a level set of g. $\nabla g(P)$ is normal to the tangent plane of the sphere. Since $\nabla f = \lambda \nabla g$, $\nabla f(P)$ is either $\vec{0}$ or is also normal to the tangent plane.

Q9

a Yes. Since (a, b) is a local maximum, (a, b) has a greater value than all nearby points (both on the constraint and not). Thus it has a greater value than all nearby points on the constraint.

b Since f is smooth and (a, b) is a local maximum, $\nabla f(a, b) = \vec{0}$. Unless $g(a, b) = \vec{0}$, the method of Lagrange multipliers demands that $\vec{0} = \nabla f(a, b) = \lambda \nabla g(a, b)$. In this case, $\lambda = 0$.

Q11

$f(x,y) = y - x^2$ is our objective function $g(x,y) = x^2 + y^2 = 4$ is our constraint.

$$\nabla f(x,y) = \langle -2x, 1\rangle$$
$$\nabla g(x,y) = \langle 2x, 2y\rangle$$

$\nabla g(x,y) = \vec{0}$ only at the origin, which is not on the constraint. Thus the maximum must satisfy $\nabla f = \lambda \nabla g$ and $g(x,y) = 4$.

$$\begin{aligned}
-2x &= \lambda 2x & 1 &= \lambda 2y & x^2 + y^2 &= 4 \\
-\lambda 2x - 2x &= 0 \\
-2x(\lambda + 1) &= 0 \\
\text{if } x &= 0 & & & 0^2 + y^2 &= 4 \\
& & & & y &= \pm 2 \\
\text{if } \lambda &= -1 & 1 &= (-1)(2y) \\
& & -\frac{1}{2} &= y \\
& & & & x^2 + \left(-\frac{1}{2}\right)^2 &= 4 \\
& & & & x^2 &= \frac{15}{4} \\
& & & & x &= \pm\frac{\sqrt{15}}{2}
\end{aligned}$$

One of these is the maximum. To find it we evalaute f at each point.

$$\begin{aligned}
f(0,2) &= 2 & &\text{maximum} \\
f(0,-2) &= -2 \\
f\left(\pm\frac{\sqrt{15}}{2}, -\frac{1}{2}\right) &= -\frac{17}{4} & &\text{minimum}
\end{aligned}$$

Q13

Let $g(x, y, z) = x^2 + y^2 + z^2$.

$$\nabla f(x, y, z) = \langle yz, xz, xy \rangle$$
$$\nabla g(x, y, z) = \langle 2x, 2y, 2z \rangle$$

$\nabla g(x, y, z) = \vec{0}$ only at the origin, which is not on the constraint. The maximum must satisfy $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$. We can exploit the symmetry to make three of the equations have the same left side.

$$\begin{aligned} yz &= \lambda 2x \\ \frac{xyz}{2\lambda} &= x^2 \\ y^2 &= x^2 \\ y &= \pm x \end{aligned} \qquad \begin{aligned} xz &= \lambda 2y \\ \frac{xyz}{2\lambda} &= y^2 \\ z^2 &= y^2 \\ z &= \pm y \end{aligned} \qquad \begin{aligned} xy &= \lambda 2z \\ \frac{xyz}{2\lambda} &= z^2 \end{aligned} \qquad x^2 + y^2 + z^2 = 36$$

$$\begin{aligned} x^2 + (\pm x)^2 + +(\pm x)^2 &= 36 \\ 3x^2 &= 36 \\ x &= \pm\sqrt{12} \end{aligned}$$

Since we multiplied both sides by x, y and z, the other possibility is that one of these variables is 0.

$$f(\text{anything with a } 0) = 0$$
$$f(\pm\sqrt{12}, \pm\sqrt{12}, \pm\sqrt{12}) = \pm 12^{3/2}$$

The maximum value is $12^{3/2}$, the minimum is $-12^{3/2}$.

Q15

Let $g(x, y, z) = 25x^2 + y^2 + 4z^2$.

$$\nabla f(x, y, z) = \langle 0, 3, 2\rangle$$
$$\nabla g(x, y, z) = \langle 50x, 2y, 8z\rangle$$

$\nabla g(x, y, z) = \vec{0}$ only at the origin, which is not on the constraint. The maximum must satisfy $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$. Two of the equations show us that $\lambda \neq 0$, $y \neq 0$ and $z \neq 0$.

$$0 = \lambda 50x \qquad 3 = \lambda 2y \qquad 2 = \lambda 8z \qquad 25x^2 + y^2 + 4z^2 = 100$$
$$0 = x$$
$$\frac{3}{2y} = \lambda$$
$$2 = \frac{3}{2y}(8z)$$
$$y = 6z$$
$$25(0) + (6z)^2 + 4z^2 = 100$$
$$40z^2 = 100$$
$$z = \pm\sqrt{\frac{5}{2}}$$
$$y = \pm 3\sqrt{10}$$

We evaluate f on these points

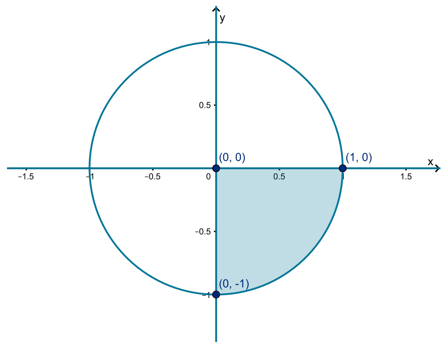
$$f\left(0, 3\sqrt{10}, \sqrt{\frac{5}{2}}\right) = 10\sqrt{10} \qquad \text{maximum}$$
$$f\left(0, -3\sqrt{10}, -\sqrt{\frac{5}{2}}\right) = -10\sqrt{10} \qquad \text{minimum}$$

Q17

f is continuous and D is closed and bounded, so a maximum exists. Furthermore, the maximum cannot lie in the interior of D, because f has no critical points. Thus it must be on the boundary circle. $g(x, y) = x^2 + y^2$ has a nonzero gradient on this circle, so $\nabla f = \lambda \nabla g$ must hold at the maximum.

Q19

Here is the domain. It is closed and bounded and f is continuous, so a maximum and minimum exist.



Now we look for critical points.

- On the interior:

$$\nabla f(x,y) = \langle 2x, -1\rangle$$

This is never $\langle 0,0\rangle$, so there are no ceritical points in the interior.

- On $x=0$, we substitute

$$\begin{aligned} f(0,y) &= 0^2 - y = -y \\ f'(y) &= -1 \end{aligned}$$

This is never 0, so there are no critical points on $x=0$.

- On $y=0$, we substitute

$$\begin{aligned} f(x,0) &= x^2 - 0 = x^2 \\ f'(x) &= 2x \\ \text{Set } 0 &= 2x \\ 0 &= x \end{aligned}$$

The point $(0,0)$ is a critical point.

- On $x^2+y^2=1$, we can substitute $x^2 = 1-y^2$

$$\begin{aligned} f(x,y) &= 1 - y^2 - y \\ f'(y) &= -2y - 1 \\ \text{Set } 0 &= -2y - 1 \\ -\frac{1}{2} &= y \\ x^2 &= 1 - \left(\frac{1}{2}\right)^2 \\ x &= \pm\sqrt{\frac{3}{4}} \end{aligned}$$

We obtain critical points $\left(\pm\sqrt{\frac{3}{4}}, -\frac{1}{2}\right)$ but only the positive value of x is in the domain.

- We also cannot rule out the endpoints of the curves which are $(0,0)$, $(1,0)$ and $(0,-1)$.

We plug in all of our potential points and evaluate.

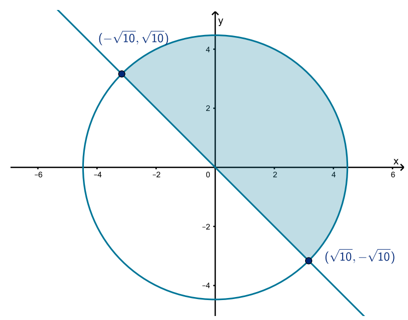
$$f(0,0) = 0 \qquad \text{minimum}$$
$$f(1,0) = 1$$
$$f(0,-1) = 1$$
$$f\left(\sqrt{\frac{3}{4}}, -\frac{1}{2}\right) = \frac{5}{4} \qquad \text{maximum}$$

The maximum value is $\frac{5}{4}$. The minimum is 0.

Q21

We solve for the intersection of $y=-x$ with $x^2+y^2=20$ by substitution:

$$\begin{aligned} x^2+(-x)^2 &= 20 \\ 2x^2 &= 20 \\ x^2 &= 10 \\ x &= \pm\sqrt{10} \\ y &= \mp\sqrt{10} \end{aligned}$$



D is closed and bounded and f is continuous so the EVT applies; one of our critical points is the maximum and one is the minimum. We look for critical points in the following places.

- In the interior

$$\nabla f(x,y) = \langle 4x^3y, x^4\rangle \text{ (never undefined)}$$
$$\text{Set } 4x^3y = 0 \qquad\qquad x^4 = 0$$
$$x = 0$$

  There are infinitely many critical points, of the form $(0,y)$.

- On $y=-x$. We use substitution

$$\begin{aligned} f(x) &= x^4(-x) = -x^5 \\ f'(x) &= -5x^4 \text{ (never undefined)} \\ \text{Set } -5x^4 &= 0 \\ x &= 0 \\ y &= 0 \end{aligned}$$

  This gives us the critical point $(0,0)$, which we already had.

