GEOMETRIC PROPERTIES OF SETS IN EUCLIDEAN SPACE

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March 2021

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Senior Thesis submitted in partial fulfillment for the honors
requirements for the Bachelor of Science degree in Mathematics

INTRODUCTION

The goal of this thesis is to cover the background of Hausdorff measure, density and tangency, and to explain how they have been used to prove many fundamental geometric results on sets of integral as well as nonintegral Hausdorff dimensions. In section 1, we review basic measure theory concepts corresponding to the Lebesgue measure. In section 2, we introduce the Hausdorff measure and Hausdorff dimension together with their unique properties, laying the framework for the rest of the paper.

In section 3, we mainly study the density, which is an intrinsic function associated to a measurable set in the Euclidean space. We examine how it is affected by the dimension of the given set and show that it is stable when passed to measurable subsets or countable unions. In section 4, we introduce the notions of tangency and rectifiability of sets in the plane. In particular, we look at how these objects behave under a regularity condition described in terms of the circular density. With a substantial amount of work, we are able to show that regularity, rectifiability and tangency are equivalent conditions for subsets of Hausdorff dimension one in the plane. In section 5, we carry the discussion on tangency and density to subsets of nonintegral dimensions in the plane, and we examine an elegant result by Marstrand which says tangents and density fail to exist at almost every point of such sets.

In general, this thesis is an expository study on some of the early results by Besicovitch and Marstrand in geometric measure theory. Falconer's book [Fal85] is used as the major source; meanwhile, other standard textbooks [Ste05], [Mat95] as well as original papers [Bes38], [Mar54] are consulted to supply a richer context. Intended readers are expected to know basic topology and real analysis; preferred though, a knowledge of measure theory is not required.

Contents

NOTATIONS

$\mathbb{R}^n$	n-dimensional Euclidean space.
$\mathbb{N}$	the set of positive integers.
$B_r(x)$	closed ball centered at x with radius r.
$S_r(x, \boldsymbol{\theta}, \phi)$	sector centered at x pointing $\boldsymbol{\theta}$ with radius r.
$\|U\|$	diameter of the set U.
E°	interior of the set E.
E^c	complement of the set E
$\mathscr{L}(\Gamma)$	length of the curve Γ.
$\mathcal{H}^s$	s-dimensional Hausdorff measure or outer measure.
$\mathcal{H}^s_\delta$	δ-outer measure used in constructing $\mathcal{H}^s$
$\mathcal{L}^n$	n-dimensional Lebesgue measure.
$D^s(E,x)$	density of E at x.
$\underline{D}^s(E,x), \overline{D}^s(E,x)$	lower, upper densities.
$\overline{D}^s_c(E,x)$	upper convex density.
$\overline{D}^s(E,x,\boldsymbol{\theta},\phi)$	upper angular density.

Acknowledgement

I would like to thank my advisor, Professor Tatiana Toro, for her continuous support throughout the process of writing this thesis, from recommending excellent reading materials to helping me find my interested topics, from guiding me through the sophisticated details to providing invaluable feedback. For those, I am forever grateful.

I would also like to thank my math friends at the University of Washington, Shea Engle, Pubo Huang, Jon Kim, Andreas Quist, Yang Yu and many others. It is through our tirelessly passionate discussions that I have become more mathematically mature.

Finally, I would like to thank my family for being my angels, my dream, and my everlasting cause to go on.

1. Measure Theory Preliminaries

Given a subset E of $\mathbb{R}^n$, one of the first questions we could ask is how big it is. As sets in $\mathbb{R}^n$ can be quite intangible, we need a careful measurement of their sizes. The basic idea is to approximate E by unions of sets whose sizes are intuitive. Here we adopt (closed) cubes as our building blocks as they have a standard notion of size or volume.

Definition 1.1. A cube C is a subset of $\mathbb{R}^n$ of the form

$$C = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$$

where $a_i, b_i \in \mathbb{R}$ and $a_i < b_i$ for each i. Define the volume of C as

$$V(C) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$$

Proceeding with this idea, we arrive at our first notion of size for E.

Definition 1.2. Define the Lebesgue outer measure of $E \subset \mathbb{R}^n$ as

$$\mathcal{L}^n(E) = \inf \left\{ \sum_{i=1}^{\infty} V(C_i) \middle| E \subset \bigcup_{i=1}^{\infty} C_i \right\}$$

i.e. the infimum is taken over all countable coverings of E by closed cubes.

Note that the countable coverings exists because $\mathbb{R}^n$ is separable. Moreover, we also see that by construction the Lebesgue outer measure is always nonnegative, possibly infinite, and defined for all subsets of $\mathbb{R}^n$. The following properties of the Lebesgue outer measure are direct consequences of the definition:

1. (Subadditivity) Given a countable collection $\{E_i\} \subset \mathbb{R}^n$,

$$\mathcal{L}^n\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mathcal{L}^n(E_i)$$

2. $\mathcal{L}^n(\emptyset) = 0$.
3. (Translation Invariant) For any $E \subset \mathbb{R}^n, x \in \mathbb{R}^n$, $\mathcal{L}^n(E + x) = \mathcal{L}^n(E)$.
4. For any closed cube $C \subset \mathbb{R}^n$, $\mathcal{L}^n(C) = V(C)$.

Note that in general any set function $\mu : \mathcal{P}(X) \to [0, \infty]$ on a nonempty set X is an **outer measure** if it satisfies properties 1 and 2. Now while properties 2, 3 and 4 are fairly intuitive for the Lebesgue outer measure, property 1 may seem unfamiliar at first, but since the E_i's are likely to have nonempty intersection, so the size of the union is partly consumed by the overlapping.

Nevertheless, when E_i's are disjoint, we do wish that the inequality in property 1 becomes an equality. Unfortunately, one can show that such an additive property, if imposed, will be inconsistent with properties 2 and 3. As it is hopeless to require the three desirable properties to hold for all subsets of $\mathbb{R}^n$, we take a step back and ask if they hold for a reasonably large collection of subsets, which are known as the Lebesgue measurable sets.

Definition 1.3. A subset $E \subset \mathbb{R}^n$ is Lebesgue measurable, or $\mathcal{L}^n$-measurable, if for every $A \subset \mathbb{R}^n$,

$$\mathcal{L}^n(A) = \mathcal{L}^n(A \cap E) + \mathcal{L}^n(A \setminus E).$$

The above condition is oftentimes known as the Carathéodory's criterion. We denote the collection of all Lebesgue measurable sets as $\mathcal{M}$ and call the Lebesgue outer measure $\mathcal{L}^n$ restricted to $\mathcal{M}$ the Lebesgue measure.

It then can be shown that $\mathcal{M}$ is a σ-algebra, namely $\mathcal{M}$ is closed under complementation and countable unions, and for a disjoint collection $\{E_i\} \subset \mathcal{M}$, we have

$$\mathcal{L}^n\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mathcal{L}^n(E_i),$$

which is called the additivity of $\mathcal{L}^n$. Moreover, $\mathcal{L}^n$ satisfies the following properties, whose proofs can be found in a standard real analysis textbook such as [Ste05]:

1. (Continuity from below) If $\{E_i\} \subset \mathcal{M}$ and $E_1 \subset E_2 \subset \cdots$, then

$$\mathcal{L}^n\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{i\to\infty} \mathcal{L}^n(E_i)$$

2. (Continuity from above) If $\{E_i\} \subset \mathcal{M}$, and $E_1 \supset E_2 \supset \cdots$, and $\mathcal{L}^n(E_j) < \infty$ for some j, then

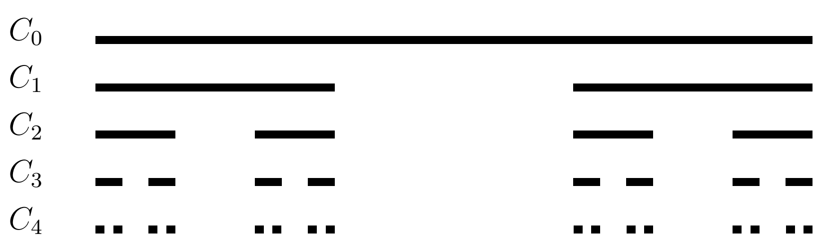
$$\mathcal{L}^n\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{i\to\infty} \mathcal{L}^n(E_i)$$

We also briefly mention one result that make Lebesgue measure $\mathcal{L}^n$ particularly nice to work with: We say a measure μ over a topological space X is called **Borel regular** if (i) all open sets are μ-measurable and (ii) each subset A of X is contained in a Borel set B for which $\mu(A) = \mu(B)$. We have the following theorem.

Theorem 1.4. *The Lebesgue measure is Borel regular.*

We shall prove Theorem 1.4 as a special case of the Theorem 2.5 as we show that the Hausdorff measure, as a natural generalization of the Lebesgue measure, is Borel regular. Finally, we conclude this section by introducing the Cantor set as a particular $\mathcal{L}^1$-measurable set that serves as a motivation for the next chapter.

Example 1.5. Start by removing the middle third open interval $(1/3, 2/3)$ from $C_0 = [0,1]$, then removing the middle third open intervals $(1/9, 2/9)$ and $(7/9, 8/9)$ of the two remaining intervals of C_1, and so forth...



The **Cantor set** C is defined as $C = \bigcap_{k=0}^{\infty} C_k$, and it has the following properties [Ste05]:

(1) C has Lebesgue measure zero.
(2) C is uncountable.
(3) C is self-similar in the sense that any part is a replica of the whole.

2. Hausdorff Measure

Lebesgue measure, powerful though, is not an ideal setting to investigate geometric properties of sets. First of all, Lebesgue measure is not sensitive enough to capture the "dimension" of a given set. By dimension we mean for each appropriate set E, there is a value $s \geq 0$ associated to E such that the quantity $\mathcal{H}^t(E)$ captures the actual size of E when $t = s$, in the sense that for any $t > s$, we have $\mathcal{H}^t(E) = 0$ and for any $t < s$, we have $\mathcal{H}^t(E) = \infty$.

Take the Cantor set in $[0, 1]$ as an example: Since it has Lebesgue measure zero, its size in dimension one is negligible. This shows the dimension of the Cantor set is less than or equal to 1. The question is can we be more accurate? In fact, we will show that the Cantor set has a well-defined Hausdorff dimension $s = \log 2/ \log 3$.

Definition 2.1. Given $\delta > 0$ and $E \subset \mathbb{R}^n$, a countable collection of $\{U_i\}$ of (arbitrary) sets in $\mathbb{R}^n$ is a δ**-cover** of E if $E \subset \bigcup_{i=1}^{\infty} U_i$ and $0 < |U_i| \leq \delta$, where

$$|U_i| = \operatorname{diam}(U_i) = \sup\{|x - y| \,:\, x, y \in U_i\}.$$

Definition 2.2. Let $E \subset \mathbb{R}^n$ and s be a non-negative real number. For $\delta > 0$, define

$$\mathcal{H}^s_\delta(E) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s \right\}$$

where the infimum is taken over all δ-cover of E. Since $\mathcal{H}^s_\delta$ increases as δ decreases, the limit $\lim_{\delta\to 0} \mathcal{H}^s_\delta(E) = \sup_{\delta>0} \mathcal{H}^s_\delta(E)$ exists (possibly infinite). Define the **Hausdorff** s**-dimensional outer measure** of E as

$$\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(E)$$

We check that $\mathcal{H}^s$ is indeed an outer measure: It is clear from the definition that $\mathcal{H}^s(\emptyset) = 0$. Moreover, for any countable collection $\{E_i\}$ of sets in $\mathbb{R}^n$, $\delta > 0$, and $\varepsilon > 0$, there exists a δ-cover $\{U_{i,j}\}_{j\geq 1}$ of E_i such that

$$\sum_{j=1}^{\infty} |U_{i,j}|^s \leq \mathcal{H}^s_\delta(E_i) + \frac{\varepsilon}{2^i}$$

Now since $\bigcup_{i,j} U_{i,j}$ is a δ-cover of $\bigcup_{i=1}^{\infty} E_i$, we have

$$\mathcal{H}^s_\delta\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mathcal{H}^s_\delta(E_i) + \varepsilon \leq \sum_{i=1}^{\infty} \mathcal{H}^s(E_i) + \varepsilon$$

Since ε is arbitrary, letting $\delta \to 0$ shows that $\mathcal{H}^s(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mathcal{H}^s(E_i)$. This shows the countable subadditivity.

Moreover, the Hausdorff s-dimensional outer measure $\mathcal{H}^s$ has the nice geometric property that it is additive on positively separated sets, namely it satisfies the following theorem:

Theorem 2.3. *If $d(E_1, E_2) > 0$, then $\mathcal{H}^s(E_1 \cup E_2) = \mathcal{H}^s(E_1) + \mathcal{H}^s(E_2)$.*

Proof. By the subadditivity, we automatically have $\mathcal{H}^s(E_1 \cup E_2) \leq \mathcal{H}^s(E_1) + \mathcal{H}^s(E_2)$. Hence, it suffices to show that $\mathcal{H}^s(E_1 \cup E_2) \geq \mathcal{H}^s(E_1) + \mathcal{H}^s(E_2)$.

Notice if $0 < \delta < d(E_1, E_2)$, then no set in a δ-cover $\{U_i\}$ of $E \cup F$ intersects both E and F. That is the δ-cover of E defined as $\mathcal{V} = \{U_i : U_i \cap E \neq \emptyset\}$ and the δ-cover of F defined as $\mathcal{W} = \{U_i : U_i \cap F \neq \emptyset\}$ are disjoint. Hence,

$$\sum_{U_j \in \mathcal{V}} |U_j|^s + \sum_{U_k \in \mathcal{W}} |U_k|^s \leq \sum_{i=1}^{\infty} |U_i|^s$$

which implies

$$\mathcal{H}^s_\delta(E) + \mathcal{H}^s_\delta(F) \leq \sum_{i=1}^{\infty} |U_i|^s$$

Since $\{U_i\}$ is an arbitrary, taking the infimum over all δ-cover of $E \cup F$ gives $\mathcal{H}^s_\delta(E) + \mathcal{H}^s_\delta(F) \leq \mathcal{H}^s_\delta(E \cup F)$. Now letting $\delta \to 0$, we obtain the desired result. $\square$

Together with the following theorem, whose proof can be found in [Fal85],

Theorem 2.4. *If ν is a metric outer measure on (X, d), then all Borel subsets of X are ν-measurable.*

We know that $\mathcal{H}^s$ is a measure when restricted to Borel sets in $\mathbb{R}^n$, in which case it is called the **s-dimensional Hausdorff measure**. Comparing the definitions of Hausdorff measure and Lebesgue measure, we see that one of the major differences is that a scaling factor "s" appears in the definition of Hausdorff outer measure. This reflects the idea that the measure of a set should scale according to its dimension. For example, if Γ is a curve of "length" L, then $r\Gamma$ has "length" rL. If C is a cube in $\mathbb{R}^n$, then the volume of rC is $r^n V(C)$. This feature of dimension is precisely captured in the definition of the Hausdorff measure in the way that if a set E is scaled by r, then $|E|^s$ is scaled by r^s. To formalize this notion of dimension, consider for any $F \subset \mathbb{R}^n$, if $|F| \leq \delta$, then if $t > s$,

$$|F|^t = |F|^{t-s}|F|^s \leq \delta^{t-s}|F|^s$$

which implies for any $F \subset \mathbb{R}^n$

$$\mathcal{H}^t_\delta(E) \leq \delta^{t-s}\mathcal{H}^s_\delta(E) \leq \delta^{t-s}\mathcal{H}^s(E)$$

If $\mathcal{H}^s(E) < \infty$, then taking $\delta \to 0$ gives $\mathcal{H}^t(E) = 0$. Similarly, if $t < s$ and $\mathcal{H}^s(E) > 0$, taking $\delta \to 0$ implies $\mathcal{H}^t(E) = \infty$. Thus, there is a unique real value $\dim(E)$, namely the **Hausdorff dimension** of E, such that $\mathcal{H}^s(E) = \infty$ if $0 \leq s < \dim(E)$, $\mathcal{H}^s(E) = 0$ if $\dim(E) < s < \infty$.

