Putnam Practice Session 11

Probability: Linearity of Expectation & Indicator Variables

Overview & Pedagogical Goals

Probability problems on the Putnam Competition frequently avoid heavy measure-theoretic machinery in favor of clever structural decompositions. Chief among these tools is the **linearity of expectation**, which allows one to calculate the expected value of a complicated random variable without knowing its full probability distribution.

This practice session focuses on five key techniques:

(1) **Indicator Random Variables**: Decomposing counting variables $X = \sum I_k$ into binary events to reduce global expectations to local probabilities.

(2) **Symmetry and Incremental Progressions**: Leveraging uniform distributions over discrete permutations or continuous domains to compute step-by-step increments.

(3) **Graph Invariants and Global Bounds**: Pairing topological inequalities (such as Euler's formula $V - E + F \geq C$) with expectation bounds on random configurations.

(4) **Continuous Variables and Order Density Integrals**: Computing expected values under ordering constraints by integrating density functions over continuous regions.

(5) **Algebraic Expansion of Matrix Determinants**: Decomposing determinants or Pfaffians into sums over permutations/matchings and evaluating independence across matrix entries.

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1 Study Guide & Core Principles

1.1 Linearity of Expectation & Indicator Decompositions

Theorem 1 (Linearity of Expectation)**.** *Let $X_1, X_2, \ldots, X_n$ be random variables defined on the same probability space, and let $c_1, c_2, \ldots, c_n \in \mathbb{R}$. Then*

$$\mathbb{E}\left[\sum_{i=1}^{n} c_i X_i\right] = \sum_{i=1}^{n} c_i \mathbb{E}[X_i].$$

Crucially, this identity holds ***regardless of whether the X_i are independent.***

When X counts the total number of occurrences of a specific feature, we introduce **indicator random variables** $I_k \in \{0, 1\}$, where $I_k = 1$ if feature k occurs and 0 otherwise. Since $\mathbb{E}[I_k] = \mathbb{P}(I_k = 1)$, we obtain:

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_{k} I_k\right] = \sum_{k} \mathbb{P}(\text{feature } k \text{ occurs}).$$

Example 1 (Expected Fixed Points of a Permutation)**.** *Let π be a permutation of $\{1, 2, \ldots, n\}$ chosen uniformly at random. What is the expected number of fixed points $\pi(i) = i$?*
Let I_i be the indicator that $\pi(i) = i$. Since π is chosen uniformly at random, $\mathbb{P}(\pi(i) = i) = \frac{1}{n}$ for each $i \in \{1, \ldots, n\}$. Thus,

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[I_i] = \sum_{i=1}^{n} \frac{1}{n} = 1.$$

Notice that the events $\pi(i) = i$ and $\pi(j) = j$ are not independent, yet linearity of expectation applies seamlessly.

1.2 Combining Expectations with Graph Invariants

When analyzing random grid colorings or geometric networks, individual components (like connected regions) are difficult to count directly. We bound them using structural invariants such as Euler's formula $V - E + F = 1 + C$ for planar graphs, or simple edge-removal inequalities:

$$C \geq |V| - |E|,$$

where $|V|$ is the number of vertices, $|E|$ the number of edges, and C the number of connected components. By taking expectations on both sides, local edge and face probabilities yield rigorous global lower bounds on $\mathbb{E}[C]$.

1.3 Continuous Variables and Order Density Integrals

When independent continuous random variables $X_1, X_2, \cdots \sim \text{Uniform}(0, 1)$ are subject to ordering constraints (e.g., $X_1 > X_2 > \cdots > X_i$), symmetry dictates that all $i!$ relative orderings are equally likely, so

$$\mathbb{P}(X_1 > X_2 > \cdots > X_i) = \frac{1}{i!}.$$

To evaluate the expected value of X_i conditioned on this ordering, observe that for a fixed value $X_i = t \in [0, 1]$, the preceding $i - 1$ variables $X_1, \ldots, X_{i-1}$ must all lie in $[t, 1]$ in decreasing order. The volume of this $(i-1)$-dimensional simplex is $\frac{(1-t)^{i-1}}{(i-1)!}$. Integrating over $t \in [0, 1]$ yields the joint expectation:

$$\mathbb{E}\left[X_i \cdot I(X_1 > X_2 > \cdots > X_i)\right] = \int_0^1 t \cdot \frac{(1-t)^{i-1}}{(i-1)!}\, dt = \frac{1}{(i+1)!}.$$

Dividing by $\mathbb{P}(X_1 > \cdots > X_i) = \frac{1}{i!}$ gives the conditional expectation $\mathbb{E}[X_i \mid X_1 > \cdots > X_i] = \frac{1}{i+1}$.