- On $x^2 + y^2 = 20$ we use Lagrange. Set $g(x, y) = x^2 + y^2$. $\nabla g(x, y) = \langle 2x, 2y\rangle$, which is never $\langle 0, 0\rangle$ on the constraint. All that remains is to solve $\nabla f = \lambda \nabla g$

$$4x^3y = \lambda 2x \qquad x^4 = \lambda 2y \qquad x^2 + y^2 = 20$$

$$\begin{aligned} 4x^3y^2 &= x(\lambda 2y) \\ 4x^3y^2 &= x(x^4) \\ 0 &= x^5 - 4x^3y^2 \\ 0 &= x^3(x - 2y)(x + 2y) \\ \text{if } 0 &= x \end{aligned}$$

$$\begin{aligned} 0^2 + y^2 &= 20 \\ y &= \pm\sqrt{20} \end{aligned}$$

$$\text{if } \pm 2y = x \qquad (\pm 2y)^2 + y^2 = 20$$

$$\begin{aligned} 5y^2 &= 20 \\ y &= \pm 2 \qquad x = \pm 4 \end{aligned}$$

  This gives us the critical points $(0, \pm\sqrt{20})$, which we already had and $(\pm 4, \pm 2)$, of which only $(4, 2)$ and $(4, -2)$ lie in the domain.

- We also cannot rule out the intersections $(-\sqrt{10}, \sqrt{10})$ and $(\sqrt{10}, -\sqrt{10})$.

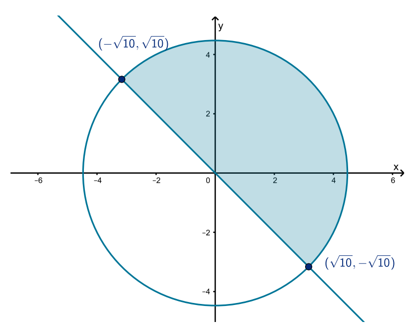
Now we plug in all the points we found

$$\begin{aligned} f(0, y) &= 0 \\ f(4, 2) &= 512 && \text{(maximum)} \\ f(4-, 2) &= -512 && \text{(minimum)} \\ f(-\sqrt{10}, \sqrt{10}) &= 100\sqrt{10} \approx 300 \\ f(\sqrt{10}, -\sqrt{10}) &= -100\sqrt{10} \approx -300 \end{aligned}$$

Q23

We solve for the intersection of $y = -x$ with $x^2 + y^2 = 20$ by substitution:

$$\begin{aligned} x^2 + (-x)^2 &= 20 \\ 2x^2 &= 20 \\ x^2 &= 10 \\ x &= \pm\sqrt{10} \\ y &= \mp\sqrt{10} \end{aligned}$$



D is closed and bounded and f is continuous so the EVT applies; one of our critical points is the maximum and one is the minimum. We look for critical points in the following places.

- In the interior

$$\nabla f(x,y) = \langle 4x^3y, x^4\rangle \text{ (never undefined)}$$
$$\text{Set } 4x^3y = 0 \qquad x^4 = 0$$
$$x = 0$$

There are infinitely many critical points, of the form $(0, y)$.

- On $y = -x$. We use substitution

$$\begin{aligned} f(x) &= x^4(-x) = -x^5 \\ f'(x) &= -5x^4 \text{ (never undefined)} \\ \text{Set } -5x^4 &= 0 \\ x &= 0 \\ y &= 0 \end{aligned}$$

This gives us the critical point $(0, 0)$, which we already had.

- On $x^2 + y^2 = 20$ we use Lagrange. Set $g(x,y) = x^2 + y^2$. $\nabla g(x,y) = \langle 2x, 2y\rangle$, which is never $\langle 0, 0\rangle$ on the constraint. All that remains is to solve $\nabla f = \lambda \nabla g$

$$4x^3y = \lambda 2x \qquad x^4 = \lambda 2y \qquad x^2 + y^2 = 20$$
$$\begin{aligned} 4x^3y^2 &= x(\lambda 2y) \\ 4x^3y^2 &= x(x^4) \\ 0 &= x^5 - 4x^3y^2 \\ 0 &= x^3(x-2y)(x+2y) \\ \text{if } 0 &= x \end{aligned}$$
$$\begin{aligned} 0^2 + y^2 &= 20 \\ y &= \pm\sqrt{20} \end{aligned}$$
$$\text{if } \pm 2y = x \qquad (\pm 2y)^2 + y^2 = 20$$
$$\begin{aligned} 5y^2 &= 20 \\ y &= \pm 2 \qquad x = \pm 4 \end{aligned}$$

This gives us the critical points $(0, \pm\sqrt{20})$, which we already had and $(\pm 4, \pm 2)$, of which only $(4, 2)$ and $(4, -2)$ lie in the domain.

- We also cannot rule out the intersections $(-\sqrt{10}, \sqrt{10})$ and $(\sqrt{10}, -\sqrt{10})$.

Now we plug in all the points we found

$$\begin{aligned} f(0,y) &= 0 \\ f(4,2) &= 512 && \text{(maximum)} \\ f(4-,2) &= -512 && \text{(minimum)} \\ f(-\sqrt{10}, \sqrt{10}) &= 100\sqrt{10} \approx 300 \\ f(\sqrt{10}, -\sqrt{10}) &= -100\sqrt{10} \approx -300 \end{aligned}$$

Q25

Since our constraint is $g(x,y) \leq c$, that means values of g smaller than c are in D, so $\nabla g(P)$ must point out of D. Since $\nabla f(P)$ also points out of D, λ must be positive.

Q27

The constraints have gradients $\nabla g = \langle 1, 1\rangle$ and $\nabla h = \langle 1, -1\rangle$. We get four equations that nicely pair off to add and solve.

$$\begin{aligned} f_x(x,y) &= \lambda + \mu & x + y &= 1 \\ +f_y(x,y) &= \lambda - \mu & +x - y &= 0 \\ f_x(x,y) + f_y(x,y) &= 2\lambda & 2x &= 1 \\ \frac{1}{2}(f_x(x,y) + f_y(x,y)) &= \lambda & x &= \frac{1}{2} \\ f_x(x,y) &= \frac{1}{2}(f_x(x,y) + f_y(x,y)) + \mu & \frac{1}{2} + y &= 1 \\ \frac{1}{2}(f_x(x,y) - f_y(x,y)) &= \mu & y &= \frac{1}{2} \end{aligned}$$

This suggets that no matter what the derivatvies of f are, there is a solution at $(x,y) = \left(\frac{1}{2}, \frac{1}{2}\right)$. This is the only point that satisfies both constraints, so it will automatically be the maximum and minimum of the constrained optimziation.

Q29

a The distance to the origin is $\sqrt{x^2+y^2+z^2}$. It is easier to minimize $f(x,y,z)=x^2+y^2+z^2$ instead, since larger numbers have larger sqaure roots. Our constraint is $g(x,y,z)=7x+6y-3z-42$.

$$\nabla f(x,y,z)=\langle 2x,2y,2z\rangle$$
$$\nabla g(x,y,z)=\langle 7,6,-3\rangle$$

$\nabla g(x,y,z)$ is never $\vec{0}$, so the minimum must satisfy $\nabla f(P)=\lambda\nabla g(P)$.

$$\begin{array}{cccc}
2x=\lambda 7 & 2y=\lambda 6 & 2z=\lambda(-3) & 7x+6y-3z-42=0\\
\frac{2x}{7}=\lambda & \frac{2y}{6}=\lambda & -\frac{2z}{3}=\lambda & \\
\frac{2x}{7}=\frac{2y}{6} & & \frac{2y}{6}=-\frac{2z}{3} & \\
x=\frac{7y}{6} & & -\frac{y}{2}=z & \\
 & & & 7\frac{7y}{6}+6y+3\frac{y}{2}-42=0\\
 & & & \frac{94}{6}y=42\\
 & & & y=\frac{126}{47}\\
x=\frac{147}{47} & & z=-\frac{63}{47} &
\end{array}$$

Since we can be reasonably sure that a closest point exists, this must be it. Our minimum is at $A=\left(\frac{147}{47},\frac{126}{47},-\frac{63}{47}\right)$.

b If $\overrightarrow{OA}$ is not normal to the plane, then there is some B in the plane such that the angle between $\overrightarrow{AO}$ and $\overrightarrow{AB}$ is acute. Travelling toward B from A will decrease the distance to O, which suggests that A was not the closest point at all.

c Since $\overrightarrow{OA}$ and $\langle 7,6,-3\rangle$ are both normal to the plane, they are scalar multiples of each other. Since $\overrightarrow{OA}=\langle 7m,6m,-3m\rangle$, the coordinates of A are $(7m,6m,-3m)$. Since A lies on the plane it should satisfy the equation of the plane.

$$7(7m)+6(6m)-3(-3m)-42=0$$
$$94m=42$$
$$m=\frac{21}{47}$$

From this, we determine $A=\left(\frac{147}{47},\frac{126}{47},-\frac{63}{47}\right)$.

Q31

Let p be the price per square meter glass. $5p$ is the cost per square meter of slate. The cost function is our objective function. $f(\ell, w, h) = 5p\ell w + 2p\ell h + 2pwh$. Our constraint is $g(\ell, w, h) = \ell wh = 20$.

$$\nabla f(\ell, w, h) = \langle p(5w + 2h), p(5\ell + 2h), 2p(\ell + w)\rangle$$
$$\nabla g(\ell, w, h) = \langle wh, \ell h, \ell w\rangle$$

$\nabla g(\ell, w, h) = \vec{0}$ only if two or three of the dimensions are 0, which does not satisfy our constraint. In fact none of the dimensions can be 0, so we can divide by them when solving $\nabla f = \lambda\nabla g$.

$$\begin{array}{cccc}
p(5w+2h) = \lambda wh & p(5\ell+2h) = \lambda \ell h & 2p(\ell+w) = \lambda \ell w & \ell wh = 20 \\
\left(\frac{5}{h}+\frac{2}{w}\right) = \frac{\lambda}{p} & \left(\frac{5}{h}+\frac{2}{\ell}\right) = \frac{\lambda}{p} & \left(\frac{2}{w}+\frac{2}{\ell}\right) = \frac{\lambda}{p} & \\
\left(\frac{5}{h}+\frac{2}{w}\right) = \left(\frac{5}{h}+\frac{2}{\ell}\right) & \left(\frac{5}{h}+\frac{2}{\ell}\right) = \left(\frac{2}{w}+\frac{2}{\ell}\right) & & \\
\frac{2}{w} = \frac{2}{\ell} & \frac{5}{h} = \frac{2}{w} & & \\
w = \ell & \frac{5}{2}w = h & & \\
 & & & (w)w\left(\frac{5}{2}w\right) = 20 \\
 & & & \frac{5}{2}w^3 = 20 \\
 & & & w = 2 \\
2 = \ell & 5 = h & &
\end{array}$$

Assuming a minimum cost exists, $(\ell, w, h) = (2, 2, 5)$ must be the minimum. The aquarium should have a $2m$ by $2m$ square base and a height of $5m$.