Next we prove that $\mathcal{H}^s$ is a regular measure (see the definition above Theorem 1.4), together with a very useful result that allows us to approximate s-sets from below by closed subsets.

Theorem 2.5. *(i) Hausdorff measure is Borel regular. (ii) Any $\mathcal{H}^s$-measurable set of finite $\mathcal{H}^s$-measure contains a closed set differing from it by arbitrarily small measure.*

Proof. (i) It follows from Theorem 2.4 that all open sets in $\mathbb{R}^n$ are $\mathcal{H}^s$-measurable. Now it remains to show that each measurable set is contained in a Borel set of equal measure.

For any subset E of $\mathbb{R}^n$, if $\mathcal{H}^s(E) = \infty$, then $\mathbb{R}^n$ contains E and has equal measure. Suppose $\mathcal{H}^s(E) < \infty$. For each $i \in \mathbb{N}$, by the definition of $\mathcal{H}^s_{1/i}(E)$, there exists an open $(2/i)$-cover $\{U_{ij}\}_j$ of E such that

$$\sum_{j=1}^{\infty} |U_{ij}|^s \leq \mathcal{H}^s_{1/i}(E) + 1/i.$$

Let $G = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} U_{ij}$, then G is a Borel set containing E. Moreover, since $\{U_{ij}\}_j$ is a $(1/i)$-cover of G, we have

$$\mathcal{H}^s_{1/i}(G) \leq \sum_{j=1}^{\infty} |U_{ij}|^s \leq \mathcal{H}^s_{1/i}(E) + 1/i.$$

Letting $i \to \infty$, we get $\mathcal{H}^s(G) = \mathcal{H}^s(E)$, which shows that $\mathcal{H}^s$ is Borel regular.

(ii) Suppose $E \subset \mathbb{R}^n$ is $\mathcal{H}^s$-measurable and $\mathcal{H}^s(E) < \infty$. It follows from part (i) that there is a sequence of open sets $\{U_i\}$ containing E with $\mathcal{H}^s(\bigcap_{i=1}^{\infty} U_i \setminus E) = 0$. Moreover, since each open set in $\mathbb{R}^n$ can be written as the union of an increasing sequence of closed sets, there exists $\{F_{ij}\}_j$ for each i such that $U_i = \bigcup_{j=1}^{\infty} F_{ij}$. Then by the continuity of $\mathcal{H}^s$,

$$\lim_{j\to\infty} \mathcal{H}^s(E \cap F_{ij}) = \mathcal{H}^s(E \cap U_i) = \mathcal{H}^s(E)$$

Hence, given $\varepsilon > 0$, there exists $j_i \in \mathbb{N}$ such that

$$\mathcal{H}^s(E \setminus F_{ij_i}) = \mathcal{H}^s(E) - \mathcal{H}^s(E \cap F_{ij_i}) < \varepsilon/2^i$$

Let $F = \bigcap_{i=1}^{\infty} F_{ij_i}$, then F is a closed set contained in E. Moreover,

$$\mathcal{H}^s(F) \geq \mathcal{H}^s(E \cap F) \geq \mathcal{H}^s(E) - \sum_{i=1}^{\infty} \mathcal{H}^s(E \setminus F_{ij_i}) > \mathcal{H}^s(E) - \varepsilon$$

as desired. $\square$

Remark. This theorem later allows us to generate results by only looking at closed subsets, which are much more desirable to work with. It also worth mentioning that when s is a positive integer n, there is a beautiful connection between the Hausdorff measure and the Lebesgue measure, namely:

Theorem 2.6. *If $E \subset \mathbb{R}^n$, then $\mathcal{L}^n(E) = c_n \mathcal{H}^n(E)$, where $c_n = \pi^{n/2}/2^n(n/2)!$.*

This indicates that the Hausdorff measure is indeed a natural generalization of the Lebesgue measure. A proof can be found in [Ste05]: It relies on the isodiametric inequality, which states that among all sets of a given diameter, the ball has the largest volume. The constant c_n arises naturally from the volume of the n-dimensional unit ball within this context. By Theorem 2.4, 2.5, and 2.6, Lebesgue measure is also a metric outer measure and Borel regular (Theorem 1.4).

Now we shift our attention to the discussion of curves by starting with a lemma that will be useful in later chapters. The lemma states how Hausdorff measure changes via a Hölder continuous or a Lipschitz map.

Lemma 2.7. *Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a surjective mapping $\phi(E) = F$ such that*

$$|\phi(x) - \phi(y)| \le c|x - y|^t \quad (x, y \in E,\ 0 < t \le 1)$$

for a positive constant c. Then $\mathcal{H}^{s/t}(F) \le c^{s/t}\mathcal{H}^s(E)$

Proof. Let $\{U_i\}$ be a δ-cover of E. For each i and any $x, y \in E \cap U_i$, we have $|\phi(x) - \phi(y)| \le c|x - y|^t \le c|U_i|^t$. This implies $|\phi(E \cap U_i)| \le c|U_i|^t$ and thus

$$\sum_{i=1}^{\infty} |\phi(E \cap U_i)|^{s/t} \le c^{s/t} \sum_{i=1}^{\infty} |U_i|^s$$

Since ϕ is surjective, $\{\phi(E \cap U_i)\}$ is a $c\delta^t$-cover of F. Taking infimum over all δ-cover of E, we have $\mathcal{H}^{s/t}_{c\delta^t}(F) \le c^{s/t}\mathcal{H}^s_\delta(E)$. Letting $\delta \to 0$, we obtain the desired result. ☐

Moreover, this theorem immediately gives another desirable property of the Hausdorff measure as the quantity $\mathcal{H}^1(\Gamma)$ equals precisely to the length of Γ. We define a **curve** as the image of a continuous injection $\psi : [a, b] \to \mathbb{R}^n$, where $[a, b] \subset \mathbb{R}$ is a closed interval. Thus, a curve is a continuum (i.e. a compact connected set). The **length** of the curve Γ is defined as

$$\mathscr{L}(\Gamma) = \sup \sum_{i=1}^{m} |\psi(t_i) - \psi(t_{i-1})| \tag{1}$$

where the supremum is taken over all partitions $a = t_0 < t_1 < \ldots < t_m = b$ of $[a, b]$.

Theorem 2.8. *If Γ is a curve, then $\mathcal{H}^1(\Gamma) = \mathscr{L}(\Gamma)$.*

Proof. Let Γ be a curve joining z and w. Let proj denote the orthogonal projection from $\mathbb{R}^n$ onto the line segment $[z, w]$ through z and w, then $[z, w] = \operatorname{proj}\Gamma$. Since projection does not increase length, we must have $|\operatorname{proj}(x) - \operatorname{proj}(y)| \le |x - y|$ for all $x, y \in \mathbb{R}^n$. Now it follows from Lemma 2.7 and Theorem 2.6 that $\mathcal{H}^1(\Gamma) \ge \mathcal{H}^1(\operatorname{proj}\Gamma) = \mathcal{H}^1([z, w]) = \mathcal{L}^1([z, w]) = |z - w|$.

Suppose Γ is defined as $\psi : [a, b] \to \mathbb{R}^n$. It follows from the above result that $\mathcal{H}^1(\psi[t, u]) \ge |\psi(u) - \psi(t)|$ for any $u, t \in [a, b]$. Then for any partition $a = t_0 < t_1 < \ldots < t_m = b$ of $[a, b]$, we have

$$\sum_{i=1}^{m} |\psi(t_i) - \psi(t_{i-1})| \le \sum_{i=1}^{m} \mathcal{H}^1(\psi[t_i, t_{i-1}]) = \mathcal{H}^1(\psi[a, b]) = \mathcal{H}^1(\Gamma).$$

Thus, taking the supremum over all partitions gives $\mathscr{L}(\Gamma) \le \mathcal{H}^1(\Gamma)$. Now for the other direction, if $\mathscr{L}(\Gamma) = \infty$, we are done. Suppose $\mathscr{L}(\Gamma) < \infty$, then let ψ

parametrize Γ by arc length, i.e. $\psi : [0, \mathscr{L}(\Gamma)] \to \mathbb{R}^n$ such that $\mathscr{L}(\psi[0,t]) = t$ for any $t \in [0, \mathscr{L}(\Gamma)]$. By construction we have

$$|t_1 - t_2| = \mathscr{L}(\psi[t_1, t_2]) \geq |\psi(t_1) - \psi(t_2)|$$

for any $t_1, t_2 \in [0, \mathcal{L}(\Gamma)]$. Since ψ is a surjection from $[0, \mathcal{L}(\Gamma)]$ to Γ, it again follows from Lemma 2.7 that $\mathcal{H}^1(\Gamma) \leq \mathcal{H}^1([0, \mathscr{L}(\Gamma)]) = \mathscr{L}(\Gamma)$. $\square$

The last yet possibly the most powerful asset we would like to secure is the Vitali Covering Theorem, which allows us to obtain global results from examining local property:

Definition 2.9. A collection $\mathcal{V}$ of closed balls in $\mathbb{R}^n$ is a **fine cover** of a set $E \subset \mathbb{R}^n$ if

$$E \subset \bigcup_{B \in \mathcal{V}} B \quad \text{and} \quad \inf\{\operatorname{diam}(B) \mid x \in B, B \in \mathcal{V}\} = 0.$$

for all $x \in E$, i.e. given any point $x \in E$, there exists $B \in \mathcal{V}$ of arbitrarily small radius that contains x.

Lemma 2.10. *Suppose $\mathcal{V}$ is a fine cover of $E \subset \mathbb{R}^n$ of closed balls of uniformly bounded radii, then there exists a countable family of disjoint balls $\{B_i\} \subset \mathcal{V}$ such that for each finite subcollection $\{B_1, B_2, \ldots, B_m\}$,*

$$E \setminus \bigcup_{i=1}^{m} B_m \subset \bigcup_{i=m+1}^{\infty} \hat{B}$$

where $\hat{B} = 5B$.

Theorem 2.11. *(Vitali covering theorem) Let E be an $\mathcal{H}^s$-measurable subset of $\mathbb{R}^n$ such that $\mathcal{H}^s(E) < \infty$ and let $\mathcal{V}$ be a fine cover of closed balls for E. Then there is a disjoint sequence of closed balls $\{B_i\} \subset \mathcal{V}$ such that either $\sum |B_i|^s = \infty$ or*

$$\mathcal{H}^s(E \setminus \bigcup_i B_i) = 0$$

Moreover, we may also require that for any $\varepsilon > 0$

$$\mathcal{H}^s(E) \leq \sum_i |B_i|^s + \varepsilon.$$

Proof. Fix $\delta > 0$; we may assume that $|B| < \delta$ for all $B \in \mathcal{V}$. Let U_1 be an open set containing E, then the collection

$$\mathcal{V}_1 = \{B \in \mathcal{V} \mid B \subset U_1\}$$

is by construction a fine cover of E. By Lemma 2.10, there exists a disjoint collection $\{B_i\} \subset \mathcal{V}_1$ such that for each finite subcollection $\{B_1, \ldots, B_m\}$,

$$E \setminus \bigcup_{i=1}^{m} B_i \subset \bigcup_{i=m+1}^{\infty} \hat{B}_i$$

If $\sum |B_i|^s < \infty$, we may choose m_1 large enough so that the finite collection $\{B_1, B_2, \ldots, B_{m_1}\}$ satisfies

$$\mathcal{H}^s_{5\delta}\left(E \setminus \bigcup_{i=1}^{m_1} B_i\right) \le \sum_{i=m_1+1}^{\infty} |\hat{B}_i|^s = 5^s \sum_{i=m_1+1}^{\infty} |B_i|^s < \theta \mathcal{H}^s(E)$$

for some $0 < \theta < \frac{1}{2}$. Letting $\delta \to 0$ and μ be the restriction of $\mathcal{H}^s$ to E, then the above result is equivalent to

$$\mu\left(U_1 \setminus \bigcup_{i=1}^{m_1} B_i\right) < \theta\mu(U_1)$$

Let $m_0 = 0$ and define inductively

$$U_{k+1} = U_k \setminus \bigcup_{i=m_{k-1}+1}^{m_k} B_i \quad \text{for each} \quad k \ge 1$$

By construction U_{k+1} is open, we may repeat the above steps and get a finite disjoint subcollection $\{B_{m_k+1}, \ldots, B_{m_{k+1}}\} \subset \mathcal{V}$ such that

$$\mu\left(U_1 \setminus \bigcup_{i=1}^{m_{k+1}} B_i\right) = \mu\left(U_{k+1} \setminus \bigcup_{i=m_k+1}^{m_{k+1}} B_i\right) < \theta\mu(U_{k+1}) < \theta^{k+1}\mu(U_1)$$

Since $m_k \to \infty$ as $k \to \infty$,

$$\mathcal{H}^s\left(E \setminus \bigcup_{i=1}^{\infty} B_i\right) = \mu\left(U_1 \setminus \bigcup_{i=1}^{\infty} B_i\right) = 0$$

which proves the first part of the theorem. Moreover, given $\varepsilon > 0$, by the definition of $\mathcal{H}^s$ as the limit of $\mathcal{H}^s_\delta$, we may choose δ at the beginning of the proof so that

$$\mathcal{H}^s(E) < \mathcal{H}^s_\delta(E) + \frac{1}{2}\varepsilon \le \sum_{i=1}^{\infty} |W_i|^s + \frac{1}{2}\varepsilon$$

for any δ-cover $\{W_i\}$ of E. Given how the collection $\{B_i\}$ has been constructed, we may find a δ-cover $\{V_i\}$ of $E \setminus \bigcup_i B_i$ such that

$$\frac{1}{2}\varepsilon = \mathcal{H}^s\left(E \setminus \bigcup_{i=1}^{\infty} B_i\right) + \frac{1}{2}\varepsilon > \sum_{i=1}^{\infty} |V_i|^s$$

Since $\{B_i\} \cup \{V_i\}$ is then a δ-cover of E and $\mathcal{H}^s(H \setminus \bigcup U_i) = 0$, it follows

$$\mathcal{H}^s(E) < \sum_{i=1}^{\infty} |B_i|^s + \sum_{i=1}^{\infty} |V_i|^s + \frac{1}{2}\varepsilon < \sum_{i=1}^{\infty} |B_i|^s + \varepsilon.$$

□

Finally, we conclude this chapter by examining the Hausdorff dimension and measure of the Cantor set from Example 1.5. Since in general calculating the Hausdorff measure and dimension of a given set is difficult, it is striking to see that these quantities associated to the Cantor set turn out to be simple.

Theorem 2.12. *The Hausdorff dimension of the Cantor set E is $s = \log 2/\log 3$.*

Proof. We show $0 < \mathcal{H}^s(C) < \infty$ for s as above. Recall from Section 1 that the Cantor set $C = \bigcap_{k=1}^{\infty} C_k$, where each C_k is a finite union of 2^k intervals of length 3^{-k}. For any $\delta > 0$, there exists a K such that $3^{-K} < \delta$. Now since C_k covers C and consists of 2^K intervals of diameter less than δ, we have

$$\mathcal{H}^s_\delta(C) \leq 2^K(3^{-K})^s = 1$$

Letting $\delta \to 0$, we have $\mathcal{H}^s(C) \leq 1$. For the other direction, consider the *Cantor-Lebesgue function* F constructed in Folland, which is the limit of a sequence $\{F_n\}$ of piecewise linear functions. Since F_n increases by at most 2^{-n} on each interval of 3^{-n}, it follows that for any $x, y \in [0, 1]$.

$$|F_n(x) - F_n(y)| \leq \left(\frac{3}{2}\right)^n |x - y|$$

Notice $|F(x) - F_n(x)| \leq 2^{-n}$ for any $x \in [0, 1]$. Together with the triangle inequality,

$$|F(x) - F(y)| \leq |F(x) - F_n(x)| + |F_n(x) - F_n(y)| + |F_n(y) - F(y)|$$

It follows that for every $n \in \mathbb{N}$,

$$|F(x) - F(y)| \leq (3^n|x - y| + 2)\, 2^{-n}$$

In particular, for fixed x and y, we may choose n so that $3^n|x - y|$ lies between 1 and 3. Then, we have $|F(x) - F(y)| \leq 5 \cdot 2^{-n} = 5 \cdot (3^{-n})^s \leq c|x - y|^s$ for some positive constants c. Thus, by Lemma 2.7 $c\mathcal{H}^s(C) \geq \mathcal{H}^1(F(C)) = \mathcal{H}^1([0, 1]) = 1$, which implies $\mathcal{H}^s(C) > 0$, and proof is complete. (We may further show that $\mathcal{H}^s(C) = 1$, but the proof is more involved and not shown here. Interested readers may consult [Fal85] for a detailed inquiry.) ☐

3. Basic Density Properties

One of the important results in the theory of Lebesgue measure is the Lebesgue density theorem, which says for a $\mathcal{L}^n$-measurable subset E of $\mathbb{R}^n$, the Lebesgue density of E at x

$$\lim_{r \to 0} \frac{\mathcal{L}^n(E \cap B_r(x))}{\mathcal{L}^n(B_r(x))},$$

exists and equals 1 if $x \in E$ a.e. and equals 0 a.e. if $x \notin E$. Motivated by this result, we look for its analogue for Hausdorff measure.