1.4 Expected Values of Random Determinants & The Pfaffian

For a matrix M with independent random entries M_{ij}, the determinant is expanded as a sum over permutations $\sigma \in S_n$:

$$\mathbb{E}[\det M] = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \mathbb{E}\left[\prod_{i=1}^{n} M_{i,\sigma(i)}\right].$$

If M is a $2n \times 2n$ skew-symmetric matrix ($M = A - A^T$), its determinant equals the square of the **Pfaffian**:

$$\text{Pf}(M) = \sum_{\alpha} \text{sgn}(\alpha) \prod_{(u,v) \in \alpha} M_{uv},$$

where $\alpha = \{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\}$ ranges over all perfect matchings of $\{1, 2, \ldots, 2n\}$ with $u_i < v_i$ for each pair.

Definition 1 (Sign of a Matching)**.** *The* ***sign of a matching*** $\operatorname{sgn}(\alpha)$ *is defined as the sign of the permutation* $\pi_\alpha \in S_{2n}$ *that lists the pairs in order:*

$$\pi_\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & 2n-1 & 2n \\ u_1 & v_1 & u_2 & v_2 & \ldots & u_n & v_n \end{pmatrix}.$$

Reordering the pairs does not change $\operatorname{sgn}(\pi_\alpha)$ *because swapping any two pairs requires an even number of transpositions.*

When taking expectations of $\det(M) = (\operatorname{Pf}(M))^2 = \sum_{\alpha,\beta} \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta) \prod_{(u,v)\in\alpha} M_{uv} \prod_{(s,t)\in\beta} M_{st}$, independence ensures that cross-terms with mismatched matchings ($\alpha \neq \beta$) contain at least one unrepeated linear factor M_{uv} with $\mathbb{E}[M_{uv}] = 0$, isolating only the diagonal terms $\alpha = \beta$.

2 Selected Putnam Problems

Here are five classical Putnam problems on probability and linearity of expectation, ordered by relative difficulty.

Problem 1 (Putnam 2006, Problem A4)

Let $S = \{1, 2, \ldots, n\}$ for an integer $n > 1$. Say a permutation π of S has a *local maximum* at $k \in S$ if:

(i) $\pi(k) > \pi(k+1)$ for $k = 1$;

(ii) $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$ for $1 < k < n$;

(iii) $\pi(k-1) < \pi(k)$ for $k = n$.

What is the average number of local maxima of a permutation of S, averaging over all permutations of S?

Problem 2 (Putnam 2023, Problem B3)

A sequence $y_1, y_2, \ldots, y_k$ of real numbers is called *zigzag* if $k = 1$, or if $y_2 - y_1, y_3 - y_2, \ldots, y_k - y_{k-1}$ are nonzero and alternate in sign. Let $X_1, X_2, \ldots, X_n$ be chosen independently from the uniform distribution on $[0, 1]$. Let $a(X_1, X_2, \ldots, X_n)$ be the largest value of k for which there exists an increasing sequence of integers $i_1 < i_2 < \cdots < i_k$ such that $X_{i_1}, X_{i_2}, \ldots, X_{i_k}$ is zigzag. Find the expected value of $a(X_1, X_2, \ldots, X_n)$ for $n \geq 2$.

Problem 3 (Putnam 2022, Problem A4)

Suppose that $X_1, X_2, \ldots$ are real numbers between 0 and 1 that are chosen independently and uniformly at random. Let $S = \sum_{i=1}^{k} X_i/2^i$, where k is the least positive integer such that $X_k < X_{k+1}$ (or $k = \infty$ if no such integer exists). Find the expected value of S.

Problem 4 (Putnam 2004, Problem A5)

An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares p and q are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

Problem 5 (Putnam 2016, Problem B4)

Let A be a $2n \times 2n$ matrix, with entries chosen independently at random. Every entry is chosen to be 0 or 1, each with probability $1/2$. Find the expected value of $\det(A - A^T)$ as a function of n, where A^T denotes the transpose of A.