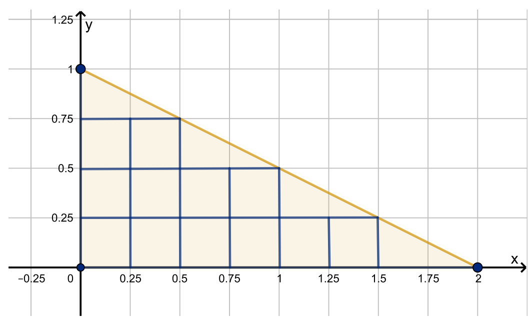
Q1

The use prisms of any shape base. The height is given by the height of the surface.

Q3

Fubini's Theorem tells us that we can compute a double integral (dA) as an iterated integral $(dydx)$.

Q5

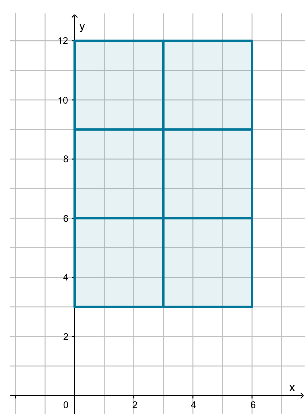


As the side lengths shrink to 0, the areas of the triangles will shrink to 0 as well, even if we add them all together. Thus the prisms above them will contribute an arbitrarily small amount of volume to our approximation. Therefore it doesn't matter whether we include them or not, when taking a limit.

Q7

$A = (3)(3) = 9$. Our test points are $(3,3)$, $(6,3)$, $(3,6)$, $(6,6)$, $(3,9)$, and $(6,9)$.

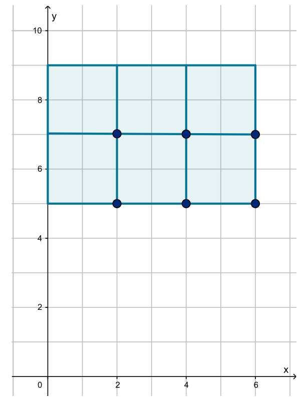
$$\begin{aligned}\iint_D f(x,y)\,dA &\approx \sum_i f(x_i^*, y_i^*)A \\ &\approx (3)(3)(9) + (6)(3)(9) + (3)(6)(9) \\ &\quad + (6)(6)(9) + (3)(9)(9) + (6)(9)(9) \\ &\approx 9(9 + 18 + 18 + 36 + 27 + 54) \\ &\approx 9(162) \\ &\approx 1458\end{aligned}$$



Q9

a $A = 2 \times 2 = 4$. The test points are marked in the diagram.

$$\begin{aligned}\int_0^6 \int_5^9 \frac{x}{y}\, dydx &\approx \sum_i f(x_i^*, y_i^*)A \\ &\approx \left(\frac{2}{5} + \frac{4}{5} + \frac{6}{5} + \frac{2}{7} + \frac{4}{7} + \frac{6}{7}\right)4 \\ &\approx \frac{48}{5} + \frac{48}{7} \\ &\approx \frac{576}{35}\end{aligned}$$



b In the first quadrant, the function $\frac{x}{y}$ increases as x increases and decreases as y increases. Thus the lower right corner, which has the greatest x and least y in each square, has the greatest value of f in each square. This suggests that using the lower right corners as test points will overestimate the true value of the integral.

Q11

The area of the $y = 2$ cross-section is obtained by setting y equal to 2 and integrating $f(x, y)$ with respect to x.

$$\begin{aligned}\text{Area} &= \int_0^5 f(x, 2)\, dx \\ &= \int_0^5 4\sin \pi x + 9\, dx \\ &= -\frac{4}{\pi}\cos(\pi x) + 9x\Big|_0^5 \\ &= -\frac{4}{\pi}(-1) + 36 + \frac{4}{\pi}(1) - 0 \\ &= 36 + \frac{8}{\pi}\end{aligned}$$

Q13

$$\begin{aligned}\iint_R y^2 \sin \pi x + 9 \; dA &= \int_0^4 \int_0^3 y^2 \sin \pi x + 9 \; dA \\ &= \int_0^5 \frac{y^3}{3} \sin \pi x + 9y \Big|_0^3 \; dx \\ &= \int_0^5 9 \sin \pi x + 27 \; dx \\ &= -\frac{9}{\pi} \cos \pi x + 27x \Big|_0^5 \\ &= -\frac{9}{\pi}(-1) + (27)(4) + \frac{9}{\pi}(1) - (27)(0) \\ &= 108 + \frac{18}{\pi}\end{aligned}$$

Q15

$$\begin{aligned}\int_4^5 \int_0^3 y e^x \; dy dx &= \int_4^5 \frac{y^2 e^x}{2} \Big|_0^3 \; dx \\ &= \int_4^5 \frac{9}{2} e^x - 0 e^x \; dx \\ &= \frac{9}{2} e^x \Big|_4^5 \\ &= \frac{9}{2} e^5 - \frac{9}{2} e^4\end{aligned}$$

Q17

$A(x) = \int_c^d f(x)g(y) \; dy$. Since x is not a variable of integration, $f(x)$ is a constant multiple, so we can rewrite $A(x) = f(x) \int_c^d g(y) \; dy$.

Q19

There are infinitely many ways to break up the integrand. The most straightforward way is $\iint_R y^2 \sin \pi x \, dA = \left(\int_0^5 \sin(\pi x) \; dx \right) \left(\int_0^3 y^2 \; dy \right)$

Q21

$$\begin{aligned}
\text{mass} &= \iint_R 3 + \sin 2x \; dA \\
&= \int_0^4 \int_0^{10} 3 + \sin 2x \; dydx \\
&= \int_0^4 3y + y\sin 2x \Big|_0^{10} \; dx \\
&= \int_0^4 30 + 10\sin 2x - 0 - 0 \; dx \\
&= 30x - 5\cos 2x \Big|_0^4 \\
&= (30)(4) - 5\cos 8 - (0)(4) + 5\cos 0 \\
&= 125 - 5\cos 8
\end{aligned}$$

Q23

We could approximate this region by prisms over R whose height is the vertical distance between $f(x, y)$ and $g(x, y)$. The approximation can be written carefully to fit the definition of a double integral.

$$\text{Volume} = \lim_{|A|\to 0} \sum_{i=1}^{n} (g(x_1^*, y_1^*) - f(x_1^*, y_i^*))A_i = \iint_R (g(x, y) - f(x, y)) \; dA$$

Q1

- Choose an inner and outer variable
- Identify the upper and lower bounds of the inner vairable, which may be functions of the outer variable. If necessary, break the region into pieces.
- Set the bounds of the outer variable to the highest and lowest values of that variable in the region.

Q3

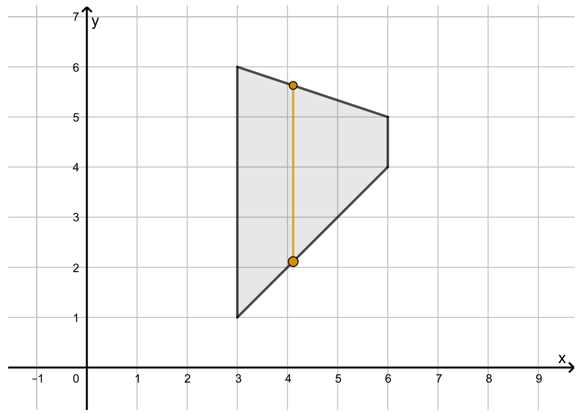
Anti-symmetry is when a function has opposite values on the other side of some line of symmetry. If a region is symmetric across this line, then the integral of the anti-symmetric function on that region is 0.

Q5

The upper y-bound is the line from $(4,0)$ to $(0,8)$. This has equation $y=-2x+8$. The lower y-bound is the line from $(4,0)$ to $(0,-2)$. This has equation $y=\frac{1}{2}x-2$.

$$\begin{aligned}
\iint_D x^2y\ dA &= \int_0^4 \int_{\frac{1}{2}x-2}^{-2x+8} x^2y\ dydx \\
&= \int_0^4 \left.\frac{x^2y^2}{2}\right|_{\frac{1}{2}x-2}^{-2x+8} dx \\
&= \int_0^4 \frac{x^2}{2}\left(4x^2-32x+64-\frac{1}{4}x^2+2x-4\right)\ dx \\
&= \int_0^4 \frac{1}{8}(15x^4-120x^3+240x^2)\ dx \\
&= \left.\frac{1}{8}(3x^5-30x^4+80x^3)\right|_0^4 \\
&= \frac{1}{8}(3072-7680+5120-0+0-0) \\
&= \frac{512}{8} \\
&= 64
\end{aligned}$$

Q7



The x bounds of D are $3 \leq x \leq 6$. At each value of x, the cross sections have a y-lower bound on the segment from $(3, 1)$ to $(6, 4)$ and a y upper bound on the segment from $(3, 6)$ to $(6, 5)$. We write equations for each.