First, we restrict our attention to a particular family of sets. A subset E of $\mathbb{R}^n$ is called an s-**set** $(0 \leq s \leq n)$ if E is $\mathcal{H}^s$-measurable and $0 < \mathcal{H}^s(E) < \infty$. Note that an s-set by definition is automatically a set of Hausdorff dimension s, and when $s = 0$, E is of no interest in studying since it is simply a set of finitely many points.

Now we define the **density** of an s-set E at a given point x, which captures the amount of masses that E has at x locally with respect to a ball. Densities will play a major role in our analysis.

Definition 3.1. The **upper** and **lower** densities of an s-set E at a point $x \in \mathbb{R}^n$ are defined respectively as

$$\overline{D}^s(E,x) = \limsup_{r\to 0} \frac{\mathcal{H}^s(E\cap B_r(x))}{(2r)^s}, \qquad \underline{D}^s(E,x) = \liminf_{r\to 0} \frac{\mathcal{H}^s(E\cap B_r(x))}{(2r)^s}$$

If $\overline{D}^s(E,x) = \underline{D}^s(E,x)$, we say that the **density** of E at x exists and denote it as $D(E,x)$. Moreover, if $D(E,x) = 1$, we say that x is a **regular** point of E, otherwise x is an **irregular** point. Lastly, an s-set is **regular** if $D(E,x) = 1$ for x $\mathcal{H}^s$-almost everywhere in E.

Characterizing regular sets and obtaining bounds for the upper and lower densities of s-sets are two major themes in section 4. In fact, we will look at one of the most fundamental as well as elegant results in geometric measure theory in section 5 that an s-set cannot be regular unless s is an integer.

We shall first examine a few density properties of s-sets for all values of s. In particular, the following two theorems can be viewed as the analogue of Lebesgue density theorem for Hausdorff measure.

Theorem 3.2. *If E is an s-set in $\mathbb{R}^n$, then $2^{-s} \le \overline{D}^s(E,x) \le 1$ for $\mathcal{H}^s$-almost all $x \in E$.*

Proof. We first prove the left hand inequality: Let $\Lambda = \{x \in E : \overline{D}^s(E,x) < 2^{-s}\}$ and for each $k \in \mathbb{N}$,

$$\Lambda_k = \left\{ x \in E : \mathcal{H}^s(E\cap B_r(x)) < \frac{kr^s}{k+1},\ 0 < r < \frac{1}{k} \right\}.$$

Then we have $\Lambda = \bigcup_{k=1}^\infty \Lambda_k$, and we show that $\mathcal{H}^s(\Lambda) = 0$ by showing $\mathcal{H}^s(\Lambda_k) = 0$ for each k: Indeed, let $\varepsilon > 0$, by the definition of $\mathcal{H}^s(\Lambda_k)$, we can find a $(1/k)$-cover $\{V_i\}$ of Λ_k such that $V_i \cap \Lambda_k \neq \emptyset$ and

$$\sum_{i=1}^\infty |V_i|^s \le \mathcal{H}^s_{1/k}(\Lambda_k) + \varepsilon \le \mathcal{H}^s(\Lambda_k) + \varepsilon$$

Now for each i we pick $x_i \in V_i \cap \Lambda_k$ and let $r_i = |V_i|$. Then it follows from elementary geometry that $V_i \cap \Lambda_k \subset B_{r_i}(x_i) \cap \Lambda_k \subset B_{r_i}(x_i) \cap \Lambda$. Thus,

$$\begin{aligned}\mathcal{H}^s(\Lambda_k) &\le \sum_{i=1}^\infty \mathcal{H}^s(V_i \cap \Lambda_k) \le \sum_{i=1}^\infty \mathcal{H}^s(B_{r_i}(x_i)\cap\Lambda) \\ &< \sum_{i=1}^\infty \frac{k}{k+1} r_i^s = \frac{k}{k+1}\sum_{i=1}^\infty |V_i|^s \le \frac{k}{k+1}(\mathcal{H}^s(\Lambda_k)+\varepsilon)\end{aligned}$$

Letting $\varepsilon \to 0$, we have $\mathcal{H}^s(\Lambda_k) = 0$ since $k \ge 1$ and $\mathcal{H}^s(\Lambda_k) \le \mathcal{H}^s(\Lambda) \le \mathcal{H}^s(E) < \infty$.

Now for the right hand inequality, for every $t > 1$ let $A_t = A = \{x \in E : \overline{D}^s(E,x) > t\}$. We show that $\overline{D}^s(E,x) \le 1$ by showing $\mathcal{H}^s(A) = 0$. Indeed, given $\varepsilon > 0$ and $\delta > 0$, by the Borel regularity of $\mathcal{H}^s$, we may find an open set U containing A such that $\mathcal{H}^s(E \cap U) < \mathcal{H}^s(A) + \varepsilon$. Moreover, by the construction of A, the collection of closed balls

$$\mathcal{V} = \{B_r(x) : x \in A,\ \mathcal{H}^s(E \cap B_r(x)) > t(2r)^s,\ 0 < r < \delta/2,\ B_r(x) \subset U\}$$

is a fine cover of A. Thus, by the Vitali Covering Theorem, there exists a sequence of disjoint balls $\{B_i\}$ such that $\mathcal{H}^s(A \setminus \bigcup_{i=1}^{\infty} B_i) = 0$. Thus,

$$\mathcal{H}^s(A) + \varepsilon > \mathcal{H}^s(E \cap U) \geq \sum_{i=1}^{\infty} \mathcal{H}^s(E \cap B_i) > t \sum_{i=1}^{\infty} |B_i|^s \geq t\mathcal{H}^s_\delta \left(A \cap \bigcup_{i=1}^{\infty} B_i \right)$$

Letting $\delta \to 0$ and $\varepsilon \to 0$, we have $\mathcal{H}^s(A) \geq t\mathcal{H}^s(A \cap \bigcup_{i=1}^{\infty} B_i)$. However, since $\mathcal{H}^s(A \setminus \bigcup_{i=1}^{\infty} B_i) = 0$, we have $\mathcal{H}^s(A) = t\mathcal{H}^s(A \cap \bigcup_{i=1}^{\infty} B_i)$, which forces $\mathcal{H}^s(A) = 0$ using $t > 1$. Since the result holds for every $t > 1$, we get the desired result. □

Remark. Because of the above theorem, upper densities are usually more useful in practice than lower densities.

Theorem 3.3. *If E is an s-set in $\mathbb{R}^n$, then $D^s(E, x) = 0$ at $\mathcal{H}^s$-almost all $x \notin E$.*

Proof. We first show that for any $t > 0$, the set $A = \{x \in \mathbb{R}^n \setminus E : \overline{D}^s(E, x) > t\}$ has measure zero. Given $\varepsilon > 0$ and $\delta > 0$, since $\mathcal{H}^s(E \cap A) = 0$, it follows from the Borel regularity of $\mathcal{H}^s$ that there exists an open set U containing A such that $\mathcal{H}^s(E \cap U) < \varepsilon$. Now by the construction of A, we know that the collection of closed balls

$$\mathcal{V} = \{B_r(x) : x \in A,\ \mathcal{H}^s(E \cap B_r(x)) > t(2r)^s,\ 0 < r < \delta/2,\ B_r(x) \subset U\}$$

is a fine cover of A. Thus, by the Vitali Covering Theorem, there exists a disjoint sequence of closed balls $\{B_i\}$ such that $\mathcal{H}^s(A \setminus \bigcup_{i=1}^{\infty} B_i) = 0$. Thus,

$$\varepsilon > \mathcal{H}^s(E \cap U) \geq \sum_{i=1}^{\infty} \mathcal{H}^s(E \cap B_i) > t \sum_{i=1}^{\infty} |B_i|^s \geq t\mathcal{H}^s_\delta \left(A \cap \bigcup_{i=1}^{\infty} B_i \right)$$

Letting $\delta \to 0$ and $\varepsilon \to 0$, using $\mathcal{H}^s(A \setminus \bigcup_{i=1}^{\infty} B_i) = 0$, we have $t\mathcal{H}^s(A) = 0$. Since $t > 0$, we have that $\mathcal{H}^s(A) = 0$, which implies $\underline{D}^s(E, x) = \overline{D}^s(E, x) = D(E, x) = 0$ at $\mathcal{H}^s$-almost all $x \notin E$. □

The two following useful corollaries that are direct consequences of the above theorems, which will allow us to pass density results to subsets and countable union:

Corollary 3.4. *Let F be a measurable subset of an s-set E. Then $\underline{D}^s(F, x) = \underline{D}^s(E, x)$ and $\overline{D}^s(F, x) = \overline{D}^s(E, x)$ for almost all $x \in F$.*

Proof. Let $H = E \setminus F$. Then,

$$\mathcal{H}^s(H \cap B_r(x)) = \mathcal{H}^s(E \cap B_r(x)) - \mathcal{H}^s(F \cap B_r(x))$$

which implies

$$\overline{D}^s(H, x) = \overline{D}^s(E, x) - \overline{D}^s(F, x), \quad \underline{D}^s(H, x) = \underline{D}^s(E, x) - \underline{D}^s(F, x)$$

Since by Theorem 3.3 $\overline{D}^s(H, x) = \underline{D}^s(H, x) = D(H, x) = 0$ for almost all $x \in F$, we have the desired results. □

Remark. Since it is easy to check that for any s-set E in $\mathbb{R}^n$, the set of regular points A and that of irregular points B are both measurable, it follows from the above theorem that $E = A \cup B$, $D(A, x) = 1$ for almost all $x \in A$ and $D(B, x) \neq 1$ for almost all $x \in B$, i.e. any s-set can be decomposed into a disjoint union of a regular set and an irregular set.

Corollary 3.5. *Let $E = \cup_j E_j$ be a countable disjoint union of s-sets with $\mathcal{H}^s(E) < \infty$. Then for any k,*

$$\underline{D}^s(E_k, x) = \underline{D}^s(E, x) \quad \textit{and} \quad \overline{D}^s(E_k, x) = \overline{D}^s(E, x)$$

for almost all $x \in E_k$.

Proof. Since $E_k \subset E$ for each k, the result follows from Corollary 3.4. □

Remark. It worth mentioning that in Theorem 3.2 the upper bound 1 is always sharp, while the lower bounded 2^{-s} is sharp for all $s \leq 1$. A sharp estimate in the case when $s > 1$ is conjectured to be $1/2$ yet still unknown. This shows that density behaves differently for different s, which we will discuss in details in Sections 4 and 5.

Before concluding this section, we introduce one more type of density that come in handy when studying the properties of s-sets, namely:

Definition 3.6. The **upper convex density** of an s-set E at x is defined as

$$\overline{D}_c^s(E, x) = \limsup_{r \to 0} \left\{ \sup \frac{\mathcal{H}^s(E \cap U)}{|U|^s} \right\}$$

where the supremum is taken over all convex sets U with $x \in U$ and $0 < |U| \leq r$.

Since any bounded set is contained in a convex set of equal diameter, the definition of upper convex density is equivalent to taking the supremum over all sets U with $x \in U$ and $0 < |U| \leq r$. Moreover, since $B_r(x)$ is convex and if $x \in U$, then $U \subset B_r(x)$ for $r = |U|$, thus the convex density is related to the normal density by

$$2^{-s} \overline{D}_c^s(E, x) \leq \overline{D}^s(E, x) \leq \overline{D}_c^s(E, x)$$

It can be shown that if E is an s-set, then $\overline{D}_c^s(E, x) = 1$ for almost all $x \in E$.[1]

4. Sets of Integral Dimension

Now we arrive at the first highlight of this paper, which is a characterization of the structure of s-sets in $\mathbb{R}^n$ in terms of tangency, density, and rectiafiability. The main question we would like to answer is that "what do integral dimensional regular sets look like?".

Considering the higher dimensional case is much more difficult to deal with, thus we will restrict our attention to the case when $s = 1$ and $n = 2$, which is almost entirely the work of Besicovitch [Bes28], [Bes38], and [Bes39]. Meanwhile, we will examine what makes the higher dimensional version of the question hard to answer.

[1] Theorem 2.3 [Fal85]; we will use this result without proving later in Section 4.

First, we introduce some definitions: Recall the length of a curve Γ, $\mathscr{L}(\Gamma)$, defined in equation (1) of Section 2, we say:

Definition 4.1. A curve $\Gamma \subset \mathbb{R}^n$ is **rectifiable** if its length $\mathscr{L}(\Gamma)$ is finite.

Remark. It follows from Theorem 2.8 that if Γ is a rectifiable curve, then $\mathcal{H}^s(\Gamma) = \infty$ if $s < 1$ and $\mathcal{H}^s(\Gamma) = 0$ if $s < 1$.

Definition 4.2. An s-set $E \subset \mathbb{R}^n$ is **rectifiable** (or 1-rectifiable) if it contained in a countable union of rectifiable curves. E is **purely unrectifiable** if E intersects any rectifiable curves in a set of $\mathcal{H}^s$-measure zero.

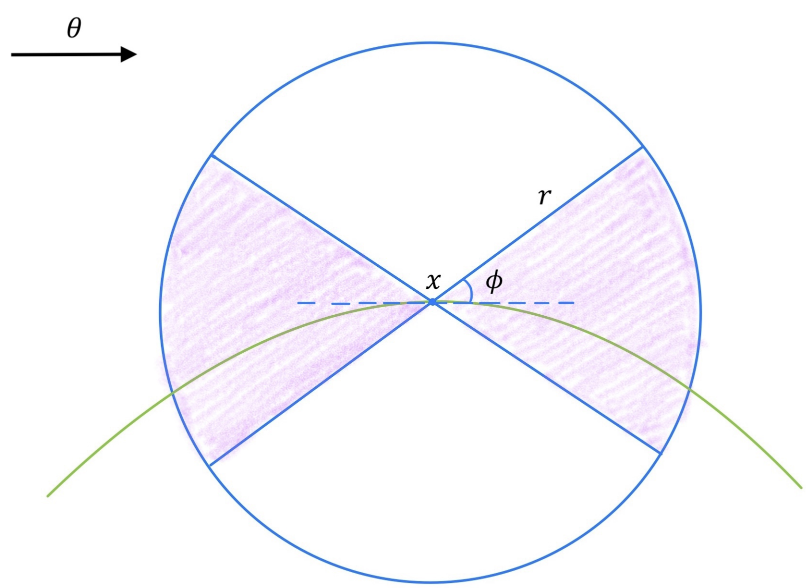
Finally, we would like to define what does it mean for an s-set E to have a tangent at x. Following the traditional train of thought, we expect the tangent to be locally the best linear approximation to the set. However, since there is no smoothness condition guaranteed on E, we look for an analogue in a measure-theoretic setting and arrive at the following definition:

Definition 4.3. An s-set E in $\mathbb{R}^n$ has a **tangent** at x in the direction of $\pm\boldsymbol{\theta}$ if $\overline{D}^s(E, x) > 0$ and for every angle $\phi > 0$,

$$\lim_{r\to 0} \frac{\mathcal{H}^s(E \cap (B_r(x) \setminus (S_r(x, \boldsymbol{\theta}, \phi) \cup S_r(x, -\boldsymbol{\theta}, \phi))))}{r^s} = 0 \tag{2}$$

where $S_r(x, \boldsymbol{\theta}, \phi)$ denotes the circular sector with vertex x, radius r, axis in the direction $\boldsymbol{\theta}$ consisting of points y for which the segment $[x, y]$ makes an angle of at most ϕ with respect to $\boldsymbol{\theta}$. We use $S_\infty(x, \boldsymbol{\theta}, \phi)$ to denote the corresponding infinite cone.