3 Solutions and Commentary

Solution to Problem 1 (Putnam 2006, A4)

Proof. Let I_k be the indicator random variable that π has a local maximum at $k \in \{1, 2, \ldots, n\}$. By linearity of expectation, the expected number of local maxima is

$$\mathbb{E}[X] = \sum_{k=1}^{n} \mathbb{E}[I_k] = \sum_{k=1}^{n} \mathbb{P}(k \text{ is a local maximum}).$$

- For $k = 1$: $\pi(1)$ is a local maximum if $\pi(1) > \pi(2)$. Since all permutations of $\{\pi(1), \pi(2)\}$ are equally likely, $\mathbb{P}(1 \text{ is a local max}) = \frac{1}{2}$.
- For $k = n$: Similarly, $\pi(n)$ is a local maximum if $\pi(n) > \pi(n-1)$, which occurs with probability $\frac{1}{2}$.
- For $1 < k < n$: $\pi(k)$ is a local maximum if $\pi(k) > \max(\pi(k-1), \pi(k+1))$. Among the 3 values $\{\pi(k-1), \pi(k), \pi(k+1)\}$, each is equally likely to be the largest. Thus $\mathbb{P}(k \text{ is a local max}) = \frac{1}{3}$.

Summing these probabilities:

$$\mathbb{E}[X] = \frac{1}{2} + \frac{1}{2} + \sum_{k=2}^{n-1} \frac{1}{3} = 1 + \frac{n-2}{3} = \frac{n+1}{3}.$$

□

Solution to Problem 2 (Putnam 2023, B3)

Proof. For $n = 2$, any two distinct values $X_1 \neq X_2$ form a zigzag sequence of length 2, so $a(X_1, X_2) = 2$ with probability 1. Thus $\mathbb{E}[a(X_1, X_2)] = 2$.

For $n \geq 3$, consider the contribution of X_n to $a(X_1, \ldots, X_n)$. The difference $\Delta_n = a(X_1, \ldots, X_n) - a(X_1, \ldots, X_{n-1})$ is 0 if X_{n-2}, X_{n-1}, X_n form a monotonic sequence (either $X_{n-2} < X_{n-1} < X_n$ or $X_{n-2} > X_{n-1} > X_n$), and 1 otherwise.

Since $X_1, \ldots, X_n$ are i.i.d. continuous random variables, all $3! = 6$ orderings of (X_{n-2}, X_{n-1}, X_n) are equally likely. Exactly 2 of these 6 orderings are monotonic, so

$$\mathbb{P}(\Delta_n = 0) = \frac{2}{6} = \frac{1}{3} \implies \mathbb{P}(\Delta_n = 1) = \frac{2}{3}.$$

By linearity of expectation, $\mathbb{E}[\Delta_n] = \frac{2}{3}$ for each $n \geq 3$. Therefore:

$$\mathbb{E}[a(X_1, \ldots, X_n)] = \mathbb{E}[a(X_1, X_2)] + \sum_{k=3}^{n} \mathbb{E}[\Delta_k] = 2 + \frac{2(n-2)}{3} = \frac{2n+2}{3}.$$

□

Solution to Problem 3 (Putnam 2022, A4)

Proof. Rewrite S as an infinite sum $S = \sum_{i=1}^{\infty} \frac{X_i}{2^i} \cdot I(i \leq k)$, where $I(i \leq k)$ is the indicator that $k \geq i$. The condition $k \geq i$ is equivalent to $X_1 \geq X_2 \geq \cdots \geq X_i$.

By linearity of expectation, $\mathbb{E}[S] = \sum_{i=1}^{\infty} \frac{1}{2^i} \mathbb{E}[X_i \cdot I(X_1 \geq X_2 \geq \cdots \geq X_i)]$.

For a fixed i, conditioning on $X_i = t \in [0,1]$, the event $X_1 \geq X_2 \geq \cdots \geq X_i = t$ requires $X_1, \ldots, X_{i-1}$ to be chosen independently in $[t,1]$ and arranged in non-increasing order. The probability density of this event is $\frac{(1-t)^{i-1}}{(i-1)!}$.

Thus, the expected value of the i-th term is:

$$\mathbb{E}[X_i \cdot I(i \leq k)] = \int_0^1 t \cdot \frac{(1-t)^{i-1}}{(i-1)!}\, dt.$$

Using the substitution $u = 1 - t$:

$$\int_0^1 (1-u)\frac{u^{i-1}}{(i-1)!}\, du = \frac{1}{(i-1)!}\left(\frac{1}{i} - \frac{1}{i+1}\right) = \frac{1}{(i+1)!}.$$

Summing over all $i \geq 1$:

$$\mathbb{E}[S] = \sum_{i=1}^{\infty} \frac{1}{2^i(i+1)!} = 2\sum_{i=1}^{\infty} \frac{(1/2)^{i+1}}{(i+1)!} = 2\left(e^{1/2} - 1 - \frac{1}{2}\right) = 2e^{1/2} - 3.$$

□

Solution to Problem 4 (Putnam 2004, A5)

Proof. View the checkerboard as a grid graph $G = (V, E)$ with $|V| = mn$ vertices. Let A be the number of edges connecting adjacent squares of the same color, and let B be the number of 2×2 monochromatic squares (4-cycles).