$$m = \frac{4-1}{6-3} \qquad m = \frac{5-6}{6-3} \qquad \text{(rise/run)}$$

$$= 1 \qquad = -\frac{1}{3}$$

$$y - 1 = 1(x-3) \qquad y - 6 = -\frac{1}{3}(x-3) \qquad \text{(point-slope form)}$$

$$y = x - 2 \qquad y = -\frac{1}{3}x + 7 \qquad \text{(solve for } y\text{)}$$

Now we can set up the iterated integral

$$\int_3^6 \int_{x-2}^{-\frac{1}{3}x+7} f(x,y)\ dydx$$

Q9

First we compute the intersections

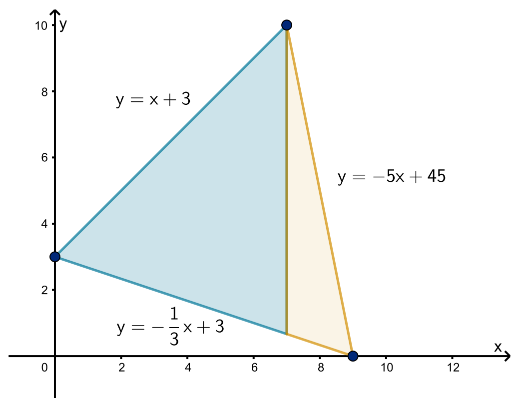
$$\begin{aligned} 6 - x^2 &= x \\ 0 &= x^2 + x - 6 \\ 0 &= (x+3)(x-2) \\ -3 \text{ or } 2 &= x \end{aligned}$$

We can use 0 as a test point to see $6 - x^2$ is the upper bound and x is the lower bound.

$$\begin{aligned} \iint_D xe^y \, dA &= \int_{-3}^{2} \int_{x}^{6-x^2} xe^y \, dy dx \\ &= \int_{-3}^{2} xe^y \Big|_{x}^{6-x^2} \, dx \\ &= \int_{-3}^{2} xe^{6-x^2} - xe^x \, dx \\ &= -\frac{1}{2}e^{6-x^2} - xe^x + e^x \Big|_{-3}^{2} && u\text{-sub and by parts} \\ &= \left(-\frac{1}{2}e^2 - 2e^2 + e^2\right) - \left(-\frac{1}{2}e^{-3} + 3e^{-3} + e^{-3}\right) \\ &= -\frac{3}{2}e^2 - \frac{7}{2}e^{-3} \end{aligned}$$

Q11

- The line from $(0,3)$ to $(9,0)$ has equation $y=-\frac{1}{3}x+3$ and is the lower bound of y for all values of x.
- The line from $(0,3)$ to $(7,10)$ has equation $y=x+3$ and is the upper bound of y for $0\leq x\leq 7$.
- The line from $(7,10)$ to $(9,0)$ has equation $y=-5x+45$ and is the upper bound of y for $7\leq x\leq 9$.



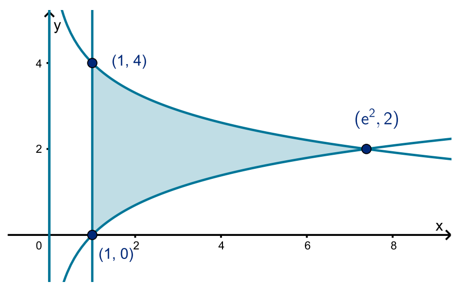
Since the upper bound changes at $x=7$ we divide T into two regions to write the integral

$$\iint_T f(x,y)\ dA=\int_0^7\int_{-\frac{1}{3}x+3}^{x+3} f(x,y)\ dydx+\int_7^9\int_{-\frac{1}{3}x+3}^{-5x+45} f(x,y)\ dydx$$

Q13

First we compute the intersections

$$\begin{aligned} \ln x &= 4 - \ln x \\ 2\ln x &= 4 \\ \ln x &= 2 \\ x &= e^2 \\ y &= \ln e^2 \\ y &= 2 \end{aligned} \qquad \begin{aligned} y &= \ln 1 \\ y &= 0 \end{aligned} \qquad \begin{aligned} y &= 4 - \ln 1 \\ y &= 4 \end{aligned}$$



- For y as the inner variable, we can use 1 as a test point to see $4 - \ln x$ is the upper bound and $\ln x$ is the lower bound.

$$\iint_D f(x,y)\ dA = \int_1^{e^2} \int_{\ln x}^{4-\ln x} f(x,y)\ dydx$$

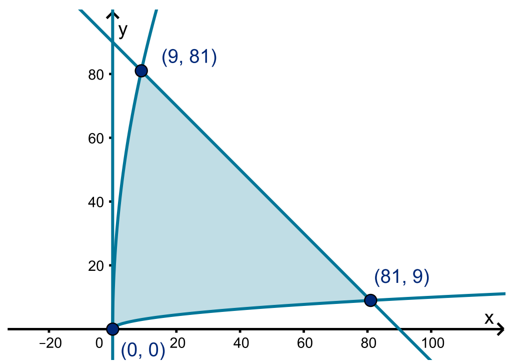
- For x as the inner variable, we invert the functions $x = e^y$ and $x = e^{4-y}$. $x = 1$ is the lower bound. The upper bounds switch at the intersection $y = 2$.

$$\iint_D f(x,y)\ dA = \int_0^2 \int_1^{e^y} f(x,y)\ dxdy + \int_2^4 \int_1^{e^{4-y}} f(x,y)\ dxdy$$

Q15

First we need to solve for the intersections

$$\begin{aligned} \sqrt{x} &= 27\sqrt{x} \\ 0 &= 26\sqrt{x} \\ 0 &= x \\ 0 &= y \end{aligned} \qquad \begin{aligned} \sqrt{x} &= 90 - x \\ x + \sqrt{x} - 90 &= 0 \\ (\sqrt{x}+10)(\sqrt{x}-9) &= 0 \\ x &= 81 \\ y &= 9 \end{aligned} \qquad \begin{aligned} 27\sqrt{x} &= 90 - x \\ x + 27\sqrt{x} - 90 &= 0 \\ (\sqrt{x}+30)(\sqrt{x}-3) &= 0 \\ x &= 9 \\ y &= 81 \end{aligned}$$



a $$\iint_D f(x,y)\, dA = \int_0^9 \int_{\sqrt{x}}^{27\sqrt{x}} f(x,y)\, dydx + \int_9^{81} \int_{\sqrt{x}}^{90-x} f(x,y)\, dydx$$

b We need to invert all of our boundary functions.

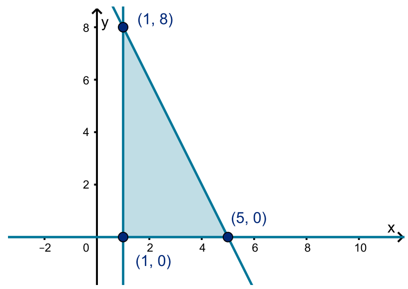
$$\begin{aligned} y &= 90 - x && \Rightarrow && x = 90 - y \\ y &= \sqrt{x} && \Rightarrow && x = y^2 \\ y &= 27\sqrt{x} && \Rightarrow && x = \frac{y^2}{729} \end{aligned}$$

We can now rewrite the integral

$$\iint_D f(x,y)\, dA = \int_0^9 \int_{y^2/729}^{y^2} f(x,y)\, dxdy + \int_9^{81} \int_{y^2/729}^{90-y} f(x,y)\, dxdy$$

Q17

This region is bounded above and below by $y = 10 - 2x$ and $y = 0$. These intersect at $x = 5$, so that explains one of the x-bounds. The other bound must be the vetical line $x = 1$.



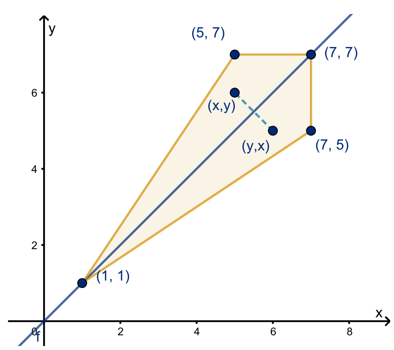
From the diagram, we see that the upper bound of x is $y = 10 - 2x$ which inverts to $x = 5 - \frac{y}{2}$.

$$\int_1^5 \int_0^{10-2x} f(x, y)\ dydx = \int_0^8 \int_1^{5-y/2} f(x, y)\ dxdy$$

Q19

The domain of integration is a circle of radius 8 centered at the origin. This is symmetric across any line through the origin. $f(x, y)$ is antisymmetric across $y = 0$ since $\sqrt[3]{\cos x \sin y} = -\sqrt[3]{\cos x \sin(-y)}$. Thus the integral over top half of the domain $(y > 0)$ cancels out the integral over the bottom half $(y < 0)$. The total integral is 0.

Q21



R is symmetric across the line $y = x$. A point and its reflection have their coordinates switched, so do make an antisymmetry argument we would need $f(x, y) = -f(y, x)$.

Q23

The region from $x = a$ to $x = b$ that lies between $y = -h(x)$ and $y = h(x)$ is symmetric across the x axis. Each point (x, y) has the mirror point $(x, -y)$, In order to apply antisymmetry, we need $f(x, y) = -f(x, -y)$.

Q25

No. The integral would become

$$\begin{aligned}\int_0^2 \int_0^{1-\frac{1}{2}y} e^{y^2}\ dxdy &= \int_0^2 xe^{y^2}\Big|_0^{1-\frac{1}{2}y}\ dy \\ &= \int_0^2 e^{y^2} - \frac{1}{2}ye^{y^2}\ dy\end{aligned}$$

This integrand cannot be integrated by substitution. There is no way to write the antiderivative that can be evaluated exactly.

Q27

Taking the inner integral leaves $\displaystyle\int_0^2 \frac{3e^{3x}}{x} - \frac{e^{3x}}{x^2}\ dx$. We don't know an antiderivative for this. Instead reverse the order of integration. Since the domain is a rectangle, we can just switch the bounds.

$$\begin{aligned}\int_0^2 \int_0^3 ye^{xy}\ dydx &= \int_0^3 \int_0^2 ye^{xy}\ dxdy \\ &= \int_0^3 e^{xy}\Big|_0^2\ dy \\ &= \int_0^3 e^{2y} - 1\ dy \\ &= \frac{1}{2}e^{2y} - y\Big|_0^3 \\ &= \frac{1}{2}e^6 - 3 - \frac{e^0}{2} + 0 \\ &= \frac{e^6 - 7}{2}\end{aligned}$$

Q29

Since the integrand is 1, this is the area of the region of integration. The region of integration is a quarter circle of radius 10. Its area is $A = \frac{\pi 10^2}{4} = 25\pi$.