We visualize this definition of tangency by the following picture:



Consider the dashed line (in the direction of $\pm\boldsymbol{\theta}$) as the tangent of E at x. Then equation (2) says for an angle $\phi > 0$, the two-sided circular sectors locally cover almost all the mass of E at x. Since it holds for any arbitrarily small angle $\phi > 0$, we see the line in the direction of $\pm\boldsymbol{\theta}$ nicely approximates E at x.

Moreover, there are two observations we can make on this definition. First, it is clear that E has at most one tangent at x; otherwise, there exists two-sided cones covering almost all the mass of E in multiple directions, which gives a contradiction if we let the corresponding ϕ's to be sufficiently small. Second, if E has a tangent in the direction of $\pm\boldsymbol{\theta}$ at x, then any subset of E containing x has the same tangent at x.

Angular densities were introduced by Besicovitch to study tangential properties of s-sets in $\mathbb{R}^n$ when $n \geq 2$. We introduce their definitions and use them later in this section:

Definition 4.4. The **upper angular density** of an s-set E at x is defined as

$$\overline{D}^s(E, x, \boldsymbol{\theta}, \phi) = \limsup_{r \to 0} \frac{\mathcal{H}^s(E \cap S_r(x, \boldsymbol{\theta}, \phi))}{(2r)^s}$$

where the use of symbols is consistent with Definition 4.3. Similarly, the **lower angular density** is defined by taking the limit infimum.

Now we are ready to look at the big theorem of this section:

Theorem 4.5. *Let E be a 1-set in $\mathbb{R}^2$. Then the following statements are equivalent:*

(1) *E is regular.*
(2) *E is countably rectifiable.*
(3) *E has a tangent almost everywhere.*

We briefly mention that this result holds for any s-set in $\mathbb{R}^n$ when s is an integer less than n. However, this generalization into higher dimensions was extremely difficult and took over 50 years. We follow the original work of Besicovitch and establish the equivalence of the three statements by showing that (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1). Not necessary though, we provide a proof for the direction (2) $\Rightarrow$ (1) because it is relatively simple and contains useful ideas.

(Rectifiability $\Rightarrow$ Regularity) [(2) $\Rightarrow$ (1)]

This rather innocent-looking lemma will be very powerful and frequently used later:

Lemma 4.6. *Let E be a continuum (a compact connected set) containing x and y. If $|x - y| = \rho$, then $\mathcal{H}^1(E \cap B_\rho(x)) \geq \rho$. In particular, $\mathcal{H}^1(E) \geq |E|$.*

Proof. Define a function $f : \mathbb{R}^n \to [0, \infty)$ by $f(z) = |z - x|$. Then by triangle inequality,

$$|f(z) - f(w)| = ||z - x| - |w - x|| \leq |(z - x) - (w - x)| = |z - w|$$

for all $z, w \in \mathbb{R}^n$. Thus f is continuous. Now we claim the set $f(E \cap B_\rho(x))$ contains the interval $[0, \rho]$: Suppose not, then there exists an r with $0 < r < \rho$ such that

$r \notin f(E \cap B_\rho(x))$. But the continuity of f implies that there exists some $\delta > 0$ such that $(r-\delta, r+\delta) \cap f(E \cap B_\rho(x)) = \emptyset$. Thus, $E = (E \cap B_r(x)) \cup (E \setminus B_r(x))$ is a decomposition of E into disjoint closed set since $E \setminus B_r(x) = E \setminus B_{r+\delta/2}(x)$ is closed, which contradicts the assumption that E is a continuum. Now applying Lemma 2.7 gives

$$\mathcal{H}^1(E \cap B_\rho(x)) \geq \mathcal{H}^1(f(E \cap B_\rho(x))) \geq \mathcal{H}^1([0,\rho]) = \rho.$$

Moreover, since E is compact and the distance function is continuous, we know that there exists $z, w \in E$ such that $|z-w| = |E|$, which implies $\mathcal{H}^1(E) \geq |E|$. □

Now we are ready to show:

Theorem 4.7. *A rectifiable curve is a regular* 1*-set.*

Proof. We first show that a rectifiable curve Γ is a 1-set: Indeed, since Γ by construction contains at least two distinct points, it follows from the previous Lemma that $\mathcal{H}^1(\Gamma) > 0$. Moreover, since Γ is rectifiable, i.e. $\mathscr{L}(\Gamma) < \infty$. It follows from Theorem 2.8 that $\mathcal{H}^1(\Gamma) < \infty$.

To show that Γ is regular, let $x \in \Gamma$ be a point on Γ other than an endpoint, then x divides Γ into two rectifiable subcurves, say Γ_+ and Γ_-. Then by Lemma 4.6, when $\rho > 0$ is small enough, we have $\mathcal{H}^1(\Gamma_+ \cap B_\rho(x)) \geq \rho$ and $\mathcal{H}^1(\Gamma_- \cap B_\rho(x)) \geq \rho$, which implies $\mathcal{H}^1(\Gamma \cap B_\rho(x)) \geq 2\rho$. By definition,

$$\underline{D}^1(\Gamma, x) = \liminf_{\rho \to 0} \frac{\mathcal{H}^1(\Gamma \cap B_\rho(x))}{2\rho} \geq 1.$$

At the same time, we also know from Theorem 3.2 that $\overline{D}^1(\Gamma, x) \leq 1$ for almost all $x \in \Gamma$, which implies $\overline{D}^1(\Gamma, x) = \underline{D}^1(\Gamma, x) = D^1(\Gamma, x) = 1$ for almost all $x \in \Gamma$. □

The following result follows immediately by applying Corollaries 3.4 and 3.5 to Theorem 4.7

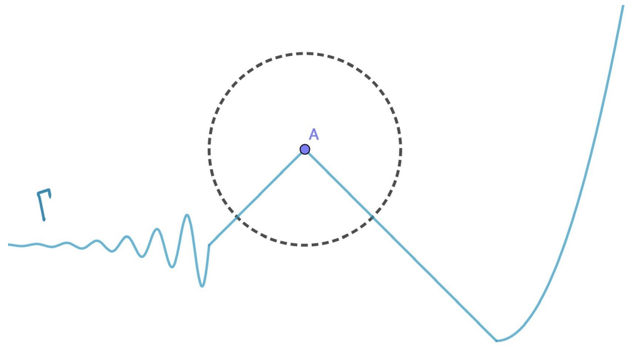
Corollary 4.8. *If a* 1*-set E is countably rectifiable, then E is regular.*

(Rectifiability $\Rightarrow$ Tangency) [(2) $\Rightarrow$ (3)]

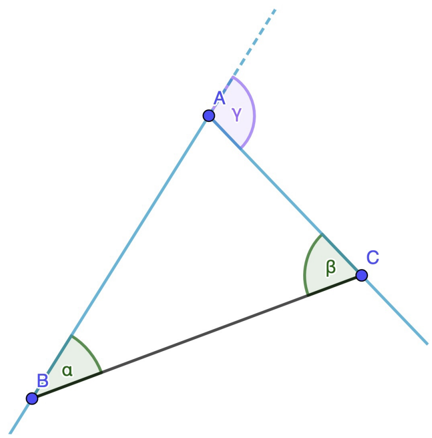
A smooth curve has tangents everywhere because at each point, the curve turns continuously and does not spike, which allows us to determine its best linear approximation at each point. However, rectifiable curves do spike, quite often sometimes, yet we would still like to examine their tangential properties. Thus, an estimate on the size of the set of spikes a rectifiable curve can have is crucial:

Lemma 4.9. *Let Γ be a rectifiable curve with endpoints x and y, and let ϕ be an angle with $0 < \pi < \pi/2$. Let E be the set of points on Γ that belong to pairs of arbitrarily small subarcs of Γ subtending chords that make a angle of more than 2ϕ with each other. Then $\mathcal{H}^1(E) \leq (\mathscr{L}(\Gamma) - |x-y|)/(1-\cos\phi)$.*

To visually translate the statement of the lemma, we take a rectifiable curve Γ:

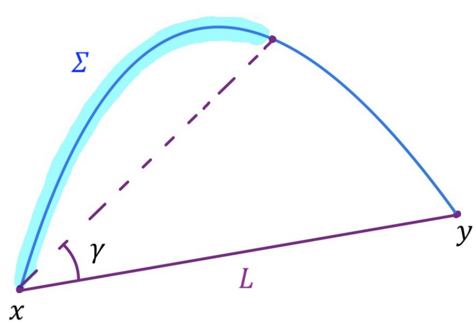


and look at the spike of Γ at A locally;



If $\gamma > 2\phi$, then the point A belongs to the set E since for arbitrarily small subarcs AB and AC, their subtending chords make a constant angle γ. Moreover, we have either $\alpha > \phi$ or $\beta > \phi$.

Proof. To prove the above lemma, let L denote the line through x and y and let $\mathcal{V}$ be the collection of closed subarcs of Γ subtending chords that make angles of more than ϕ with L. See the figure below:



That is if $\gamma > \phi$, then the subarc $\Sigma \subset \Gamma$ is included in the collection $\mathcal{V}$. Then we see for any $x \in E$, at least half of the arbitrarily small subarcs associated with x belong to $\mathcal{V}$, which implies $\mathcal{V}$ is a fine cover of E by closed subarcs. Now for any $\varepsilon > 0$, by the Vitali Covering Theorem [2], we may find a finite collection $\Gamma_1, \ldots, \Gamma_m$ of disjoint subarcs of Γ belonging to $\mathcal{V}$ such that

$$\mathcal{H}^1(E) \leq \sum_{i=1}^{m} |\Gamma_i| + \varepsilon \leq \sum_{i=1}^{m} \mathcal{H}^1(\Gamma_i) + \varepsilon = \sum_{i=1}^{m} \mathscr{L}(\Gamma_i) + \varepsilon \tag{3}$$

Suppose $\Gamma'_0, \Gamma'_1, \ldots, \Gamma'_m$ are the complementary arcs and $\phi'_0, \phi'_1, \ldots, \phi'_m$ are the corresponding angles between the subtending chords and the line L. If we denote the corresponding angles between the subtending chords of $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$ and L as $\phi_1, \phi_2, \ldots, \phi_m$ respectively, then projecting them orthogonally onto the line L, we see that since $\pi/2 > \phi_i > \phi > 0$,

$$|x - y| \leq \sum_{i=1}^{m} \cos(\phi_i) \mathscr{L}(\Gamma_i) + \sum_{i=0}^{m} \cos(\phi'_i) \mathscr{L}(\Gamma'_i) \leq \cos\phi \sum_{i=1}^{m} \mathscr{L}(\Gamma_i) + \sum_{i=0}^{m} \mathscr{L}(\Gamma'_i). \tag{4}$$

Since

$$\sum_{i=1}^{m} \mathscr{L}(\Gamma_i) + \sum_{i=0}^{m} \mathscr{L}(\Gamma'_i) = \mathscr{L}(\Gamma),$$

together with (3) and (4), we get

$$\mathcal{H}^1(E) - \varepsilon \leq \sum_{i=1}^{m} \mathscr{L}(\Gamma_i) \leq (\mathscr{L}(\Gamma) - |x - y|)/(1 - \cos\phi).$$

Since ε is arbitrary, we obtain the desired result. □

This upper bound on $\mathcal{H}^1(E)$ allows us to neatly "chop off" each of the spikes by a straight line and concludes that the set of all the spikes are invisible:

Corollary 4.10. *If $\phi > 0$ and E is the set of points on a rectifiable curve Γ that belong to pairs of arbitrarily small subarcs of Γ subtending chords that make an angle of more than 2ϕ with each other, then $\mathcal{H}^1(E) = 0$.*

Proof. Given $\varepsilon > 0$ and suppose $\psi : [a, b] \to \mathbb{R}^2$ defines the curve Γ parametrized by arc length, it follows from the definition of $\mathscr{L}(\Gamma)$ that there exists a partition $a = x_0 < x_1 < \ldots < x_m = b$ such that

$$\mathscr{L}(\Gamma) \leq \sum_{i=1}^{m} |x_i - x_{i-1}| + \varepsilon$$

[2] Here we use a variant of Theorem 2.11 where the fine cover is allowed to be any closed sets. A proof can be found in [Fal85] Theorem 1.10.

Let Γ_i denote the subarc between $\psi(x_i)$ and $\psi(x_{i-1})$. The applying Lemma 4.9 to each subarc Γ_i gives

$$\begin{aligned}\mathcal{H}^1(E) = \sum_{i=1}^{m}\mathcal{H}^1(E\cap\Gamma_i) &\le \sum_{i=1}^{m}(\mathscr{L}(\Gamma_i) - |x_i - x_{i-1}|)/(1-\cos\phi)\\ &= (\mathscr{L}(\Gamma) - \sum_{i=1}^{m}|x_i - x_{i-1}|)/(1-\cos\phi) < \varepsilon/(1-\cos\phi).\end{aligned}$$

Since ε is arbitrary, we may conclude that $\mathcal{H}^1(E) = 0$. □

Now we are finally ready to show the direction that rectifiability implies existence of tangents almost everywhere:

Theorem 4.11. *A rectifiable curve* Γ *has a tangent at almost all of its points.*

Proof. Since Γ is a continuum having at least two distinct points, say x and y. If $|x-y| = \rho$, then it follows from Lemma 4.6. that $\overline{D}^1(\Gamma, z) \ge 1/2 > 0$ for almost all $z \in \Gamma$.

Now to check the limit condition of a tangent, namely equation (2), holds, let $\psi : [a,b] \to \mathbb{R}^n$ be the function that defines Γ parametrized by arc length. It follows from Corollary 4.10 that for almost all $x \in \Gamma$, given $\phi > 0$, there exists a unit vector $\boldsymbol{\theta}$ and $\varepsilon > 0$ such that,

$$\psi(u) \in S_\infty(x, \boldsymbol{\theta}, \phi) \cup S_\infty(x, -\boldsymbol{\theta}, \phi)$$

whenever $|u - t| < \varepsilon$, where $\psi(t) = x$. Moreover, we claim that there exists $\rho > 0$ such that if $|u-t| \ge \varepsilon$, then $\psi(u) \notin B_\rho(x)$: Suppose not, then there exists a sequence $\{u_i\} \subset [a,b]$ such that $|u_i - t| \ge \epsilon$ for each i and $\phi(u_i) \to x$. On the other hand, by the sequential compactness of $[a,b]$, there exists $u \in [a,b]$ such that $u_i \to u$ by passing to a subsequence if necessary. However, this shows that $\psi(t) = x = \psi(u)$ with $|u-t| \ge \varepsilon$, which contradicts the injectivity of Γ. Thus,

$$\Gamma \cap (B_\rho(x) \setminus (S_\rho(x, \boldsymbol{\theta}, \phi) \cup S_\rho(x, -\boldsymbol{\theta}, \phi))) = \emptyset$$

which implies equation (2) holds for almost all $x \in \Gamma$, i.e. Γ has a tangent at almost all of its points. □

Corollary 4.12. *If a* 1*-set* E *is countably rectifiable, then* E *has a tangent at almost all of its points.*

Proof. It follows from Theorem 4.7 that E is regular, which implies $\overline{D}^1(E, x) > 0$ for almost all $x \in E$. Moreover, it follows from Corollaries 3.4 and 3.5 that equation (2) holds for almost all $x \in E$. □

Before we dive into the remaining directions, let us take a look at a beautiful and very unique geometric property of continuum with finite one dimensional Hausdorff measure. We first need a lemma:

Lemma 4.13. *A continuum* E *with* $\mathcal{H}^1(E) < \infty$ *is arcwise connected.*

Proof. For any $z, w \in E$, since E is connected, we claim that z and w are ε-chainable, i.e. for every $\varepsilon > 0$ there exists a finite sequence of points $z = x_0, x_1, \dots, x_m = w$ in E such that $|x_i - x_{i-1}| < \varepsilon$ for all $1 \le i \le m$: Indeed, denote the set of points in E that are ε-chainable to z as A. Since z itself is in A, we know A is nonempty. A is open because for any point $u \in A$, the ε-neighborhood of u belongs to A. A is also closed because for any limit point u' of A, there exists $u \in A$ such that $|u - u'| < \varepsilon$, thus A contains all of its limit points. Hence, $A = E$.

