Math 346, HW5 Solution

3.6.1 (b)

We can introduce two artificial variables x_4 and x_5 to the constraint, and minimize their sum:

$$\begin{array}{ll} \text{minimize} & x_4 + x_5 \\ \text{subject to} & x_1 + x_2 + x_4 = 1 \\ & 2x_1 + x_2 - x_3 + x_5 = 3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{array}$$

Note that the objective function is equal to $x_4 + x_5 = (1 - x_1 - x_2) + (3 - 2x_1 - x_2 + x_3) = 4 - 3x_1 - 2x_2 + x_3$. Here are the results using simplex algorithm (in the tableau form):

$$\left[\begin{array}{c|ccccc|c} & x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline x_4 & 1 & 1 & 0 & 1 & 0 & 1 \\ x_5 & 2 & 1 & -1 & 0 & 1 & 3 \\ \hline & -3 & -2 & 1 & 0 & 0 & -4 \end{array}\right]$$

$$\left[\begin{array}{c|ccccc|c} & x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline x_1 & 1 & 1 & 0 & 1 & 0 & 1 \\ x_5 & 0 & -1 & -1 & -2 & 1 & 1 \\ \hline & 0 & 1 & 1 & 3 & 0 & -1 \end{array}\right]$$

The simplex algorithm stops with a non-degenerate BFS such that $x_5 > 0$. Therefore there is no nonnegative solution satisfying the original constraints.

3.6.1(a)

To solve nonnegative solutions for the given constraints, we may also use the Big-M methods by choosing an arbitrary objective function, for example one can choose objective function to be x_1 (0 is also fine). Now by introducing the artificial variables x_4, x_5, we would like to solve the following LP:

$$\begin{array}{ll} \text{minimize} & x_1 + M(x_4 + x_5) \\ \text{subject to} & x_1 - x_2 + x_4 = 1 \\ & 2x_1 + x_2 - x_3 + x_5 = 3 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{array}$$

In order to apply the simplex algorithm, first we rewrite the objective function as a function only in the non-basic variables x_1, x_2, x_3:

$$\begin{aligned} x_1 + M(x_4 + x_5) &= x_1 + M(1 - x_1 + x_2) + M(3 - 2x_1 - x_2 + x_3) \\ &= 4M + (1 - 3M)x_1 + Mx_3 \end{aligned}$$

	x_1	x_2	x_3	x_4	x_5	
x_4	1	-1	0	1	0	1
x_5	2	1	-1	0	1	3
	$1-3M$	0	M	0	0	$-4M$

The $1 - 3M$ term in the last row (objective function) is negative, so we can do minimum ratio test for the first column, and pivot the entry 1:

	x_1	x_2	x_3	x_4	x_5	
x_1	1	-1	0	1	0	1
x_5	0	3	-1	-2	1	1
	0	$1-3M$	M	$3M-1$	0	$-M-1$

The $1 - 3M$ term is still negative, we apply simplex method one more round:

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	$-1/3$	$1/3$	$1/3$	$4/3$
x_2	0	1	$-1/3$	$-2/3$	$1/3$	$1/3$
	0	0	$1/3$	$M-1/3$	$M-1/3$	$-4/3$

Now all the artificial variables are out of the basis, this gives us a feasible solution to the original LP, with $(x_1, x_2, x_3) = (4/3, 1/3, 0)$.

3.6.2(c)

We use the two-phase method. In the first phase, we minimize the sum of artificial variables:

$$\begin{array}{ll} \text{minimize} & x_5 + x_6 \\ \text{subject to} & 4x_1 + x_2 + x_3 + 4x_4 + x_5 = 8 \\ & x_1 - 3x_2 + x_3 + 2x_4 + x_6 = 16 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{array}$$

We can choose x_5 and x_6 as the initial basis. The objective function is thus equal to $x_5 + x_6 = (8 - 4x_1 - x_2 - x_3 - 4x_4) + (16 - x_1 + 3x_2 - x_3 - 2x_4) = 24 - 5x_1 + 2x_2 - 2x_3 - 6x_4$.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_5	4	1	1	4	1	0	8
x_6	1	-3	1	2	0	1	16
	-5	2	-2	-6	0	0	-24

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	1	1/4	1/4	1	1/4	0	2
x_6	−1	−7/2	1/2	0	−1/2	1	12
	1	7/2	−1/2	0	3/2	0	−12

	x_1	x_2	x_3	x_4	x_5	x_6	
x_3	4	1	1	4	1	0	8
x_6	−3	−4	0	−2	−1	1	8
	3	4	0	2	2	0	−8

Note that this is a non-degenerate optimal solution with the artificial variable $x_6 > 0$, therefore the original LP is infeasible (so we do not have to continue the second phase).