Q31

It is a function of y. It gives the area of a cross section of the solid under $z = f(x, y)$ above the region between $x = f(y)$ and $x = g(y)$. For each y, the cross-section is perpendicular to the y-axis at that value of y.

Q1

We integrate the density function over the region in $\mathbb{R}^2$ that consists of the ordered pairs (x, y) of the outcomes in our set.

Q3

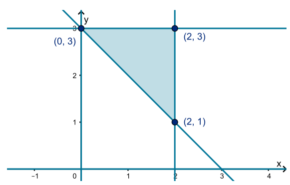
Independence means that the outcome of one variable does not depend on the outcome of the other.

Q5

It computes the probability that x is between 0 and 1 and y is between 0 and x.

Q7

Let the region above $x + y = 3$ be denoted by D.



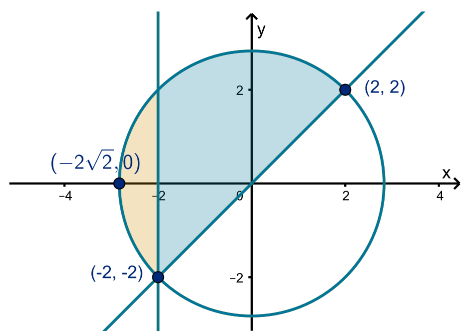
$$
\begin{aligned}
P(X+Y \geq 3) &= \iint_D f_{X,Y}(x,y)\ dA \\
&= \int_0^2 \int_{3-x}^3 \frac{y^2}{18}\ dydx + \iint_{\text{rest of } D} 0\ dA \\
&= \int_0^2 \left.\frac{y^3}{54}\right|_{3-x}^3 dx \\
&= \int_0^2 \frac{27 - 27 + 27x - 9x^2 + x^3}{54}\ dx \\
&= \int_0^2 \frac{27x - 9x^2 + x^3}{54}\ dx \\
&= \left.\frac{x^2}{4} - \frac{x^3}{18} + \frac{x^4}{216}\right|_0^2 \\
&= \frac{4}{4} - \frac{8}{18} + \frac{16}{216} \\
&= 1 - \frac{4}{9} + \frac{2}{27} \\
&= \frac{17}{27}
\end{aligned}
$$

Q9

a Only values (x, y) in the disk $x^2 + y^2 \leq 8$ are possible.

b The probability density function is larger when x is small and y is large. Thus the most likely outcome is in the upper left region of the disk and the least likely outcomes are in the lower right region of the disk.

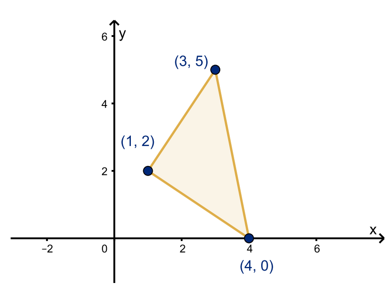
c The region of possible outcomes where $Y > X$ does not have a single bottom curve so we divide it in two. We solve for the points on the circle where $x = 0$ and where $y = x$.



$$P(Y > X) = \int_{-2\sqrt{2}}^{2} \int_{-\sqrt{8-x^2}}^{\sqrt{8-x^2}} \frac{4 - x + y}{32\pi}\, dydx + \int_{-2}^{2} \int_{x}^{\sqrt{8-x^2}} \frac{4 - x + y}{32\pi}\, dydx$$

Q11

$f_X(x) = \displaystyle\int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy$. This is only nonzero, if the vertical line at x meets the nonzero inputs of the density function. This is the case for any x in $[1, 4]$.



Q13

Since the density function is only nonzero in the unit disk, we have for each x that

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f_{X,Y}(x,y)\ dydx$$

These cross sections are longer in the y-direction near $x = 0$ and shorter near $x = 1$ or $x = -1$. In order to have a constant value, the height of the cross sections (which is the height of $z = f_{X,Y}(x,y)$) must be larger, on average, near $x = 1$ and $x = -1$ and smaller near $x = 0$.

Q15

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \\ &= \int_{-\infty}^{\infty} 0\ dy && \text{if } x < 0 \text{ or } x > 1 \\ &= \int_{0}^{1} x + y\ dy && \text{if } 0 \le x \le 1 \\ &= xy + \frac{y^2}{2}\Big|_0^1 \\ &= x + \frac{1}{2} \end{aligned}$$

$$f_X(x) = \begin{cases} x + \frac{1}{2} & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x < 0 \text{ or } x > 1 \end{cases}$$

Q17

T is the region between $y = 0$ and $y = 1 - x$ from $x = 0$ to $x = 1$.

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \\ &= \int_{-\infty}^{\infty} 0dy && \text{if } x < 0 \text{ or } x > 1 \\ &= \int_{0}^{1-x} 6xdy && \text{if } 0 \le x \le 1 \\ &= 6xy\Big|_0^{1-x} \\ &= 6x - 6x^2 \end{aligned}$$

$$f_X(x) = \begin{cases} 6x - 6x^2 & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x < 0 \text{ or } x > 1 \end{cases}$$

Q19

Since X and Y are independent, we can write $f_{X,Y}(x,y) = g(x)h(y)$ The given information becomes

$$g(3)h(7) = 0.1 \qquad g(5)h(7) = 0.15 \qquad g(5)h(2) = 0.21$$

Dividing the first two equations gives $\frac{g(3)}{g(5)} = \frac{2}{3}$.

$$\begin{aligned} f_{X,Y}(3,2) &= g(3)h(2) \\ &= g(5)h(2)\frac{g(3)}{g(5)} \\ &= (0.21)\left(\frac{2}{3}\right) \\ &= 0.14 \end{aligned}$$

Q21

The range of possible Y values must be the same no matter what value X takes. Thus D has a straight top and bottom. It must be a rectangle.

Q23

The region between the circle of radius 1 and the circle of radius 2 centered at the origin.

Q25

Let $Z_1 = g_1(X,Y)$ and $Z_2 = g_2(X,Y)$. Then

$$\begin{aligned} E[Z_1 + Z_2] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (g_1(x,y) + g_2(x,y)) f_{X,Y}(x,y)\ dydx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(x,y) f_{X,Y}(x,y)\ dydx + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_2(x,y) f_{X,Y}(x,y)\ dydx \\ &= E[Z_1] + E[Z_2] \end{aligned}$$

by the sum rule of integals.

Q27

$$\begin{aligned}
E[Y] &= \iint_{\mathbb{R}^2} y f_{X,Y}(x,y)\ dA \\
&= \int_0^{1000} \int_0^x y f_{X,Y}(x,y)\ dydx + \iint_{\text{everywhere else}} y f_{X,Y}(x,y)\ dA \\
&= \int_0^{1000} \int_0^x y \frac{12-0.012x}{1000} \left(\frac{y}{x^2} - \frac{y^2}{x^3} \right)\ dydx \\
&= \int_0^{1000} \frac{12-0.012x}{1000} \int_0^x \frac{y^2}{x^2} - \frac{y^3}{x^3}\ dydx \\
&= \int_0^{1000} \frac{12-0.012x}{1000} \left(\frac{y^3}{3x^2} - \frac{y^4}{4x^2} \right) \Big|_0^x\ dx \\
&= \int_0^{1000} \frac{12-0.012x}{1000} \left(\frac{x}{3} - \frac{x}{4} \right)\ dx \\
&= \int_0^{1000} \frac{12-0.012x}{1000} \frac{x}{12}\ dx \\
&= \int_0^{1000} \frac{x-0.001x^2}{1000}\ dx \\
&= \frac{x^2}{2000} - \frac{0.001x^3}{3000} \Big|_0^{1000} \\
&= 500 - \frac{1000}{3} \\
&= \frac{500}{3}
\end{aligned}$$

Q29

Darmok's expected arrival is the expected value of X.

$$\begin{aligned}
E[X] &= \iint_{\mathbb{R}^2} x f_{X,Y}(x,y)\ dA \\
&= \int_0^4 \int_0^4 x f_{X,Y}(x,y)\ dydx + \iint_{\text{everywhere else}} x f_{X,Y}(x,y)\ dA \\
&= \int_0^4 \int_0^4 \frac{x^2}{32}\ dydx + 0 \\
&= \left(\int_0^4 \frac{x^2}{32}\ dx\right)\left(\int_0^4\ dy\right) \\
&= \left(\frac{x^3}{96}\Big|_0^4\right)\left(y\Big|_0^4\right) \\
&= \left(\frac{64}{96} - 0\right)(4-0) \\
&= \frac{8}{3}
\end{aligned}$$

Darmok's expected arrival time is $\frac{8}{3}$ hours after noon, or $2:40PM$.

