A tight bound for Green's arithmetic triangle removal lemma

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Let p be a fixed prime. A triangle in $\mathbb{F}_p^n$ is an ordered triple (x, y, z) of points satisfying $x + y + z = 0$. Let $N = p^n$, (the size of $\mathbb{F}_p^n$). Green proved an arithmetic triangle removal lemma which says that for every $\epsilon > 0$ and prime p, there is a $\delta > 0$ such that if X, Y, and Z are subsets of $\mathbb{F}_p^n$ and the number of triangles in $X \times Y \times Z$ is at most δN^2 (so a δ fraction of all possible triangles), then we can delete ϵN elements from X, Y, and Z and remove all triangles. Green posed the problem of improving the quantitative bounds on the arithmetic triangle removal lemma, and, in particular, asked whether a polynomial bound holds. Despite considerable attention, prior to our work, the best known bound, due to Fox, showed that $1/\delta$ can be taken to be an exponential tower of twos of height logarithmic in $1/\epsilon$.

In this talk, we will discuss our solution to Green's problem. We prove an essentially tight bound for Green's arithmetic triangle removal lemma in $\mathbb{F}_p^n$. We show that for any fixed p, a polynomial bound holds, and further determine the best possible exponent for each prime p. The proof uses Kleinberg, Sawin, and Speyer's essentially sharp bound on multicolored sum-free sets, which builds on the recent breakthrough on the cap set problem by Croot-Lev-Pach, and the subsequent work by Ellenberg-Gijswijt, Blasiak-Church-Cohn-Grochow-Naslund-Sawin-Umans, and Alon.

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