Moreover, we may assume that $|x_i - x_j| \ge \varepsilon/2$ when $|i - j| \ge 2$ by deleting points of the chain if necessary. Thus, no point of $\mathbb{R}^n$ lies in more than two of the balls $B_{\frac{1}{2}\varepsilon}(x_i)$ for all i. Assuming $m \ge 2$ and applying Lemma 4.6, we have

$$2\mathcal{H}^1(E) \ge \sum_{i=0}^{m} \mathcal{H}^1(E \cap B_{\frac{1}{2}\varepsilon}(x_i)) \ge \frac{1}{2} m\varepsilon.$$

Let Γ_ε denote the polygonal curve joining $x_0, x_1, \dots, x_m$, then Γ_ε is necessarily not self-intersecting, and

$$\mathscr{L}(\Gamma_\varepsilon) = \sum_{i=1}^{m} |x_i - x_{i-1}| \le m\varepsilon \le 4\mathcal{H}^1(E).$$

Now let $\psi_\varepsilon : [0,1] \to \mathbb{R}^n$ be the parametrization of Γ_ε by arc length such that the part of Γ_ε connecting $z = \psi_\varepsilon(0)$ and $\psi_\varepsilon(t)$ has length $t\mathscr{L}(\Gamma_\varepsilon)$. Then for any $0 \le t_1 < t_2 \le 1$,

$$|\psi_\varepsilon(t_1) - \psi_\varepsilon(t_2)| \le \mathscr{L}(\psi_\varepsilon[t_1, t_2]) \le 4|t_2 - t_1|\mathcal{H}^1(E)$$

Hence, the sequence of functions $\{\psi_{1/j}\}$ is equicontinuous and uniformly bounded (because $\{\Gamma_{1/j}\}$ is contained in a bounded subset of $\mathbb{R}^n$). By the Arzela-Ascoli theorem, there exists a subsequence $\{\psi_{1/j_k}\}$ and a continuous function ψ such that ψ_{1/j_k} converges uniformly to ψ on $[0,1]$.

Let $\psi[0,1] = \Gamma$, and we show that Γ is the desired curve in E joining z and w: Indeed, $\Gamma(0) = \lim_{k\to\infty} \Gamma_{1/n_k}(0) = z$ and similarly $\Gamma(1) = w$. Moreover, for any $x \in \Gamma$ and $\delta > 0$, there exists a $K \in \mathbb{N}$ such that $1/j_K < \delta/2$, thus the curve $\Gamma_{1/j_K} \cap B_{\delta/2}(x) \ne \emptyset$. Thus, x is a limit point of E. Since E is closed, we have $x \in E$ and therefore $\Gamma \subset E$. □

Lemma 4.14. *Any compact arcwise connected set E with $\mathcal{H}^1(E) < \infty$ consists of a countable union of rectifiable curves union a set of $\mathcal{H}^1$-measure zero.*

Proof. Define a sequence of curves $\{\Gamma_i\}$ inductively: Let Γ_1 be the curve in E joining two of the most distant points in E. Such a curve exists because of the continuity of the distance function, compactness of E, and arcwise connectedness of E.

Suppose $\Gamma_1, \Gamma_2, \dots, \Gamma_k$ have been defined. Let x be a point in E at maximum distance, say d_k, from $\bigcup_{i=1}^k \Gamma_i$. If $d_k = 0$ for some k, then $E = \bigcup_{i=1}^k \Gamma_i$ because E is closed. Since by Theorem 2.8 $\mathscr{L}(\Gamma_i) = \mathcal{H}^1(\Gamma_i) \le \mathcal{H}^1(E) < \infty$, we know Γ_i is rectifiable for each i, and we are done.

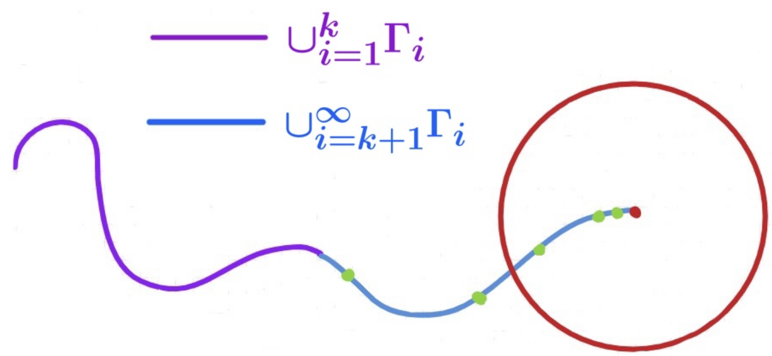
If $d_k > 0$, let Γ_{k+1} be a curve in E joining x to $\bigcup_{i=1}^{k} \Gamma_i$ with Γ_{k+1} disjoint from $\bigcup_{i=1}^{k} \Gamma_i$ except for an endpoint. By Lemma 4.6, $\mathcal{H}^1(\Gamma_{k+1}) \geq d_k$. Hence,

$$\sum_{i=1}^{\infty} d_i \leq \sum_{i=1}^{\infty} \mathcal{H}^1(\Gamma_i) \leq \mathcal{H}^1(E) < \infty \tag{5}$$

which implies $d_i \to 0$ as $i \to \infty$. Thus, for any $y \in E$, the distance between y and $\bigcup_{i=1}^{\infty} \Gamma_i$ is zero, implying $E = \overline{\bigcup_{i=1}^{\infty} \Gamma_i}$. Now for each k, consider the following collection of closed balls

$$\mathcal{V}_k = \left\{ B_r(x) : x \in E \setminus \bigcup_{i=1}^{k} \Gamma_i, \;\; B_r(x) \cap \bigcup_{i=1}^{k} \Gamma_i = \emptyset \right\}$$

Since $\bigcup_{i=1}^{k} \Gamma_i$ is closed, we have that $\mathcal{V}_k$ is a fine cover of $E \setminus \bigcup_{i=1}^{k} \Gamma_i$. For any $B_r(x) \in \mathcal{V}_k$, since x is in E and thereby in the closure of $\bigcup_{i=1}^{\infty} \Gamma_i$, it follows that there are points of $\bigcup_{i=1}^{\infty} \Gamma_i$ arbitrarily close to x (see picture below). Such points must be connected via a sequence of arcs in $\Gamma_{i=k+1}^{\infty} \Gamma_i$ to a point of $\bigcup_{i=1}^{k} \Gamma_i$ necessarily outside $B_r(x)$.



Thus by Lemma 4.6,

$$\frac{1}{2}|B_r(x)| \leq \mathcal{H}^1\left(B_r(x) \cap \bigcup_{i=k+1}^{\infty} \Gamma_i \right) \tag{6}$$

Now given any $\varepsilon > 0$, by the Vitali Covering Theorem there exists a sequence of disjoint $\{B_j\} \subset \mathcal{V}$ such that

$$\mathcal{H}^1\left(E \setminus \bigcup_{i=1}^{k} \Gamma_i \right) \leq \sum_{j=1}^{\infty} |B_j| + \varepsilon \leq 2 \sum_{j=1}^{\infty} \mathcal{H}^1\left(B_j \cap \bigcup_{i=k+1}^{\infty} \Gamma_i \right) + \varepsilon$$

where the last inequality follows from (6). Moreover, since B_j's are disjoint, we have

$$\sum_{j=1}^{\infty} \mathcal{H}^1\left(B_j \cap \bigcup_{i=k+1}^{\infty} \Gamma_i\right) = \mathcal{H}^1\left(\bigcup_{j=1}^{\infty} B_j \cap \bigcup_{i=k+1}^{\infty} \Gamma_i\right) \leq \mathcal{H}^1\left(\bigcup_{i=k+1}^{\infty} \Gamma_i\right)$$

Altogether we have

$$\mathcal{H}^1\left(E \setminus \bigcup_{i=1}^{k} \Gamma_i\right) \leq 2\mathcal{H}^1\left(\bigcup_{i=k+1}^{\infty} \Gamma_i\right) + \varepsilon \leq 2 \sum_{i=k+1}^{\infty} \mathcal{H}^1(\Gamma_i) + \varepsilon$$

By (5), letting $k \to \infty$ and $\varepsilon \to 0$ shows $\mathcal{H}^1(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i) = 0$, as desired. □

Theorem 4.15. *Let E be a continuum in $\mathbb{R}^n$ with $\mathcal{H}^1(E) < \infty$. Then E consists of a countable union of rectifiable curves, together with a set of $\mathcal{H}^1$-measure zero.*

Proof. Combining Lemma 4.13 and Lemma 4.14 gives the result. □

Remark. This unique feature of 1-sets makes tangency and density results much easier to obtain, and there is no such analogue in the case when s is an integer greater than 1. An immediate consequence of Theorem 4.15 is that any purely unrectifiable set intersects a continuum of finite $\mathcal{H}^1$-measure in a set of measure zero.

(Regularity ⇒ Rectifiability) [(1) ⇒ (2)]

In this part of the proof, we show that a regular 1-set E is countably rectifiable by establishing that any measurable subset of E of positive measure can be exhausted by pieces of rectifiable curves, i.e. any appreciable amount of E cannot behave like a purely unrectifiable set. We achieve this result by showing a purely unrectifiable set has lower density strictly less than 1 almost everywhere.

We follow the footsteps of Besicovitch to unravel one of the most complicated as well as remarkable proofs. First of all, we require a purely topological result on the removal of the interiors of discs from continua. We say that a collection of discs is **semidisjoint** if no member of the collection is contained in any other.

Lemma 4.16. *Let E be a continuum in $\mathbb{R}^2$. Suppose that $\{B_i\}_{i=1}^{\infty}$ is a countable semidisjoint collection of closed discs with each center contained in E and such that $|B_i| \geq d$ for only finitely many i for any $d > 0$. Then if Γ_i is the perimeter of B_i,*

$$F = \left(E \setminus \bigcup_{i=1}^{\infty} B_i\right) \cup \bigcup_{i=1}^{\infty} \Gamma_i$$

is a continuum.

Proof. We first show that F is closed. Note $F = (E \setminus \bigcup_i B_i^\circ) \cup \bigcup_i \Gamma_i$ where B_i° is the interior of B_i. Thus, it suffices to show that the closure of $\bigcup_i \Gamma_i$ is contained in F. Let $x \in \overline{\bigcup_i \Gamma_i}$ but $x \notin E \setminus \bigcup_i B_i^\circ$, since E is closed, we know that $x \in \bigcup_i B_i^\circ$, which implies $x \in B_k^\circ$ for some k. Let $d = \operatorname{dist}(x, \Gamma_k) > 0$. If $\Gamma_j \cap B_{d/2}(x) \neq \emptyset$, we know $B_j \cap B_{d/2}(x) \neq \emptyset$. Moreover, since B_j is not contained in B_k by assumption, we also know that $B_j \cap (\mathbb{R}^2 \setminus B_k) \neq \emptyset$. Hence, Γ_j, as the perimeter of B_j, touches both $B_{d/2}(x)$ and the complement of B_k, which implies $|\Gamma_j| \geq \operatorname{dist}(B_{d/2}(x), \Gamma_k) = \frac{1}{2}d$, so only finitely many circles Γ_j can meet $B_{d/2}(x)$. Since $x \in \overline{\bigcup_i \Gamma_i}$, we may conclude that $x \in \overline{\bigcup_{i=1}^k \Gamma_i} = \bigcup_{i=1}^k \Gamma_i \subset F$ for some k, as desired.

Since $\{B_i\}$ is a sequence of closed balls with decreasing radii, we have that $\sup |B_i| < \infty$. Moreover, since each of B_i's is centered in the bounded set E, we have $E \cup \bigcup_{i=1}^\infty B_i$ is also bounded. Thus, F is closed and bounded and thereby compact. Now it remains to show that F is connected. Suppose for the sake of contradiction that $F = F_1 \cup F_2$ where F_1 and F_2 are disjoint nonempty closed sets. Let

$$E_1 = F_1 \cup \bigcup_{i:\Gamma_i \subset F_1} B_i\,, \quad E_2 = F_2 \cup \bigcup_{i:\Gamma_i \subset F_2} B_i$$

Since each Γ_i is contained in either E_1 or E_2, we know that $E = E_1 \cup E_2$. Moreover, we also have $E_1 \cap E_2 = \emptyset$: Indeed, since any ball B_1 from E_1 clearly cannot intersect F_2, and similarly any ball B_2 from E_2 cannot intersect F_1, it suffices to check that

$$\left(\bigcup_{i:\Gamma_i \subset F_1} B_i \right) \cap \left(\bigcup_{i:\Gamma_i \subset F_2} B_i \right) = \emptyset$$

Suppose not, that is, there exist $B_j \subset E_1$ and $B_k \subset E_2$ such that $B_j \cap B_k \neq \emptyset$. Since B_j and B_k are semidisjoint, it follows that $\Gamma_j \cap \Gamma_k \neq \emptyset$, which contradicts the assumption that F_1 and F_2 are disjoint. Thus, we see that E_1 and E_2 are disjoint.

Finally, we show that E_1 is closed. If $x \in \overline{\bigcup_{\{i:\Gamma_i \subset F_1\}} B_i}$, then x is either the limit of a sequence of points in $\bigcup_{i=1}^k B_i$ for some k, in which case $x \in \bigcup_{i=1}^k B_i \subset E_1$. Or x is the limit of a sequence of discs with boundaries in the closed set F_1 and radii approaching zero, which by definition implies x is a limit point of F_1 and thus $x \in F_1$. In either case, we see that $x \in E_1$. By exactly the same argument, we have that E_2 is also closed. Hence, E_1 and E_2 give rise to a separation of E, contradicting the assumption that E is connected. $\square$

What we present next is the main ingredient to a result that provides an upper bound for the lower densities of a purely unrectifiable set. Let $R(x,y)$ denote the **common region** of the circle-pair with centers x and y, so that $R(x,y) = B_d^\circ(x) \cap B_d^\circ(y)$ where $d = |x-y|$. Roughly speaking, the next lemma shows that if the common regions with centers in a 1-set E contain a good amount of E, then E can be seen by rectifiable curves.