Q31

Let $g(x,y) = xy$. Then

$$\begin{aligned}
E[XY] &= E[g(X,Y)] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\ dydx \\
&= \int_0^1 \int_0^1 (xy)(x+y)\ dydx + \iint_{\text{elsewhere}} (xy)(0) dydx \\
&= \int_0^1 \int_0^1 x^2y + xy^2\ dydx \\
&= \int_0^1 \frac{x^2y^2}{2} + \frac{xy^3}{3}\Big|_0^1\ dx \\
&= \int_0^1 \frac{x^2}{2} + \frac{x}{3}\ dx \\
&= \frac{x^3}{6} + \frac{x^2}{6}\Big|_0^1 \\
&= \frac{1}{6} + \frac{1}{6} \\
&= \frac{1}{3}
\end{aligned}$$

Q33

Let $g(x,y) = x^2y^2$. Then

$$\begin{aligned}
E[X^2Y^2] &= E[g(X,Y)] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\ dydx \\
&= \int_0^1\int_0^1 (x^2y^2)(4xy-2x-2y+2)\ dydx + \iint_{\text{elsewhere}} (x^2y^2)(0)dydx \\
&= \int_0^1\int_0^1 4x^3y^3 - 2x^3y^2 - 2x^2y^3 + 2x^2y^2\ dydx \\
&= \int_0^1 x^3y^4 - \frac{2x^3y^3}{3} - \frac{x^2y^4}{2} + \frac{2x^2y^3}{3}\Big|_0^1\ dx \\
&= \int_0^1 x^3 - \frac{2x^3}{3} - \frac{x^2}{2} + \frac{2x^2}{3}\ dx \\
&= \int_0^1 \frac{x^3}{3} + \frac{x^2}{6}\ dx \\
&= \frac{x^4}{12} + \frac{x^3}{18}\Big|_0^1 \\
&= \frac{1}{12} + \frac{1}{18} \\
&= \frac{5}{36}
\end{aligned}$$

Q35

The unit disc has area π. The average value is

$$\begin{aligned}
f_{\text{ave}} &= \frac{1}{\pi}\iint_D f(x,y)\ dA \\
&= \frac{1}{\pi}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2y\ dydx \\
&= \frac{1}{\pi}\int_{-1}^{1} y^2\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \\
&= \frac{1}{\pi}\int_{-1}^{1} (1-x^2) - (1-x^2)\ dx \\
&= 0
\end{aligned}$$

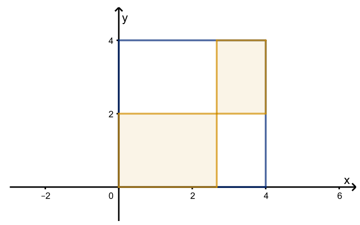
Q37

The area of the triangle T is $\frac{1}{2}bh = \frac{1}{2}(4)(8) = 16$.

$$\begin{aligned}
f_{\text{ave}} &= \frac{1}{16}\iint_D f(x,y)\ dA \\
&= \frac{1}{16}\int_0^4 \int_0^{8-2x} xy\ dydx \\
&= \frac{1}{16}\int_0^4 \left.\frac{xy^2}{2}\right|_0^{8-2x} dx \\
&= \frac{1}{16}\int_0^4 \frac{x(8-2x)^2}{2} - 0\ dx \\
&= \frac{1}{16}\int_0^4 2x^3 - 16x^2 + 32x\ dx \\
&= \left.\frac{x^4}{2} - \frac{16x^3}{3} + 16x^2\right|_0^4 \\
&= \frac{256}{2} - \frac{1024}{3} + 256 \\
&= \frac{128}{3}
\end{aligned}$$

Q39

a The function is positive when both factors are positive or both factors are negative.



b Since X and Y are independent, they factor into marginal density functions $f_X(x) = \frac{x}{8}$ and $f_Y(y) = \frac{1}{4}$. We can decompose the expected value integral as a product.

$$
\begin{aligned}
E[g(X)] &= \int_0^4 \int_0^4 (x - E[X])\,(y - E[Y]) f_X(x) f_Y(y)\; dy dx && (1)\\
&= \left(\int_0^4 (x - E[X])\, f_X(x)\; dx\right)\left(\int_0^4 (y - E[Y]) f_Y(y)\; dy\right) && (2)\\
&= \left(\int_0^4 x f_X(x)\; dx - \int_0^4 E[X] f_X(x)\; dx\right)\left(\int_0^4 y f_Y(y)\; dy - \int_0^4 E[Y] f_Y(y)\; dy\right) && (3)\\
&= \left(\int_0^4 x f_X(x)\; dx - E[X]\int_0^4 f_X(x)\; dx\right)\left(\int_0^4 y f_Y(y)\; dy - E[Y]\int_0^4 f_Y(y)\; dy\right) && (4)\\
&= \left(E[X] - E[X]\int_0^4 f_X(x)\; dx\right)\left(E[Y] - E[Y]\int_0^4 f_Y(y)\; dy\right) && (5)\\
&= (E[X] - E[X](1))\,(E[Y] - E[Y](1)) && (6)\\
&= (0)(0) && (7)\\
&= 0 && (8)
\end{aligned}
$$

$(1) \to (2)$ product decomposition

$(2) \to (3)$ sum rule

$(3) \to (4)$ constant multiple rule

$(4) \to (5)$ expected value formula

$(5) \to (6)$ PDFs integrate to 1

Q41

a $y = 6x - x^2$ intersects to x-axis at $x = 0$ and $x = 6$, so the x values present in D are $[0, 6]$. These are the possible outcomes of X.

b Let A be the area of D. Since $f_X(x) = \int_0^{6x-x^2} \frac{1}{A}\, dy$, this is largest where $6x - x^2$ is largest. So X is more likely to be near 3 and less likely to be near 0 or 6.

Q43

a No $\frac{12x-x^2+10y-y^2}{4880}$ cannot be expressed as a product of functions $g(x)h(y)$. We have a theorem that states that who random variables are independent, if and only if their density function decomposes this way.

b

$$
\begin{aligned}
P(X \leq 5) &= \int_{-\infty}^{5} \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dydx \\
&= \int_0^5 \int_0^{10} \frac{12x - x^2 + 10y - y^2}{4880}\, dydx + \iint_{\text{everywhere else}} 0\, dA \\
&= \frac{1}{4880} \int_0^5 12xy - x^2y + 5y^2 - \frac{y^3}{3}\Big|_0^{10}\, dx \\
&= \frac{1}{4880} \int_0^5 120x - 10x^2 + \frac{500}{3}\, dx \\
&= \frac{1}{4880} \left(60x^2 - \frac{10}{3}x^3 + \frac{500}{3}x\right)\Big|_0^5 \\
&= \frac{1}{4880} \left(1500 - \frac{1250}{3} + \frac{2500}{3}\right) \\
&= \left(\frac{1}{4800}\right)\left(\frac{5750}{3}\right) \\
&= \frac{5750}{14400}
\end{aligned}
$$

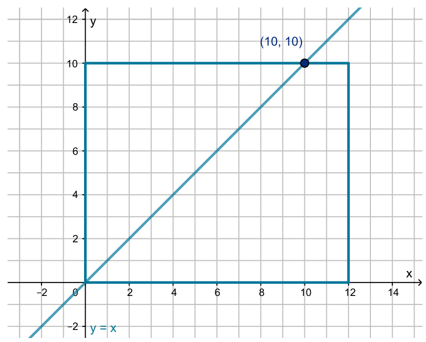
c If $X > Y$ you pay $25Y$ cents, since she brings your the snacks after Y minutes. If $Y > X$ you pay $25X$ cents, because you take her place after you get the tickets. If $X = Y$ these functions agree. Thus the payment function is

$$
g(x, y) = \begin{cases} 25y & \text{if } y \leq x \\ 25x & \text{if } y > x \end{cases}
$$

The expected value of g is

$$
E[g(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y)\, dydx
$$

$f_{X,Y}$ is only positive in the rectangle $0 \le x \le 12$, $0 \le y \le 10$. g is piecewise, so we need to divide this domain along $y = x$. The first integral is easier to write as $dxdy$, but can be split into two pieces instead, if you prefer.



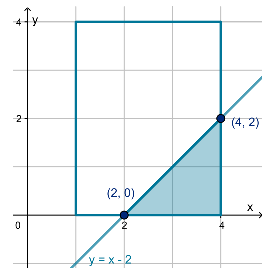
$$E[g(x,y)] = \int_0^{10}\int_y^{12} 25y\left(\frac{12x - x^2 + 10y - y^2}{4880}\right)dxdy + \int_0^{10}\int_x^{10} 25x\left(\frac{12x - x^2 + 10y - y^2}{4880}\right)dydx$$

Q45

a Darmok arrives between 1PM and 4PM. Jalad arrives between noon and 4PM.

b Darmok arrives more than 2 hours after Jalad if $X \ge Y + 2$.

$$
\begin{aligned}
P(X \geq Y+2) &= \iint_{y \leq x-2} f_{X,Y}(x,y)\ dA \\
&= \int_2^4 \int_0^{x-2} \frac{y}{6x^2}\ dydx \\
&= \int_2^4 \left. \frac{y^2}{12x^2} \right|_0^{x-2} dx \\
&= \int_2^4 \frac{x^2-4x+4}{12x^2}\ dx \\
&= \int_2^4 \frac{1}{12} - \frac{1}{3x} + \frac{1}{3x^2}\ dx \\
&\left(\frac{x}{12} - \frac{\ln|x|}{3} - \frac{1}{3x} \right) \Bigg|_2^4 \\
&= \left(\frac{4}{12} - \frac{\ln 4}{3} - \frac{1}{12} \right) - \left(\frac{2}{12} - \frac{\ln 2}{3} - \frac{1}{6} \right) \\
&= \frac{4}{12} - \frac{1}{12} - \frac{2}{12} + \frac{2}{12} - \frac{\ln 4}{3} + \frac{\ln 2}{3} \\
&= \frac{1}{4} - \frac{\ln 2}{3}
\end{aligned}
$$



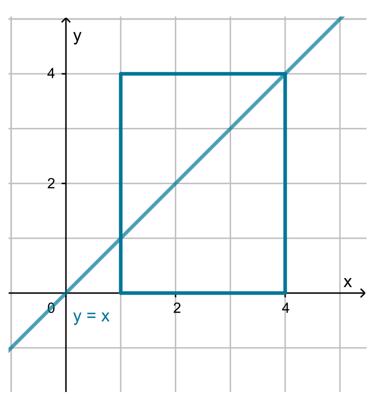
c If Darmok arrives after Jalad ($X \geq Y$) then they are together for $6 - X$ hours. If Jalad arrives after Darmok, they are together for $6 - Y$ hours. Thus our function is

$$
g(x,y) = \begin{cases} 6-x & \text{if } y \leq x \\ 6-y & \text{if } y > x \end{cases}
$$

The expected value of $g(X,Y)$ is given by

$$
E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\ dydx
$$

But this is only nonzero where f is nonzero. On that rectangle, we need to break the domain into two pieces, one for each branch of g.