Any planar graph with V vertices and E edges has at least $|V| - |E|$ connected components. Furthermore, removing the bottom-right edge from every 2×2 monochromatic square does not disconnect any region. Thus, the number of connected components C satisfies:

$$C \geq |V| - A + B = mn - A + B.$$

Taking expectations yields $\mathbb{E}[C] \geq mn - \mathbb{E}[A] + \mathbb{E}[B]$.

- The total number of internal edges is $m(n-1) + n(m-1) = 2mn - m - n$. Each edge is monochromatic with probability $\frac{1}{2}$, so $\mathbb{E}[A] = \frac{1}{2}(2mn - m - n) = mn - \frac{m+n}{2}$.
- The total number of 2×2 blocks is $(m-1)(n-1)$. Each block is monochromatic with probability $(1/2)^3 = \frac{1}{8}$, so $\mathbb{E}[B] = \frac{1}{8}(m-1)(n-1) = \frac{mn-m-n+1}{8}$.

Substituting these expectations back into the inequality gives:

$$\mathbb{E}[C] \geq mn - \left(mn - \frac{m+n}{2}\right) + \frac{mn - m - n + 1}{8} = \frac{mn + 3m + 3n + 1}{8} > \frac{mn}{8}.$$

□

Solution to Problem 5 (Putnam 2016, B4)

Proof. Let $M = A - A^T$. M is a $2n \times 2n$ skew-symmetric matrix whose off-diagonal entries $M_{ij} = A_{ij} - A_{ji}$ $(i < j)$ are independent random variables taking values in $\{-1, 0, 1\}$ with probabilities $1/4, 1/2, 1/4$. Thus $\mathbb{E}[M_{ij}] = 0$ and $\mathbb{E}[M_{ij}^2] = \frac{1}{4}(-1)^2 + 0 + \frac{1}{4}(1)^2 = \frac{1}{2}$.

Recall that the determinant of a $2n \times 2n$ skew-symmetric matrix equals the square of its Pfaffian:

$$\det(M) = (\mathrm{Pf}(M))^2 = \left(\sum_{\alpha} \mathrm{sgn}(\alpha) \prod_{(u,v)\in\alpha} M_{uv} \right)^2 = \sum_{\alpha,\beta} \mathrm{sgn}(\alpha)\,\mathrm{sgn}(\beta) \prod_{(u,v)\in\alpha} M_{uv} \prod_{(s,t)\in\beta} M_{st},$$

where α and β range over all perfect matchings of $\{1, 2, \ldots, 2n\}$ written as pairs (u, v) with $u < v$. The sign $\mathrm{sgn}(\alpha)$ is the sign of the permutation $\begin{pmatrix} 1 & 2 & \ldots & 2n \\ u_1 & v_1 & \ldots & v_n \end{pmatrix}$.

By linearity of expectation, $\mathbb{E}[\det(M)] = \sum_{\alpha,\beta} \mathrm{sgn}(\alpha)\,\mathrm{sgn}(\beta)\mathbb{E}\left[\prod_{(u,v)\in\alpha} M_{uv} \prod_{(s,t)\in\beta} M_{st}\right]$.

If $\alpha \neq \beta$, there exists at least one edge $(u, v) \in \alpha$ that is not in β. Since M_{uv} appears with power 1 and is independent of all other entries, $\mathbb{E}[M_{uv}] = 0$ forces the entire expectation to vanish.

If $\alpha = \beta$, we have $\mathrm{sgn}(\alpha)\,\mathrm{sgn}(\beta) = 1$, and the product becomes $\prod_{(u,v)\in\alpha} M_{uv}^2$. Since the n edges in a matching are disjoint, their entries are independent:

$$\mathbb{E}\left[\prod_{(u,v)\in\alpha} M_{uv}^2 \right] = \prod_{(u,v)\in\alpha} \mathbb{E}[M_{uv}^2] = \left(\frac{1}{2}\right)^n = 2^{-n}.$$

Since there are $\frac{(2n)!}{2^n n!}$ perfect matchings of $\{1, 2, \ldots, 2n\}$, we conclude:

$$\mathbb{E}[\det(A - A^T)] = \frac{(2n)!}{2^n n!} \cdot 2^{-n} = \frac{(2n)!}{4^n n!}.$$

□