Lemma 4.17. *Let E be a 1-set in $\mathbb{R}^2$ and suppose that $\alpha > 0$. Let E_0 be a compact subset of E with $\mathcal{H}^1(E_0) > 0$, such that $\mathcal{H}^1(E \cap R(x_1,x_2)) \geq \alpha|x_1 - x_2|$ for all $x_1, x_2 \in E_0$. Then there exists a continuum H such that $0 < \mathcal{H}^1(H \cap E) \leq \mathcal{H}^1(H) < \infty$.*

Proof. By Theorem 3.2 we know that $\frac{1}{2} \leq \overline{D}^1(E,x) \leq 1$. Now by Theorem 2.5, there exists a closed subset $F \subset E_0$ with $\mathcal{H}^1(F) > 0$ and $\rho_1 > 0$ such that

$$\mathcal{H}^1(B_r(x) \cap E) \leq 2 \cdot 2r \quad \text{if } x \in F \quad \text{and} \quad 0 < r \leq \rho_1$$

Since $\overline{D}^1((E \setminus F), x) = 0$ and $\overline{D}^1(F, x) \geq 1/2$ by Theorem 3.3 and 3.2 respectively, we may find a point $y \in F$ and a positive number ρ with $0 < \rho \leq \rho_1/10$ such that

$$\mathcal{H}^1((E \setminus F) \cap B_r(y)) < 2r \cdot 10^{-3}\alpha \quad (0 < r \leq 3\rho) \tag{7}$$

and

$$\mathcal{H}^1(F \cap B_\rho(y)) \geq 2\rho \cdot \frac{1}{4} = \frac{\rho}{2} \tag{8}$$

By reducing ρ if necessary, we may also assume that the perimeter Γ of $B_\rho(y)$ contains some point of F. Let $\mathcal{V}$ be the family of closed discs

$$\mathcal{V} = \{B_r(x) : x \in F \cap B_\rho(y),\ 0 < r < 2\rho,\ \mathcal{H}^1((E \setminus F) \cap B_r(x)) \geq \alpha r\}$$

If $\mathcal{V} \neq \emptyset$, then by the classic Vitali Covering Theorem, we may find a sequence of closed balls $\{B_i\} \subset \mathcal{V}$ such that

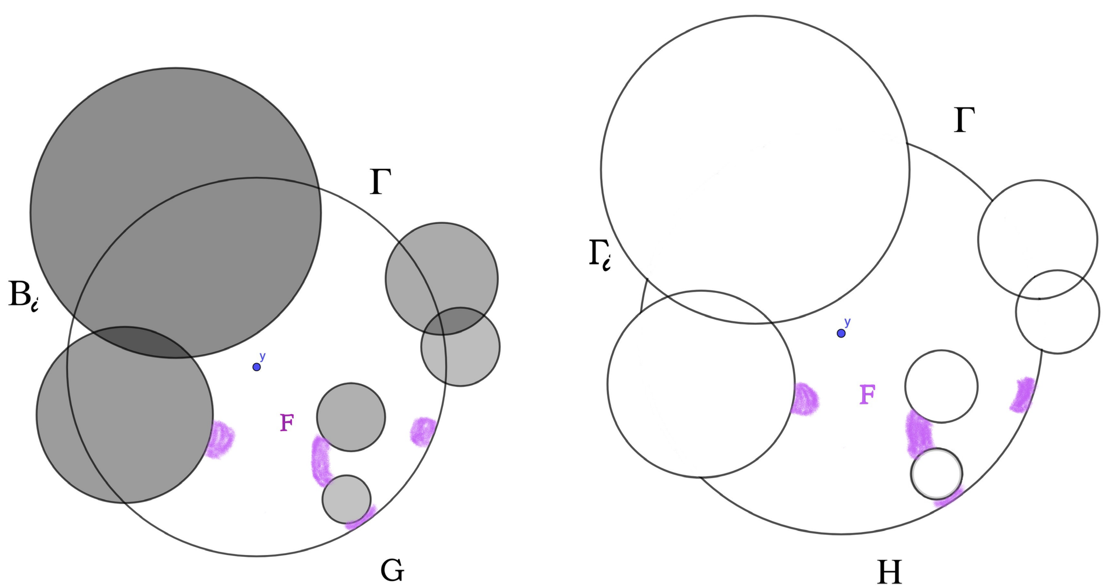
$$\bigcup_{B \in \mathcal{V}} B \subset \bigcup_{i=1}^{\infty} \hat{B}_i \quad \text{where} \quad \hat{B}_i = 5B_i$$

Moreover, by eliminating $\hat{B}_i$'s that are completely contained in one of the others, we may assume that the collection $\{\hat{B}_i\}$ to be semidisjoint. Let Γ_i denote the perimeter of $\hat{B}_i$,

$$G = (F \cap B_\rho(y)) \cup \Gamma \cup \bigcup_{i=1}^{\infty} \hat{B}_i$$

and

$$H = \left(G \setminus \bigcup_{i=1}^{\infty} \hat{B}_i\right) \cup \bigcup_{i=1}^{\infty} \Gamma_i$$



For the reminder of the proof, we divide it into six parts and show that H is indeed the desired continuum. Note that if $\mathcal{V} = \emptyset$, then $G = (F \cap B_\rho(y)) \cup \Gamma$ and $H = G$, so the results below follow directly.

(a) G is closed: Since $F \cap B_\rho(y)$ and Γ are closed, it suffices to show that $\overline{\bigcup_i \hat{B}_i} \subset G$. For any $z \in \overline{\bigcup_i \hat{B}_i}$, then either $z \in \bigcup_{i=1}^k \hat{B}_i$ for some k, in which case $z \in G$. Or z is the limit of points from a subsequence of the discs $\{\hat{B}_i\}_i$. But since B_i's are disjoint and their centers are in the compact set $F \cap B_\rho(y)$, it follows that $\bigcup_{i=1}^\infty B_i \cup (F \cap B_\rho(y)$ is bounded. On summing areas we have $\sum_{i=1}^\infty |B_i|^2 = \mathcal{H}^s(\bigcup_{i=1}^\infty B_i) < \infty$, which implies $|\hat{B}_i| = 5|B_i| \to 0$ as $i \to \infty$. Hence, z is the limit point of the centers of B_i's, so $z \in F \cap B_\rho(y)$ since $F \cap B_\rho(y)$ is closed. In either case, $z \in G$, and we conclude that G is closed.

(b) G is connected: Suppose for the sake of contradiction that $G = G_1 \cup G_2$ where G_1 and G_2 are nonempty disjoint closed sets. Since Γ is a connected subset of G, Γ is a subset of either G_1 or G_2, so assume that $\Gamma \subset G_1$. Note that for each $\hat{B}_i$ that contains points on or outside Γ, since its center is in $B_\rho(y)$, we know that it has to meet Γ and hence is contained in G_1. Thus, G_2 cannot contain points on or outside Γ, which implies $G_2 \subset B_\rho^\circ(y)$.

Let $G_1' = G_1 \cup (\mathbb{R}^2 \setminus B_\rho^\circ(y))$, then we have that G_1' is closed and disjoint from G_2. Now G_1' contains Γ which by construction contains points of F, and G_2, since it is nonempty, contains either some $\hat{B}_i$ or meet $F \cap B_\rho(y)$. In either case, G_2 also contains points of F. Let $x_1 \in G_1' \cap F$ and $x_2 \in G_2 \cap F$ be the points that minimize the distance $r = |x_1 - x_2|$. This infimum is attained and is positive (because $G_1' \cap F$ and $G_2 \cap F$ are closed and disjoint). As $\Gamma \subset G_1'$ contains a point of F, we also know that $r < 2\rho$. Moreover, the common region $R(x_1, x_2)$ is disjoint from F, for otherwise r can be further reduced. Thus, by the assumption of the lemma,

$$\begin{aligned} 0 < \alpha r \le \mathcal{H}^1(E \cap R(x_1, x_2)) &= \mathcal{H}^1((E \setminus F) \cap R(x_1, x_2)) \\ &\le \mathcal{H}^1((E \setminus F) \cap B_r(x_2)) \end{aligned}$$

Since $x_2 \in F \cap B_\rho(y)$ and $r < 2\rho$, we see that $B_r(x_2) \in \mathcal{V}$, and thus $B_r(x_2) \subset \bigcup_{i=1}^\infty \hat{B}_i$. However, $B_r(x_2) = (B_r(x_2) \cap G_1') \cup (B_r(x_2) \cap G_2)$ is a decomposition of $B_r(x_2)$ into nonempty disjoint closed sets, which is absurd. Thus, we have shown that G is connected.

(c) H is a continuum: We have shown that G is compact and connected, thereby a continuum. Recall from part (a) that $|\hat{B}_i| \to 0$ as $i \to \infty$, thus by Lemma 4.16, H is also a continuum.

(d) $\sum_{i=1}^\infty |\hat{B}_i| \le \frac{1}{10}\rho$: By the construction of $\mathcal{V}$, the disjointness of B_i's, and (7), we have that

$$\begin{aligned} \sum_{i=1}^\infty |\hat{B}_i| = 5 \sum_{i=1}^\infty |B_i| &\le \frac{10}{\alpha} \sum_{i=1}^\infty \mathcal{H}^1((E \setminus F) \cap B_i) \\ &\le \frac{10}{\alpha} \mathcal{H}^1((E \setminus F) \cap B_{3\rho}(y)) \le \frac{10}{\alpha} \cdot 6\rho 10^{-3} \alpha \le \frac{1}{10}\rho \end{aligned}$$

(e) $\mathcal{H}^1(H) < \infty$: Since $H \subset E \cup \Gamma \cup \bigcup_{i=1}^{\infty} \Gamma_i$, using part (d) gives

$$\mathcal{H}^1(H) \leq \mathcal{H}^1(E) + 2\pi\rho + \pi \sum_{i=1}^{\infty} |\hat{B}_i| < \infty$$

(f) $\mathcal{H}^1(H \cap E) > 0$: From the definition of H, we have

$$\begin{aligned}
\mathcal{H}^1(H \cap E) &\geq \mathcal{H}^1\left(\left(G \setminus \bigcup_{i=1}^{\infty} \hat{B}_i\right) \cap E\right) - \mathcal{H}^1\left(E \cap \bigcup_{i=1}^{\infty} \hat{B}_i\right) \\
&\geq \mathcal{H}^1(F \cap B_\rho(y)) - \sum_{i=1}^{\infty} \mathcal{H}^1(E \cap \hat{B}_i) \\
&\geq \mathcal{H}^1(F \cap B_\rho(y)) - 2\sum_{i=1}^{\infty} |\hat{B}_i| \quad (\text{since } |\hat{B}_i| < 20\rho \leq 2\rho_1) \\
&\geq \frac{1}{2}\rho - \frac{1}{5}\rho = \frac{3}{10}\rho > 0
\end{aligned}$$

where the last inequality follows from (8) and part (d). □

Now the result on the lower densities of purely unrectifiable sets is a direct consequence of the geometry of circle-pairs. The main idea is that a circle-pair with centers at points of high lower density and convex density must contain a subset of positive measure in its common region.

Theorem 4.18. *Let E be a purely unrectifiable set in $\mathbb{R}^2$. Then $\underline{D}^1(E, x) \leq \frac{3}{4}$ for almost all $x \in E$.*

Proof. Suppose there exists some $\alpha > 0$ such that the set $E_1 = \{x : \underline{D}^1(E, x) > \frac{3}{4} + \alpha\}$ has positive measure. Thus, by the definition of lower density, we may find a compact 1-set $E_2 \subset E_1$ of positive measure and $\rho > 0$ such that

$$\mathcal{H}^1(E \cap B_r(x)) > \left(\frac{3}{4} + \alpha\right) 2r \quad (x \in E_2,\ 0 < r \leq \rho)$$

and since the upper convex density $\overline{D}_c^1(E, x) = 1$ for almost all $x \in E$, we may at the same time require that

$$\mathcal{H}^1(E \cap U) < (1 + \alpha)|U| \quad (x \in E_2 \cap U,\ 0 < |U| \leq 3\rho,\ U \text{ convex})$$

Let E_0 be a compact subset of E_2 with $0 < \mathcal{H}^1(E_0) < \infty$ and $|E_0| \leq \rho$. If $x_1, x_2 \in E_0$, then $r = |x_1 - x_2| \leq \rho$. Since E is purely unrectifiable, any circle intersects E in a set of measure zero, which gives

$$\begin{aligned}
\mathcal{H}^1(E \cap R(x_1, x_2)) &\geq \mathcal{H}^1(E \cap B_r(x_1)) + \mathcal{H}^1(E \cap B_r(x_2)) - \mathcal{H}^1(E \cap \overline{CH(x_1, x_2)}) \\
&\geq 2 \cdot \left(\frac{3}{4} + \alpha\right) 2r - (1 + \alpha)3r = \alpha r
\end{aligned}$$

where $CH(x_1, x_2)$ denotes the convex hull of $B_r(x_1) \cup B_r(x_2)$. Hence, by applying Lemma 4.17 to E_0, we know there exists a continuum H such that $0 < \mathcal{H}^1(H \cap E) \leq$

$\mathcal{H}^1(H) < \infty$. But Theorem 4.15 tells that H is a countable union of rectifiable curves up to a set of measure zero, the conclusion $\mathcal{H}^1(H \cap E) > 0$ contradicts the assumption that E is purely unrectifiable. □

Remark. It worth mentioning that $\frac{3}{4}$ had been the best known upper bound for many decades until Preiss and Tiser improved it to $\frac{2+\sqrt{46}}{12}$ in 1992. Although Besicovitch conjectured the best upper bound to be $\frac{1}{2}$, no further improvement has been made after Preiss and Tiser. This outstanding conjecture is also known as the Besicovitch $\frac{1}{2}$-conjecture.

Finally, we have all the tools we need to show that a regular 1-set is countably rectifiable. As noted at the beginning of this direction, the following procedure attempts to exhaust any subset of positive measure with rectifiable curves, which highly resembles the proof of Lemma 4.14.

Theorem 4.19. *A regular* 1*-set* E *is countably rectifiable.*

Proof. Since E is regular, we have $\underline{D}^1(E, x) = 1$ for almost all $x \in E$. Thus, it follows from Theorem 4.17 and Corollary 3.4 that any measurable subset of E of positive measure cannot be purely unrectifiable, thereby intersecting some rectifiable curve in a set of positive measure. Based on this fact, we define a sequence of rectifiable curves $\{\Gamma_i\}$. Let Γ_1 be that

$$\mathcal{H}^1(\Gamma_1 \cap E) \geq \frac{1}{2} \sup\{\mathcal{H}^1(\Gamma \cap E) : \Gamma \text{ is rectifiable}\}$$

where we know that the supremum exists and is positive. Now if $\Gamma_1, \ldots, \Gamma_k$ have been defined and $E_k = E \setminus \bigcup_{i=1}^{k} \Gamma_i$ has positive measure (otherwise we are done), let Γ_{k+1} be a rectifiable curve with

$$\mathcal{H}^1(\Gamma_{k+1} \cap E_k) \geq \frac{1}{2} \sup\{\mathcal{H}^1(\Gamma \cap E_k) : \Gamma \text{ is rectifiable}\}$$

Suppose the process does not terminate in finitely many steps. We have the following estimate

$$\infty > \mathcal{H}^1(E) \geq \sum_{k=1}^{\infty} \mathcal{H}^1(\Gamma_{k+1} \cap E_k)$$

Thus, $\mathcal{H}^1(\Gamma_{k+1} \cap E_k) \to 0$ as $k \to \infty$. Now we claim that $\mathcal{H}^1(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i) = 0$: Indeed, suppose $\mathcal{H}^1(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i) > 0$, then by Theorem 4.17 again, we know there exists a rectifiable curve Γ such that

$$\mathcal{H}^1\left(\Gamma \cap \left(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\right)\right) = d > 0$$

However, as $\mathcal{H}^1(\Gamma_{k+1} \cap E_k) \to 0$, there exists some K such that $\mathcal{H}^1(\Gamma_{K+1} \cap E_K) < d/2$, in which case

$$\frac{1}{2}d > \frac{1}{2}\sup\{\mathcal{H}^1(\Gamma \cap E_k) : \Gamma \text{ is rectifiable}\}$$
$$\geq \frac{1}{2}\sup\left\{\mathcal{H}^1\left(\Gamma \cap \left(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\right)\right) : \Gamma \text{ is rectifiable}\right\} \geq \frac{1}{2}d$$

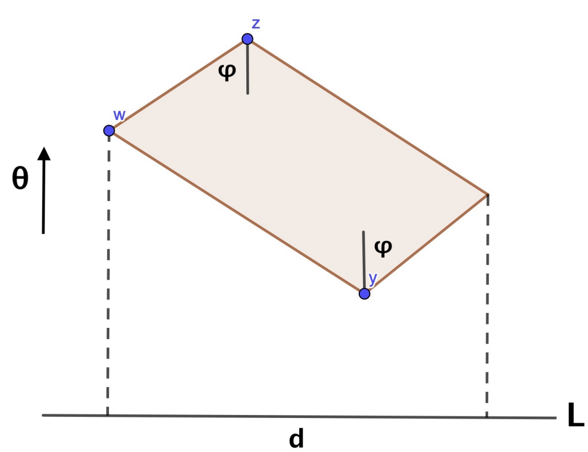
a contradiction. Thus, $\mathcal{H}^1(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i)$, and we obtained the desired result. □

(Tangency ⇒ Regularity) [(3) ⇒ (1)]

The only missing piece to proving the remarkable Theorem 4.5 is to show that if E is a 1-set in $\mathbb{R}^2$ with tangents almost everywhere, then E is regular, namely $D(E, x) = 1$ for almost all $x \in E$. We achieve this by showing the contrapositive, i.e. if E is irregular, then E has no tangent almost everywhere.