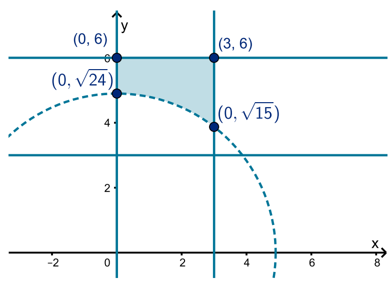
$$E[g(X,Y)] = \int_1^4 \int_0^x (6-x)\frac{y}{6x^2}\ dydx + \int_1^4 \int_x^4 (6-y)\frac{y}{6x^2}\ dydx$$

Q47

The actual value of the integral is $\left.\frac{x^3}{3}\right|_0^6 = 72$. The two equal subintervals are $[0,3]$ and $[3,6]$. let X be the test point of $[0,3]$ and Y be the test point of $[3,6]$. Since they are independent, we can multiply their individual density functions to obtain a joint density function.

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{9} & \text{if } 0 \le x \le 3 \text{ and } 3 \le y \le 6 \\ 0 & \text{otherwise} \end{cases}$$

The approximation of $\int_0^6 x^2\ dx$ is $3X^2 + 3Y^2$. We will compute the probability that this is greater than 72. To set up the bounds of integration, we draw the region where $f_{X,Y}(x,y) \neq 0$ and $x^2 + y^2 > 24$.



$$\begin{aligned} P(3X^2 + 3Y^2 > 72) &= P(X^2 + Y^2 > 24) \\ &= \int_0^3 \int_{\sqrt{24-x^2}}^6 \frac{1}{9}\ dydx \end{aligned}$$

Q1

We can break them into three iterated integrals with differentials dx, dy and dz (in any order)

Q3

The bounds of the inner integral are given by the functions that bound the region in the direction of that variable.

Q5

The region has volume $(3)(8)(4) = 96$. Each prism will have volume $V = \frac{96}{12} = 8$.

Q7

No. The order of the bounds corresponds to the variables in the differential. In the first integral we have $0 \leq x \leq 4$. In the secind we have $0 \leq x \leq 7$.

Q9

$$\begin{aligned}\int_0^2\int_0^2\int_0^3 (x+y)z\ dzdydx &= \int_0^2\int_0^2 \left.\frac{(x+y)z^2}{2}\right|_0^3\ dydx\\ &= \int_0^2\int_0^2 \frac{9x}{2} + \frac{9y}{2}\ dydx\\ &= \int_0^2 \left.\frac{9xy}{2} + \frac{9y^2}{4}\right|_0^2\ dx\\ &= \int_0^2 9x + 9 - 0 - 0\ dx\\ &= \left.\frac{9x^2}{2} + 9x\right|_0^2\\ &= \frac{36}{2} + 18\\ &= 36\end{aligned}$$

Q11

a When applying the Fundamental Theorem of Calculus to the first integral, the zs are replaced by z_0 and z_1. In the second function, the ys are additionally replaced with y_0 and y_1.

b $\int_{z_0}^{z_1} f(x,y,z)\,dz$ is the area below $w = f(x,y,z)$ above each (x,y) cross section. $\int_{y_0}^{y_1}\int_{z_0}^{z_1} f(x,y,z)\,dzdy$ is the volume below $w = f(x,y,z)$ above each x cross section.

Q13

We integrate w over z from 0 to 3, substituting $x = 4$ and $y = 1$.

$$\begin{aligned} \text{Area} &= \int_0^3 4 + e^z \ dz \\ &= 4z + e^z \Big|_0^3 \\ &= 12 + e^3 - 0 - e^0 \\ &= 11 + e^3 \end{aligned}$$

Q15

We use the triple integral formula for volume.

$$\begin{aligned} \text{Volume} &= \iiint_P 1 \ dV \\ &= \int_0^\ell \int_0^w \int_0^h \ dzdydx \\ &= \int_0^\ell \int_0^w z\Big|_0^h \ dydx \\ &= \int_0^\ell \int_0^w h \ dydx \\ &= \int_0^\ell hy\Big|_0^w \ dx \\ &= \int_0^\ell hw \ dx \\ &= hwx\Big|_0^\ell \\ &= \ell wh \end{aligned}$$

Q17

They all arrive by $12:15$, if X, Y and Z are all between 0 and $\frac{1}{4}$.

$$\begin{aligned}
P\left(0 \leq X, Y, Z \leq \frac{1}{4}\right) &= \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \frac{12}{11}(1 - x^2yz)\ dzdydx \\
&= \frac{12}{11} \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} z - \frac{x^2yz^2}{2}\Big|_0^{\frac{1}{4}}\ dydx \\
&= \frac{12}{11} \int_0^{\frac{1}{4}} \int_0^{\frac{1}{4}} \frac{1}{4} - \frac{x^2y}{32}\ dydx \\
&= \frac{12}{11} \int_0^{\frac{1}{4}} \frac{y}{4} - \frac{x^2y^2}{64}\Big|_0^{\frac{1}{4}}\ dx \\
&= \frac{12}{11} \int_0^{\frac{1}{4}} \frac{1}{16} - \frac{x^2}{1024}\ dx \\
&= \frac{12}{11} \frac{x}{16} - \frac{x^3}{3072}\Big|_0^{\frac{1}{4}}\ dx \\
&= \frac{12}{11}\left(\frac{1}{64} - \frac{1}{196,608}\right) \\
&= \frac{12}{11}\frac{3071}{196,608} \\
&= \frac{3071}{180,224}
\end{aligned}$$

Q19

a R is a ball of radius 5, centered at the origin.

b We use z for the inner variable. $x^2+y^2+z^2 \leq 25$ solves to $z = \pm\sqrt{25-x^2-y^2}$. These bounds intersect at

$$\begin{aligned}\sqrt{25-x^2-y^2} &= -\sqrt{25-x^2-y^2}\\ 0 &= 25-x^2-y^2\\ x^2+y^2 &= 25\end{aligned}$$

The xy bounds are a circle of radius 5. The y bounds solve to $y = \pm\sqrt{25-x^2}$. The circle has x values from -5 to 5.

$$\iiint_R f(x,y,z)\ dV = \int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}}\int_{-\sqrt{25-x^2-y^2}}^{\sqrt{25-x^2-y^2}} f(x,y,z)\ dzdyd$$

c The value of this integral is the volume of the sphere. You may recall

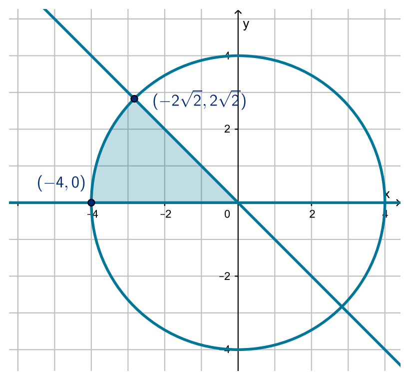
$$\begin{aligned}V &= \frac{4}{3}\pi r^3\\ &= \frac{4}{3}\pi 5^3\\ &= \frac{500\pi}{3}\end{aligned}$$

Q21

a The z lower bound is $z = 9$ and the z upper bound is $z = 25 - x^2 - y^2$. To find the x and y bounds we graph $y \leq -x$ and $y \geq 0$ and also solve

$$\begin{aligned} 9 &= 25 - x^2 - y^2 \\ x^2 + y^2 &= 16 \end{aligned}$$

Which is a circle of radius 4. Our x and y bounds are obtained from our diagram. We can solve for the intersection points.



This region does not have a single top curve, so we change the order to $dxdy$. $x^2 + y^2 = 16$ solves to $x = -\sqrt{16 - y^2}$.

$$\int_0^{2\sqrt{2}} \int_{-\sqrt{16-y^2}}^{-y} \int_9^{25-x^2-y^2} x \; dzdxdy$$

b We can see from our diagram that the domain consists of negative values of x. Thus the integrand x is negative over the entire domain, and the integral will be negative.

Q23

The two given graphs will be the upper and lower bounds of z. We solve for xy bounds.

$$\begin{aligned} x^2+y^2 &= 2y \\ x^2+y^2-2y &= 0 \\ x^2+y^2-2y+1 &= 1 && \text{complete the square} \\ x^2+(y-1)^2 &= 1 && \text{factor} \end{aligned}$$

This is a circle of radius 1 centered at $(0,1)$. We use $(0,1)$ as a test point and see that $z=x^2+y^2$ is lower than $z=2y$ on R. To write the xy bounds, we solve for y.

$$\begin{aligned} x^2+(y-1)^2 &= 1 \\ (y-1)^2 &= 1-x^2 \\ y-1 &= \pm\sqrt{1-x^2} \\ y &= 1\pm\sqrt{1-x^2} \end{aligned}$$

We can set the upper and lower y bounds equal and solve for the x bounds, or just know that the circle extends from $x=-1$ to $x=1$.

$$\iiint_R (y-1)\;dV = \int_{-1}^{1}\int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}}\int_{x^2+y^2}^{2y}(y-1)\;dzdydx$$

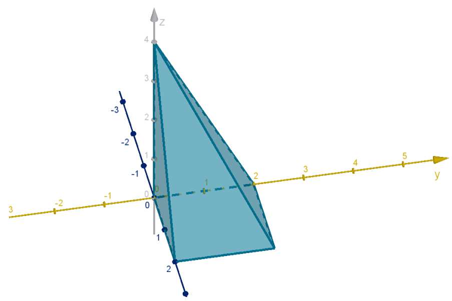
Q25

The bounds of z are x^2 and 16. Setting these equal gives $x = \pm 4$. The xy bounds are a rectangle $-4 \leq x \leq 4$ and $2 \leq y \leq 6$. We can test the z bounds using $(4, 2)$ and see $16 > x^2$ on R.

$$\begin{aligned}
\iiint_R x + z \; dV &= \int_{-4}^{4} \int_{2}^{6} \int_{x^2}^{16} x + z \; dzdydx \\
&= \int_{-4}^{4} \int_{2}^{6} xz + \frac{z^2}{2} \bigg|_{x^2}^{16} \; dydx \\
&= \int_{-4}^{4} \int_{2}^{6} 16x + 128 - x^3 - \frac{x^4}{4} \; dydx \\
&= \int_{-4}^{4} (16x + 128 - x^3 - \frac{x^4}{4})y \bigg|_{2}^{6} \; dx \\
&= \int_{-4}^{4} (16x + 128 - x^3 - \frac{x^4}{4})(6) - (16x + 128 - x^3 - \frac{x^4}{4})(2) \; dx \\
&= \int_{-4}^{4} 64x + 512 - 4x^3 - x^4 \; dx \\
&= 32x^2 + 512x - x^4 - \frac{x^5}{5} \bigg|_{-4}^{4} \\
&= 512 + 2048 - 256 - \frac{1024}{5} - 512 + 2048 + 256 - \frac{1024}{5} \\
&= \frac{52,736}{15}
\end{aligned}$$

