A generalisation of Mantel's theorem to uniformly dense hypergraphs

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A well known theorem of Mantel asserts that the Turán density of the triangle is $1/2$. More recently, Glebov, Král, and Volec proved that every large 3-uniform hypergraph with the property that all linearly sized subsets of its vertex set induce subhypergraphs of density $> 1/4$ contains four vertices spanning at least three edges. The constant $1/4$ appearing here is optimal as a random tournament construction shows.

When thinking about a possible common generalisation of these two facts to k-uniform hypergraphs, one easily observes that the tournament construction generalises to a hypergraph having the density 2^{1-k} in a very uniform sense but not containing the hypergraph $F^{(k)}$ consisting of $(k+1)$ vertices and three edges.

Our main result specifies a precise sense of 'uniformly dense' such that the generalised tournament construction does indeed have the largest density among $F^{(k)}$-free hypergraphs that are uniformly dense in this sense. Such variations of Turán's problem were first suggested by Erdős and Sós. For $k = 2$ our density notion coincides with the usual density and for $k = 3$ it is the same as the "vertex uniform density" considered in the result of Glebov, Král, and Volec mentioned above. Our proof relies on the regularity method for hypergraphs.

This is joint work with V. Rödl and M. Schacht.