The main idea is to show that the sum of the upper angular densities of an irregular 1-set along two opposite directions cannot be too small, thereby forcing the limit in equation (2) to be positive. To implement this idea, we first need two geometric results:

Lemma 4.20. *Let $\boldsymbol{\theta}$ be a unit vector in $\mathbb{R}^2$ perpendicular to a line L. Let P be a parallelogram with sides making angles φ to directions $\pm\boldsymbol{\theta}$ and let y and z be opposite vertices of P, as in the following figure. Then $|y - z| \leq d/\sin\varphi$, where d is the length of projection of P onto L.*



Proof. Let w be a third vertex of P. Then by the triangle inequality, $|y - z| \leq |y - w| + |w - z|$. Together with the relation that $d = (|y - w| + |z - w|)\sin\varphi$, we get $|y - z| \leq d/\sin\varphi$. □

A set with the property in the next lemma is sometimes called a **Lipschitz set**.

Lemma 4.21. *Let E be a bounded subset of $\mathbb{R}^2$ such that if $x, y \in E$ the segment $[x, y]$ makes an angle of, at most, $\phi < \frac{1}{2}\pi$ with a fixed line L. Then E is a subset of a rectifiable curve.*

Proof. Upon taking the closure if necessary, we may assume that E is closed and thereby compact. Let $\Pi(t)$ be the line perpendicular to L at distance t from some origin, and let a and b be the extreme values of t for which $\Pi(t)$ intersects E. It follows from the compactness of E that such a and b exist.

Then $\Pi(t)$ can contain at most one point $y \in E$ for each t, for otherwise the two points in $\Pi(t)$ for a given t give a line segment that makes an angle of $\pi/2$ with L. Let $\psi(t)$ denote the point y if it exists, otherwise if $a < t < b$, let $\psi(t)$ be the point of $\Pi(t)$ on the line segment joining the points of E nearest to $\Pi(t)$ on either side. By construction, we see that $\psi(t) : [a, b] \to \mathbb{R}^n$ is a curve joining points of E. Furthermore, since the segment $[\psi(t_1), \psi(t_2)]$ makes an angle of at most ϕ with L for any $t_1 \neq t_2$, we have $|\psi(t_1) - \psi(t_2)| \leq |t_1 - t_2|/\cos\phi$, which implies $\mathscr{L}(\psi([a, b])) \leq L/\cos\phi < \infty$. This shows that $\psi([a, b])$ is a rectifiable curve containing E, as desired. $\square$

Now we are ready to show the main ingredient of this direction of the proof. The idea is to reduce an irregular set of points to a subset of a rectifiable curve which necessarily has measure zero.

Theorem 4.22. *Let E be an irregular 1-set in $\mathbb{R}^2$. Then, given $\boldsymbol{\theta}$ and $0 < \varphi < \pi/2$,*

$$\overline{D}^1(E, x, \boldsymbol{\theta}, \varphi) + \overline{D}^1(E, x, -\boldsymbol{\theta}, \varphi) \geq \frac{1}{6}\sin\varphi$$

for almost all $x \in E$.

Proof. Take $\rho, \delta_+, \delta_- > 0$, and let $F_0 = F_0(\delta_+, \delta_-, \rho)$ be the set of x in E such that both

$$\mathcal{H}^1(E \cap S_r(x, \boldsymbol{\theta}, \varphi)) \leq 2r\delta_+ \tag{9}$$

and

$$\mathcal{H}^1(E \cap S_r(x, -\boldsymbol{\theta}, \varphi)) \leq 2r\delta_- \tag{10}$$

for all $r \leq \rho$. It is clear that F_0 is measurable; we show that if F_0 has positive measure, then $\delta_+ + \delta_-$ cannot be too small, which gives a contradiction since δ_+ and δ_- can be chosen to be arbitrarily small.

If $\mathcal{H}^1(F_0) > 0$, by the regularity of $\mathcal{H}^1$, we may find a closed subset $F_1 \subset F_0$ with positive measure. We may also take F_1 to be bounded, and thus F_1 is compact. Furthermore, given any $\eta > 0$, by the result on upper convex density (Definition 3.6) we may find a closed convex set U with $0 < |U| \leq \rho$ such that

$$\mathcal{H}^1(F_1 \cap U) > (1 - \eta)|U| \tag{11}$$

Let $F = F_1 \cap U$, and we work inside F from now on. As F is compact, we may choose y_1 and z_1 to be the most distant pair of points in F which have their connecting segment at an angle of not more than φ with $\boldsymbol{\theta}$, i.e.

$$r_1 = |y_1 - z_1| = \sup\{|y-z| : z \in F \cap S_\infty(y, \boldsymbol{\theta}, \varphi),\ y \in F \cap S_\infty(z, -\boldsymbol{\theta}, \varphi)\}$$

See figure on the next page; then the maximality of r_1 implies that

$$F \cap S_\infty(y_1, \boldsymbol{\theta}, \varphi) = F \cap S_{r_1}(y_1, \boldsymbol{\theta}, \varphi)$$

and

$$F \cap S_\infty(z_1, -\boldsymbol{\theta}, \varphi) = F \cap S_{r_1}(z_1, -\boldsymbol{\theta}, \varphi)$$

Since $|U| \le \rho$, we have that $r_1 \le \rho$. Then from (9) and (10) we may conclude that

$$\begin{aligned} \mathcal{H}^1(F \cap S_\infty(y_1, \boldsymbol{\theta}, \varphi)) &= \mathcal{H}^1(F \cap S_{r_1}(y_1, \boldsymbol{\theta}, \varphi)) \\ &\le \mathcal{H}^1(E \cap S_{r_1}(y_1, \boldsymbol{\theta}, \varphi)) \le 2r_1\delta_+ \end{aligned} \tag{11a}$$

Similarly,

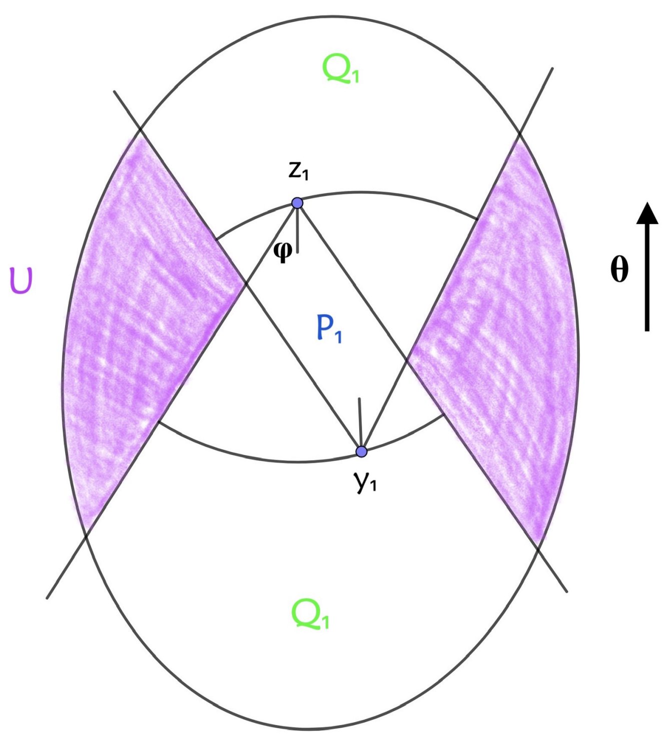
$$\mathcal{H}^1(F \cap S_\infty(z_1, -\boldsymbol{\theta}, \varphi)) \le 2r_1\delta_- \tag{12}$$

Let P_1 be the closed parallelogram

$$P_1 = S_\infty(y_1, \boldsymbol{\theta}, \varphi) \cap S_\infty(z_1, -\boldsymbol{\theta}, \varphi)$$

and let Q_1 be the open region

$$Q_1 = \operatorname{int}\{S_\infty(y_1, \boldsymbol{\theta}, \varphi) \cup S_\infty(z_1, -\boldsymbol{\theta}, \varphi)\}$$



From inequality (11a) and (12), we have

$$\mathcal{H}^1(F \cap Q_1) \leq 2r_1(\delta_+ + \delta_-) \tag{$\star$}$$

Since Q_1^c is the union of two disjoint convex closed subsets of $\mathbb{R}^2$, we know that $U \setminus Q_1$ is compact and has at most two components. If $U \setminus Q_1$ contains a pair of points of F in the same component with joining segment at angle of at most φ to $\boldsymbol{\theta}$, we may by exactly the same construction get a parallelogram P_2 disjoint from Q_1 and an open region Q_2. Formally, for $j \in \mathbb{N}$ we may find points y_j and z_j of F lying in the same component of $U \setminus \bigcup_{i=1}^{j-1} Q_i$ with $r_j = |y_j - z_j|$ maximal, and with

$$\mathcal{H}^1(F \cap Q_j) \leq 2r_j(\delta_+ + \delta_-), \tag{13}$$

where

$$Q_j = \text{int}\{S_\infty(y_j, \boldsymbol{\theta}, \varphi) \cup S_\infty(z_j, -\boldsymbol{\theta}, \varphi)\}$$

Continue this process indefinitely unless no suitable pair of points is left. Now let L be a line perpendicular to $\boldsymbol{\theta}$. We estimate the value of $\sum r_j$ following Besicovitch's original argument [Bes38].

If the sequence of parallelograms $\{P_j\}$ consists of only one term P_1, then $\sum r_j = r_1 \leq |U|$. If the sequence $\{P_j\}$ consists of more than one term, then we define a set H of segments as follows: Denote the left vertex of P_j (refer to the previous figure) as a_j and right vertex as b_j. If the points a_j and b_j are inside U, then the line segments $[a_j, z_j], [z_j, b_j]$ are included in H. If the point a_j is outside U and b_j inside, then the larger of the segments $[z_j, b_j], [b_j, y_j]$ is included in H. Notice there exists at most one such parallelogram because if it exists for some j, then $U \setminus \bigcup_{i=1}^{j} Q_j$ will only have one (right) component. Similarly, if the point b_j is outside and a_j inside, then the larger of $[a_j, z_j], [a_j, y_j]$ is included in H. It is clear that none of the P_j's can have two vertices outside U if $j \geq 2$. It follows from triangle inequality that

$$r_j = |z_j - y_j| \leq |a_j - z_j| + |b_j - z_j|$$

Let $k_j = \max\{|a_j - z_j|, |b_j - z_j|\}$, then it follows from elementary geometry that if $\varphi \geq \pi/3$, then $r_j \leq k_j$. If $\varphi < \pi/3$,

$$r_j \leq k_j + \left(1 - \frac{1}{2\cos\varphi}\right) r_j \leq k_j + \left(1 - \frac{1}{2\cos\varphi}\right) |U|$$

Now we may write

$$\sum r_j = \sum{}' r_j + \sum{}'' r_j$$

where $\sum'$ sums over those values corresponding to a_j and b_j both contained in U and $\sum''$ over those corresponding to one of a_j or b_j being outside U (there are at most two terms in $\sum''$). From the above estimates, we have

$$\sum{}' r_j \leq \sum{}' |a_j - z_j| + |z_j - b_j|$$

and

$$\sum{}'' r_j \leq \sum{}'' k_j \quad \text{if } \varphi \geq \pi/3,$$

$$\sum\nolimits'' r_j < \sum\nolimits'' k_j + (2 - \sec\varphi)|U| \quad \text{if } \varphi < \pi/3$$

Furthermore, by the construction of H we have

$$\mathcal{H}^1(H) = \sum\nolimits' (|a_j - z_j| + |z_j - b_j|) + \sum\nolimits'' k_j$$

Altogether we get

$$\mathcal{H}^1(H) \geq \sum\nolimits' r_j + \sum\nolimits'' k_j \geq \sum\nolimits' r_j + \sum\nolimits'' r_j = \sum r_j \quad \text{when } \varphi \geq \pi/3$$

$$\mathcal{H}^1(H) > \sum r_j - (2 - \sec\varphi)|U| \quad \text{when } \varphi < \pi/3$$

Since the projections of the parallelograms $\{P_j\}$ onto L are segments with disjoint interiors and H is entirely contained in U, by Lemma 4.20 we get $\mathcal{H}^1(H) \leq \frac{|U|}{\sin\varphi}$, which implies

$$\sum r_j \leq |U| \csc\varphi, \quad \text{when } \varphi \geq \pi/3$$

$$\sum r_j < (\csc\varphi - \sec\varphi + 2)\,|U|, \quad \text{when } \varphi < \pi/3$$

But since

$$\csc\varphi - \sec\varphi + 2 < 3\csc\varphi, \quad \text{when } 0 < \varphi < \pi/3$$

We get

$$\sum r_j = \sum |z_j - y_j| \leq 3|U| \csc\varphi \quad \text{for all } 0 < \varphi < \pi/2 \tag{14}$$

Now let y and z be distinct points of $F \setminus \bigcup_{j=1}^{\infty} Q_j$. If y and z lie on opposite sides of some Q_j, then the line segment $[y, z]$ makes an angle of at least φ with $\boldsymbol{\theta}$. If y and z lie on the same side and if $[y, z]$ makes an angle of less than φ with $\boldsymbol{\theta}$, then y and z would have been selected as y_j and z_j for some j since $|y_j - z_j| \to 0$ by (14). Hence, in either case, we know that the line segment joining any pair of points of $F \setminus \bigcup_{j=1}^{\infty} Q_j$ makes an angle with L of at most $\pi/2 - \varphi$. By Lemma 4.21, $F \setminus \bigcup_{j=1}^{\infty} Q_j$ is contained in a rectifiable curve. As E is irregular, E intersects any rectifiable curve in a set of measure zero, for otherwise a subset of E of positive measure would have been regular by Corollary 4.8. Thus, $\mathcal{H}^1\left(F \setminus \bigcup Q_j\right) = 0$ Together with (11), (13) and (14) we get

$$\begin{aligned}
\mathcal{H}^1(F) &= \mathcal{H}^1\left(F \cap (\cup_{j=1}^{\infty} Q_j)\right) + \mathcal{H}^1\left(F \setminus \cup_{j=1}^{\infty} Q_j\right) \\
&\leq \sum_{j=1}^{\infty} \mathcal{H}^1(F \cap Q_j) \leq 2\sum_{j=1}^{\infty} r_j(\delta_+ + \delta_-) \quad (\text{by } (\star)) \\
&\leq \frac{6|U|(\delta_+ + \delta_-)}{\sin\varphi} \leq \frac{6(\delta_+ + \delta_-)}{(1-\eta)\sin\varphi}\mathcal{H}^1(F).
\end{aligned}$$

Given any $\eta > 0$, there exists some F of positive measure that this holds, thus $\delta_+ + \delta_- \geq \frac{1}{6}\sin\varphi$. Now if we pick δ_+ and δ_- so that $\delta_+ + \delta_- < \frac{1}{6}\sin\varphi$, then $\mathcal{H}^1(F) = 0$ for all such F, which implies F_0 has measure zero. Namely, the set of x in E such that

$$\mathcal{H}^1(E \cap S_r(x, \boldsymbol{\theta}, \varphi)) + \mathcal{H}^1(E \cap S_r(x, -\boldsymbol{\theta}, \varphi)) \leq 2r \cdot \frac{1}{6}\sin\varphi$$

has measure zero. Thus, upon dividing $2r$ on both sides and taking limit supremum, we get

$$\overline{D}^1(E, x, \boldsymbol{\theta}, \varphi) + \overline{D}^1(E, x, -\boldsymbol{\theta}, \varphi) \geq \frac{1}{6} \sin \varphi$$

for almost all $x \in E$, as desired. □

Corollary 4.23. *A plane irregular* 1*-set has no tangent almost everywhere.*

Proof. It follows from the Definition 4.3 that if an irregular 1-set $E \subset \mathbb{R}^2$ has a tangent at x in the direction of $\pm\boldsymbol{\theta}$, then for any $0 < \varphi < \pi/2$ both

$$D^1\left(E, x, \boldsymbol{\theta} + \frac{\pi}{2}, \frac{\pi}{2} - \varphi\right) = 0$$

and

$$D^1\left(E, x, \boldsymbol{\theta} - \frac{\pi}{2}, \frac{\pi}{2} - \varphi\right) = 0$$

hold. However, this is impossible by Theorem 4.22, and the proof is complete. □

This completes the proof of the grand Theorem 4.5. It worth mentioning that the generalization into $s > 1$ and $n > 2$ where $s, n \in \mathbb{N}$ was not completed until 47 years after Besicovitch grounding work, using generalizations such as m-rectifiable sets and approximate tangent planes as well as many new ideas. Interested readers may consult papers by Mattila [Mat75]. Furthermore, Theorem 4.5 can be extended to more general measures, in which case we no longer require the density to be exactly 1 but only require it to exist, positive and finite, which yields the desired results of rectifiability and tangency.