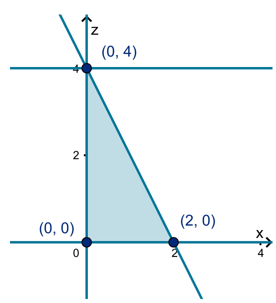
Q27

a P looks like this:



There are two upper surfaces, so we would need to split this into two integrals if we use dz as the inner integral.

b The upper planes have slope-intercept equations $z = 4 - 2x$ and $z = 4 - 2y$. The other bounds were $x = 0$, $y = 0$ and $z = 0$. We can use x or y as the inner integral. If we choose y, the bounds are $y = 0$ and $y = \frac{4-z}{2}$. These intersect at $z = 4$. To write xz-bounds, we draw the remaining bounds:



$$\iiint_P f(x, y, z)\ dV = \int_0^2 \int_0^{4-2x} \int_0^{\frac{4-z}{2}} f(x, y, z)\ dydzdx$$

Q29

a Setting the z bounds equal gives $y = 4$. The xy bounds are enclosed by $x = 0$, $y = x^2$ and $y = 4$.

$$\iiint_R f(x,y,z)\ dV = \int_0^2 \int_{x^2}^4 \int_0^{4-y} f(x,y,z)\ dzdydx$$

b Setting the y bounds equal gives $z = 4 - x^2$. The xz bounds are enclosed by $x = 0$, $z = 0$ and $z = 4 - x^2$.

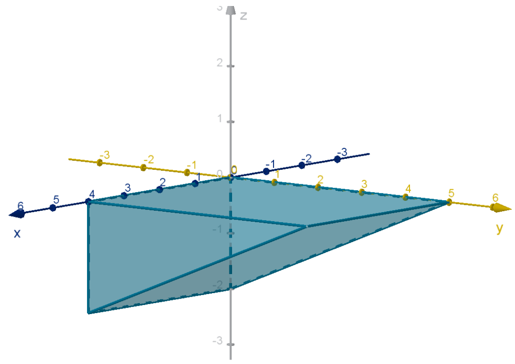
$$\iiint_R f(x,y,z)\ dV = \int_0^2 \int_0^{4-x^2} \int_{x^2}^{4-z} f(x,y,z)\ dydzdx$$

c Setting the x bounds equal gives $y = 0$. The yz bounds are enclosed by $y = 0$, $z = 0$ and $z = 4 - y$.

$$\iiint_R f(x,y,z)\ dV = \int_0^4 \int_0^{4-y} \int_0^{\sqrt{y}} f(x,y,z)\ dxdzdy$$

Q31

First we draw P



P is bounded by $x = 0$, $x = 4$, $y = 0$, $z = 0$, and $z = \frac{2}{5}y - 2$, which can also solve to $y = \frac{5}{2}z + 5$.

- If x is the inner variable, the bounds are $x = 4$ and $x = 0$. The yz region is a triangle.

$$\iiint_P g(x, y, z)\ dV = \int_0^5 \int_{\frac{2}{5}y-2}^0 \int_0^4 g(x, y, z)\ dxdzdy$$

- If y is the inner variable, the bounds are $y = 0$ and $y = \frac{5}{2}z + 5$. These intersect at $z = -2$. The xz bounds are a rectangle.

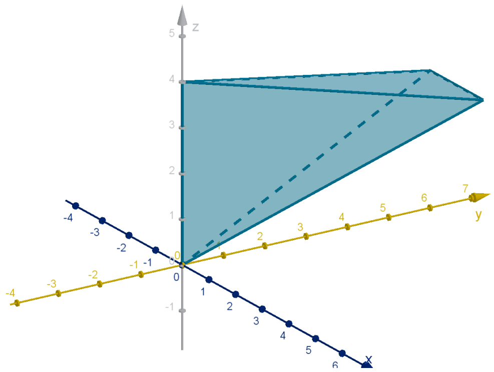
$$\iiint_P g(x, y, z)\ dV = \int_0^4 \int_{-2}^0 \int_0^{\frac{5}{2}z+5} g(x, y, z)\ dydzdx$$

- If z is the inner variable, the bounds are $z = 0$ and $z = \frac{2}{5}y - 2$. These intersect at $y = 5$. The xy bounds are a rectangle.

$$\iiint_P g(x, y, z)\ dV = \int_0^4 \int_0^5 \int_{\frac{2}{5}y-2}^0 g(x, y, z)\ dzdydx$$

Q33

If we travel through T in the x direction, we always enter and exit on the same face, so we can use these faces as the inner bounds of a single integral for T. Similarly, if we travel through T in the z direction, we always enter and exit on the same face. If we travel through T in the y direction, then depending on our position, there are two faces that could be the lower y bound and two faces that could be the upper y bound. We cannot write T as a single integral with y as the inner variable.



Q35

This is the region above $z = 0$ and below $z = x - y$. We also have $x^2 \leq y \leq x$, though $y \leq z$ is extraneous. It comes from the intersection of $z = 0$ and $z = x - y$. Similarly, the bounds of x are extraneous, since 0 and 1 are the intersections of $y = x$ and $y = x^2$. In summary, the region is

$$R = \{(x, y, z) \ : \ z \geq 0, z \leq x - y, y \geq x^2\}$$

With x as the inner variable, the relevent bounds solve to $x \geq y + z$ and $x \leq \sqrt{y}$. These intersect where $z = \sqrt{y} - y$. The other z-bound is $z = 0$. Setting these equal gives $y = 0$ or $y = 1$. The integral is

$$\int_0^1 \int_0^{\sqrt{y}-y} \int_{y+z}^{\sqrt{y}} f(x, y, z) \ dxdzdy$$

Q37

$$\begin{aligned}
&\int_3^4 \int_0^8 \int_{-1}^1 y^2 \sin x - e^{y+z} \ dzdydx \\
&= \int_3^4 \int_0^8 \int_{-1}^1 y^2 \sin x \ dzdydx - \int_3^4 \int_0^8 \int_{-1}^1 e^y e^z \ dzdydx \\
&= \int_3^4 \sin x \ dx \int_0^8 y^2 \ dy \int_{-1}^1 \ dz - \int_3^4 \ dx \int_0^8 e^y \ dy \int_{-1}^1 e^z \ dz
\end{aligned}$$

Q39

a Only two of these inequalities involve z so z is a good choice for inner integral. The upper and lower bounds are $x^2+y^2 \le z \le x^2+x+4$.

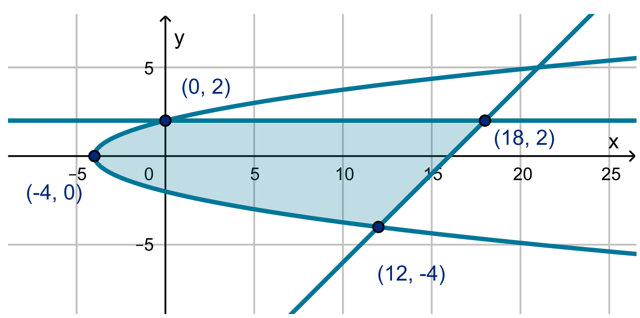
For xy bounds we have $x-16 \le y \le 2$, we also have the intersection of the top and bottom surface:

$$x^2+y^2 \le x^2+x+4$$
$$y^2-4 \le x$$

We graph these below. To solve for the intersections we substitute. The only interesting intersection is $y = x-16$ with the parabola:

$$y^2-4 = x \qquad\qquad y = x-16$$
$$y+16 = x$$

$$y^2-4 = y+16$$
$$y^2-y-20 = 0$$
$$(y+4)(y-5) = 0$$
$$\text{if } y=-4 \text{ then } x=12$$
$$\text{if } y=5 \text{ then } x=21$$



We can break this into three $dydx$ integrals or write it as one $dxdy$ integral:

$$\int_{-4}^{2}\int_{y^2-4}^{y+16}\int_{x^2+y^2}^{x^2+x+4} xyz\; dzdxdy$$

b We would like to apply the extreme value theorem. There are three things to verify.

- $f(x,y,z) = xyz$ is continuous because it is a polynomial.
- D is closed because its inequalities are not strict ($\le, \ge$).
- D is bounded in the xy-plane, according to our picture. The z values are also bounded, because x^2+y^2 and x^2+x+4 are continuous functions over a closed and bounded 2-dimensional domain. Thus D is bounded.

The extreme value theorem therefore applies, and f is guaranteed to have a maximum on D.

Q41

We will use the volume formula. For this we need the bounds of integration for R. Rotating $y = f(x)$ around the x-axis gives a circle parallel to the yz-plane of radius $f(x)$. The equation of this surface is $y^2 + z^2 = (f(x))^2$. To set up an integral, solve for z:

$$\begin{aligned} y^2 + z^2 &= (f(x))^2 \\ z^2 &= (f(x))^2 - y^2 \\ z &= \pm\sqrt{(f(x))^2 - y^2} \end{aligned}$$

For xy bounds, we have $x = a$ and $x = b$. We also have the intersections of the z-bounds, which are

$$\begin{aligned} \sqrt{(f(x))^2 - y^2} &= -\sqrt{(f(x))^2 - y^2} \\ (f(x))^2 - y^2 &= 0 \\ (f(x))^2 &= y^2 \\ \pm f(x) &= y \end{aligned}$$

We can now apply the volume formula.

$$\begin{aligned} \text{Volume} &= \iiint_R dV \\ &= \int_a^b \int_{-f(x)}^{f(x)} \int_{-\sqrt{(f(x))^2-y^2}}^{\sqrt{(f(x))^2-y^2}} dz dy dx \end{aligned}$$