3.7.2

(a) Note that the sum of the first two constraints give $x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} = 900$, while the sum of the last three constraints give the same expression. This gives a relationship between the five equations, which shows that one of the equations is redundant.

(b) We would like to solve the following linear program (phase I):

$$\begin{array}{ll} \text{minimize} & a_1 + a_2 + a_3 + a_4 + a_5 \\ \text{subject to} & x_{11} + x_{12} + x_{13} + a_1 = 350 \\ & x_{21} + x_{22} + x_{23} + a_2 = 550 \\ & x_{11} + x_{21} + a_3 = 275 \\ & x_{12} + x_{22} + a_4 = 325 \\ & x_{13} + x_{23} + a_5 = 300 \\ & x_{ij}, a_i \geq 0. \end{array}$$

We start with a_i's as the basic variable and write the objective function in the non-basic variable, therefore we could start with the following tableau:

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	a_1	a_2	a_3	a_4	a_5	
a_1	1	1	1	0	0	0	1	0	0	0	0	350
a_2	0	0	0	1	1	1	0	1	0	0	0	550
a_3	1	0	0	1	0	0	0	0	1	0	0	275
a_4	0	1	0	0	1	0	0	0	0	1	0	325
a_5	0	0	1	0	0	1	0	0	0	0	1	300
	−2	−2	−2	−2	−2	−2	0	0	0	0	0	−1800

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	a_1	a_2	a_3	a_4	a_5	
a_1	0	1	1	−1	0	0	1	0	−1	0	0	75
a_2	0	0	0	1	1	1	0	1	0	0	0	550
x_{11}	1	0	0	1	0	0	0	0	1	0	0	275
a_4	0	1	0	0	1	0	0	0	0	1	0	325
a_5	0	0	1	0	0	1	0	0	0	0	1	300
	0	−2	−2	0	−2	−2	0	0	2	0	0	−1250

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	a_1	a_2	a_3	a_4	a_5	
x_{12}	0	1	1	-1	0	0	1	0	-1	0	0	75
a_2	0	0	0	1	1	1	0	1	0	0	0	550
x_{11}	1	0	0	1	0	0	0	0	1	0	0	275
a_4	0	0	-1	1	1	0	-1	0	1	1	0	250
a_5	0	0	1	0	0	1	0	0	0	0	1	300
	0	0	0	-2	-2	-2	2	0	0	0	0	-1100

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	a_1	a_2	a_3	a_4	a_5	
x_{12}	0	1	0	0	1	0	0	0	0	1	0	325
a_2	0	0	1	0	0	1	1	1	-1	-1	0	300
x_{11}	1	0	1	0	-1	0	1	0	0	-1	0	25
x_{21}	0	0	-1	1	1	0	-1	0	1	1	0	250
a_5	0	0	1	0	0	1	0	0	0	0	1	300
	0	0	-2	0	0	-2	0	0	0	2	2	-600

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	a_1	a_2	a_3	a_4	a_5	
x_{12}	0	1	0	0	1	0	0	0	0	1	0	325
a_2	-1	0	0	0	1	1	0	1	-1	0	0	275
x_{13}	1	0	1	0	-1	0	1	0	0	-1	0	25
x_{21}	1	0	0	1	0	0	0	0	1	0	0	275
a_5	-1	0	0	0	1	1	-1	0	0	1	1	275
	2	0	0	0	-2	-2	2	0	2	0	0	-550

	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	a_1	a_2	a_3	a_4	a_5	
x_{12}	1	1	0	0	0	-1	0	-1	1	1	0	50
x_{22}	-1	0	0	0	1	1	0	1	-1	0	0	275
x_{13}	0	0	1	0	0	1	1	1	-1	-1	0	300
x_{21}	1	0	0	1	0	0	0	0	1	0	0	275
a_5	0	0	0	0	0	0	-1	-1	1	1	1	0
	0	0	0	0	0	0	2	2	0	0	0	0

The first phase of the simplex method stops here. Note that the artificial variable a_5 is still in the basis, and we cannot take it out from the basis by replacing it by a non-artificial variable. Therefore we can conclude that the constraints are redundant (in particular, the fifth constraint can be written as a linear combination of the other constraints).