Math 346, HW4 Solution

3.3.3

Since the problem is already stated in the canonical form with basic variable x_2 and x_3, we can start with the initial BFS $(0, 3, 2, 0, 0)$. Note that in the objective function the variables x_1 and x_4 have negative coefficient. We can pick one of them entering the basis, suppose we pick x_4 (using Dantzig's pivot rule). The minimum ratio test on the column for x_4 gives $\min\{\frac{2}{6}, \frac{3}{3}\} = \frac{2}{6}$. So in the next step, x_4 enters the basis and replace x_3, meaning that we need to get the canonical form for the basis $\{x_4, x_2\}$, that is

$$\begin{array}{ll} \text{minimize} & -\frac{2}{3}x_1 + \frac{1}{3}x_3 + 2x_5 - \frac{2}{3} \\ \text{subject to} & \frac{1}{6}x_1 + \frac{1}{6}x_3 + x_4 + \frac{1}{2}x_5 = \frac{1}{3} \\ & -\frac{7}{2}x_1 + x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_5 = 2 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{array}$$

Now notice that x_1 has negative coefficient in the new objective function. It enters the basis, and replace x_4 because it corresponds to the only positive a_{ij} which is $\frac{1}{6}$. Now we could obtain the canonical form for basic variables x_1, x_2:

$$\begin{array}{ll} \text{minimize} & x_3 + 4x_4 + 4x_5 - 2 \\ \text{subject to} & x_1 + x_3 + 6x_4 + 3x_5 = 2 \\ & x_2 + 3x_3 + 21x_4 + 10x_5 = 9 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0. \end{array}$$

Now all the coeffcients of non-basic variables in the objective function are non-negative. So we stop at an optimal solutions $(x_1, x_2, x_3, x_4, x_5) = (2, 9, 0, 0, 0)$, with optimum value -2.

3.4.5

Suppose r is the index minimizing $\frac{b_i}{a_{is}}$ over all i such that $a_{is} > 0$. After we pivot on a_{rs}, in the new tableau (or new canonical form), the constant terms b_i^* becomes $b_i - \frac{a_{is}b_r}{a_{rs}}$ for the basic variables except r, and $b_r^* = \frac{b_r}{a_{rs}}$, Recall that in a degenerate basic feasible solution, then the constant term b_i has to be zero for some basic variable x_i. Since the new basis is $\{x_1, \cdots, x_{r-1}, x_{r+1}, \cdots, x_m, x_s\}$. So this pivot operation produces a degenerate BFS iff one of the b_i^* computed above is equal to 0.

3.5.2(b)

We first introduce the slack variables to convert it into the standard form (also we change the objective function so that it is now a minimization problem).

$$\begin{array}{ll} \text{minimize} & -x_1 - 2x_2 + x_3 \\ \text{subject to} & x_2 + 4x_3 + x_4 = 36 \\ & 5x_1 - 4x_2 + 2x_3 + x_5 = 60 \\ & 3x_1 - 2x_2 + x_3 + x_6 = 24 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0. \end{array}$$

Note that x_4, x_5, x_6 naturally forms a basis that gives a BFS. Now we create the simplex tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	0	1	4	1	0	0	36
x_5	5	-4	2	0	1	0	60
x_6	3	-2	1	0	0	1	24
	-1	-2	1	0	0	0	0

In the next step, x_2 enters the basis and by the minimum ratio test x_4 leaves the basis (so we pivot on the entry 1)

	x_1	x_2	x_3	x_4	x_5	x_6	
x_2	0	1	4	1	0	0	36
x_5	5	0	18	4	1	0	204
x_6	3	0	9	2	0	1	96
	-1	0	9	2	0	0	72

Now x_1 enters the basis and replace x_6, since $\frac{96}{3} < \frac{204}{5}$.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_2	0	1	4	1	0	0	36
x_5	0	0	3	$\frac{2}{3}$	1	$-\frac{5}{3}$	44
x_1	1	0	3	$\frac{2}{3}$	0	$\frac{1}{3}$	32
	0	0	12	$\frac{8}{3}$	0	$\frac{1}{3}$	104

Now all the non-basic variables have nonnegative coefficients in the objective function, so we have obtained an optimal solution $(x_1, x_2, x_3, x_4, x_5, x_6) = (32, 36, 0, 0, 44)$, with optimum objective value -104. So for the original maximization problem, the maximum is 104.

3.5.6(b)

Again we first convert the linear program into the standard form and create the simplex tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_5	2	-1	1	1	1	0	60
x_6	3	4	2	-2	0	1	150
	1	-3	-6	0	0	0	0

x_3 enters the basis and the minimum ratio test shows x_5 leaves the basis since $\frac{60}{1} < \frac{150}{2}$.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_3	2	-1	1	1	1	0	60
x_6	-1	6	0	-4	-2	1	30
	13	-9	0	6	6	0	360

x_2 enters the basis and x_6 leaves the basis.

	x_1	x_2	x_3	x_4	x_5	x_6	
x_3	$\frac{11}{6}$	0	1	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{6}$	65
x_2	$-\frac{1}{6}$	1	0	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{6}$	5
	$\frac{23}{2}$	0	0	0	3	$\frac{3}{2}$	405

Now we obtain one optimal BFS: $(0, 5, 65, 0, 0, 0)$, with optimum objective value -405. Note that in the current objective function $z = \frac{23}{2}x_1 + 3x_5 + \frac{3}{2}x_6$. The non-basic variables x_1, x_5, x_6 all have strictly positive coefficients. So if another different optimum BFS exists, x_4 would be in the basis. Now we do minimum ratio test for x_4. Then it is not hard to see that x_3 has to leave the basis. Pivoting on $\frac{1}{3}$, we have:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	$\frac{11}{2}$	0	3	1	2	$\frac{1}{2}$	195
x_2	$\frac{7}{2}$	1	2	0	1	$\frac{1}{2}$	135
	$\frac{23}{2}$	0	0	0	3	$\frac{3}{2}$	405

This basis gives us another optimal BFS: $(0, 135, 0, 195, 0, 0)$.