As we have discussed the case when the dimension s of a set $E \subset \mathbb{R}^n$ is an integer, the natural question to ask is what happens when s is not an integer, i.e. E is fractal-like, and how does its nonintegral dimensionality affect rectifiability and tangency properties? The answer is surprisingly elegant.

5. Sets of Nonintegral Dimension

In this section, we examine how the nonintegral Hausdorff dimension of a set affects its density and tangency property. There will be no analogue of Theorem 4.5 as we expect these sets behave very differently from their integral counterparts. Thus, we discuss their density and tangency properties separately. The fundamental result in terms of density is the following result of Marstrand [Mar64]:

Theorem 5.1. *An s-set in $\mathbb{R}^n$ is irregular unless $s \leq n$ is an integer.*

Since the higher dimensional case when $s > 2$ are as always much more difficult to deal with, We will only look at two relatively simple cases, namely when $s \in (0, 1)$ and when $s \in (1, 2)$. In fact, we will first look at a stronger result that density does not exist almost everywhere for s-sets when $s \in (0, 1)$, and the work is entirely due to Marstrand [Mar54]. Interestingly, the following problem (now a theorem) was given to Marstrand by Besicovitch.

Theorem 5.2. *If E is an s-set with $0 < s < 1$ the density fails to exist at almost every point of E. In particular, E is irregular.*

Proof. Suppose for the sake of contradiction that E has a measurable subset of positive measure where the density exists and by Theorem 3.2 is at least $2^{-s} > 1/2$. Now by the Borel regularity of $\mathcal{H}^s$, we may find $r_1 > 0$ and a closed subset $F \subset E$ of positive measure such that if $x \in F$, then $D(E, x)$ exists and

$$\mathcal{H}^s(E \cap B_r(x)) > \frac{1}{2}(2r)^s > \frac{1}{2}r^s \quad \text{for all } \rho \leq r_1 \tag{15}$$

Let y be a point of F such that $D(E \setminus F, y) = 0$ (by Theorem 3.3 almost all points in F suffice). Let $D(E, y) = D(F, y) = c2^{-s}$ where $0 < c < \infty$. Given any $\varepsilon > 0$, we may find $r_2 < r_1$ such that

$$(c - \varepsilon)\rho^s < \mathcal{H}^s(F \cap B_\rho(y)) \leq \mathcal{H}^s(E \cap B_\rho(y)) < (c + \varepsilon)\rho^s \tag{16}$$

for all $\rho < 2r_2$. Now let α be a number with $0 < \alpha < 1$. From the above inequality we get

$$\mathcal{H}^s(E \cap B_{r_2}(y)) > (c - \varepsilon)r_2^s, \quad \mathcal{H}^s(E \cap B_{(1+\alpha)r_2}(y)) < (c + \varepsilon)(1 + \alpha)^s r_2^s$$

Thus, if we let $A((1 + \alpha)r_2, r_2)$ denote the annular region $B_{(1+\alpha)r_2}(y) \setminus B_{r_2}(y)$, then

$$\mathcal{H}^s(E \cap A((1 + \alpha)r_2, r_2)) < (c + \varepsilon)(1 + \alpha)^s r_2^s - (c - \varepsilon)r_2^s \tag{17}$$

By a similar argument applied to $\mathcal{H}^s(F \cap B_\rho(y))$ in (16), assuming all annular regions A throughout this proof are centered at y, we get that

$$\begin{aligned}\mathcal{H}^s(F \cap A((1 + 2\alpha/3)r_2,&(1 + \alpha/3)r_2)) \\ &> (c - \varepsilon)(1 + 2\alpha/3)^s r_2^s - (c + \varepsilon)(1 + \alpha/3)^s r_2^s\end{aligned}$$

Note the right hand side is positive when ε is sufficiently small. This implies there exists a point z in $A((1+2\alpha/3)r_2, (1+\alpha/3)r_2)$, and thus the ball $B_{\alpha r_2/3}(z)$ is contained in the annular region $A((1 + \alpha)r_2, r_2)$. By (15) we get

$$\mathcal{H}^s(E \cap A((1 + \alpha)r_2, r_2)) > \mathcal{H}^s(E \cap B_{\alpha r_2/3}(z)) > \frac{1}{2 \cdot 3^s}(\alpha r_2)^s > \frac{1}{6}(\alpha r_2)^s$$

Together with (17), we get

$$(c + \varepsilon)(1 + \alpha)^s - (c - \varepsilon) > \alpha^s/6$$

Since the hold inequality holds for all $\varepsilon > 0$, we have

$$c(1 + \alpha)^s - c \geq \alpha^s/6 > 0$$

However, the Taylor expansion of $(1 + \alpha)^s$ gives $(1 + \alpha)^s = 1 + O(\alpha)$, which gives a contradiction as $\alpha \to 0$, and the proof is complete. □

Before we look into the density result for s-sets when $1 < s < 2$, we divert our attention temporarily to tangency properties, as we will see that they are helpful in establishing results on density.

The question of tangents of an s-set when $0 < s < 1$ is not of particular interest, as the set is too sparse to be well defined. When $1 < s < 2$, we may apply the ideas of the proof of Theorem 4.22: Recall that Besicovitch proof depends on the fact that the s-set E in consideration intersects with rectifiable curves in a set of measure zero. But this is clearly true when $s > 1$ because for any rectifiable curve $\Gamma \subset \mathbb{R}^n$, Theorem 2.8 implies that $\mathcal{H}^s(\Gamma) = 0$. This allows us to reproduce the result that the sum of angular s-densities of opposite directions is strictly positive for any $0 < \varphi < \pi/2, \boldsymbol{\theta}$ and almost all $x \in E$, which implies for such sets no tangent exists almost everywhere.

Instead of going into the details of proof outlined above, we show the following stronger result, which will also help us answer questions on the density of such sets:

Theorem 5.3. *If E is an s-set in $\mathbb{R}^2$ with $1 < s < 2$, then at almost all points of E no weak tangent exists.*

We define a **weak tangent** of E in the direction $\boldsymbol{\theta}$ at x if $\underline{D}^s(E, x) > 0$ and for every $\phi > 0$,

$$\liminf_{r\to 0} \frac{\mathcal{H}^s(E \cap (B_r(x) \setminus (S_r(x, \boldsymbol{\theta}, \phi) \cup S_r(x, -\boldsymbol{\theta}, \phi))))}{r^s} = 0$$

which is equivalent to say that for every $0 < \phi < \pi/2$ and $\varepsilon > 0$, there exist arbitrarily small values of r such that

$$\mathcal{H}^s(E \cap S_r(x, \boldsymbol{\theta} + \pi/2, \pi/2 - \phi)) + \mathcal{H}^s(E \cap S_r(x, -\boldsymbol{\theta} + \pi/2, \pi/2 - \phi)) < \varepsilon r^s$$

Note that if the above inequality holds for all sufficiently small r, then the weak tangent becomes a tangent. In general, a set can have more than one weak tangents at a point.

To prove Theorem 5.3, we use the following lemma which says the upper angular density of an s-set when $1 < s < 2$ is always bounded above, which implies cones are not enough to approximate these sets in any direction.

Lemma 5.4. *Let E be an s-set in $\mathbb{R}^2$ with $1 < s < 2$. Then for almost all $x \in E$,*

$$\overline{D}^s(E, x, \boldsymbol{\theta}, \phi) \leq 6 \cdot 7^s \phi^{s-1}$$

for all $\boldsymbol{\theta}$ and all $\phi \leq \frac{\pi}{2}$.

Proof. Fix $\rho > 0$ and let

$$F = \{x \in E : \mathcal{H}^s(E \cap B_r(x)) < 2^{s+1} r^s \text{ for all } r \leq \rho\} \tag{18}$$

By Theorem 3.3, we know $\mathcal{H}^s(E \setminus F) = 0$, so we work inside F. Choose $x \in F$ and any $\boldsymbol{\theta}$ and ϕ with $0 < \phi \leq \frac{\pi}{2}$. For each $i \in \mathbb{N}$, let A_i be the intersection of annulus and sector

$$A_i = S_{ir\phi}(x, \boldsymbol{\theta}, \phi) \setminus S_{(i-1)r\phi}(x, \boldsymbol{\theta}, \phi)$$

so that $S_r(x, \boldsymbol{\theta}, \phi) \subset \bigcup_{i=1}^m A_i$ where $m \leq \phi^{-1} + 1 \leq 3\phi^{-1}$. The diameter of each A_i is at most the distance between a pair of opposite vertices travelled along the boundary through the bigger arc, namely

$$|A_i| \leq \phi r + 2\phi(ir\phi) \leq \phi r + 2\phi r(m\phi) \leq 7\phi r.$$

Thus, $|A_i| \le \rho$ if $r < \rho/14$. For each i, if there exists $x_i \in F \cap A_i$, then A_i is contained in the ball $B_{7\phi r}(x_0)$. Hence, summing over all i's gives

$$\begin{aligned}\mathcal{H}^s(F \cap S_r(x, \boldsymbol{\theta}, \phi)) &\le \sum_{i=1}^{m} \mathcal{H}^s(A_i \cap F) \\ &\le m\mathcal{H}^s(B_{7\phi r}(x_i) \cap E) \le m2^{s+1}(7r\phi)^s \le 3\phi^{-1}2^{s+1}(7r\phi)^s\end{aligned}$$

where the third inequality follows from (18). This is equivalent to

$$(2r)^{-s}\mathcal{H}^s(F \cap S_r(x, \boldsymbol{\theta}, \phi)) \le 6 \cdot 7^s\phi^{s-1}$$

whenever $r < \rho/14$. Thus $\overline{D}^s(F, x, \boldsymbol{\theta}, \phi) \le 6 \cdot 7^s\phi^{s-1}$ at almost all $x \in F$. Since $\overline{D}^s(E \setminus F, x) = 0$ for almost all $x \in F$, it follows that $\overline{D}^s(E, x, \boldsymbol{\theta}, \phi) \le 6 \cdot 7^s\phi^{s-1}$ for all $x \in F$. Moreover, since F and E differ by a set of measure zero, we have that the result holds for almost all $x \in E$. □

Now Theorem 5.3 is an easy corollary of the above lemma, but let us fill in the details: It follows from Lemma 5.4 that

$$\begin{aligned}\liminf_{r\to 0} \; & \frac{\mathcal{H}^s(E \cap (B_r(x) \setminus (S_r(x, \boldsymbol{\theta}, \phi) \cup S_r(x, -\boldsymbol{\theta}, \phi))))}{r^s} \\ &\ge \underline{D}^s(E, x) - (\overline{D}^s(E, x, \boldsymbol{\theta}, \phi) + \overline{D}^s(E, x, -\boldsymbol{\theta}, \phi)) \\ &\ge \underline{D}^s(E, x) - 12 \cdot 7^s\phi^{s-1}\end{aligned}$$

holds for all $\boldsymbol{\theta}$, $0 < \phi < \pi/2$ and almost all $x \in E$. Now if $\underline{D}^s(E, x) = 0$, then by definition no weak tangent exists at x. If $\underline{D}^s(E, x) > 0$, as $\phi \to 0$, we have that the limit infimum is positive for all ϕ sufficiently small, which again implies that E does not have a weak tangent at x.

We are now ready to revisit the density property of s-sets when $1 < s < 2$. In particular, we want to show that such a set is irregular. We accomplish this result by a contradiction argument. We will show that any such s-set E in $\mathbb{R}^2$ has a weak tangent at each of its regular points, which clearly contradicts Theorem 5.3. Naturally, we wish to look for results in terms of angular density, and the following theorem comes into play.

Theorem 5.5. *Let E be an s-set in $\mathbb{R}^2$ with $1 < s < 2$. Then if $\phi < \frac{1}{2}\pi$ the lower angular density $\underline{D}^s(E, x, \boldsymbol{\theta}, \phi) = 0$ for some $\boldsymbol{\theta}$ for almost all $x \in E$.*

Proof. Fix $\alpha, \rho > 0$ and define

$$F_0 = \{x : \mathcal{H}^s(E \cap S_r(x, \boldsymbol{\theta}, \phi)) > \alpha r^s \text{ for all } r \le \rho \text{ and for all } \boldsymbol{\theta}\}$$

Now it suffices to show that $\mathcal{H}^s(F_0) = 0$. Suppose for the sake of contradiction that $\mathcal{H}^s(F_0) > 0$, then we may find $\rho_1 \le \rho$ and a closed subset F of positive measure such that if $x \in F$ and $r < \rho_1$, then

$$\mathcal{H}^s(E \cap B_r(x)) < 2^{s+1}r^s$$

by Theorem 3.2. Moreover, let y be a point of F such that $D(E \setminus F, y) = 0$. Hence, given $\varepsilon > 0$, we may find $\rho_2 < \rho_1$ such that

$$\mathcal{H}^s((E \setminus F), y) < \varepsilon(2r)^s \tag{19}$$

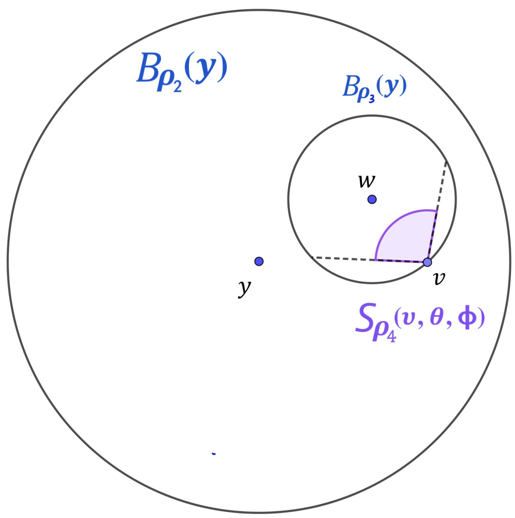
for all $r \leq \rho_2$. Now we work inside the disk $B_{\rho_2}(y)$. We claim that there are points in $B_{\rho_2/2}(y)$ distant from the set F. Indeed, suppose that there exists some $\gamma \leq \rho_2/2$ such that all points of $B_{\rho_2/2}(y)$ are within γ distance from F. Then if $x \in B_{\rho_2/2}(y)$, there is a point $z \in F$ such that z is contained in $B_\gamma(x)$. Moreover, since $z \in F \subset F_0$,

$$\alpha\gamma^s < \mathcal{H}^s(E \cap S_\gamma(z, \boldsymbol{\theta}, \phi)) \leq \mathcal{H}^s(E \cap B_\gamma(z)) \leq \mathcal{H}^s(E \cap B_{2\gamma}(x)) \tag{20}$$

for all $\boldsymbol{\theta}$, and the last inequality follows from triangle inequality. Now if $\gamma < \rho_2/4$, then $B_{\rho_2}(y)$ contains at most $(\rho_2/\gamma)^2/4$ disjoint discs with centers in $B_{\rho_2/2}(y)$ and radii 2γ. Hence, summing over all these discs with (20) gives

$$(\rho_2/\gamma)^2 \alpha\gamma^s/4 < \mathcal{H}^s(E \cap B_{\rho_2}(y)) < 2^{s+1}\rho_2^s$$

which implies $\gamma > c\rho_2$ where c only depends on α and s. Since γ can be arbitrarily small, if we choose $\gamma \leq c\rho_2$, it follows that there is a disc of radius γ contained in $B_{\rho_2}(y)$ and containing no points of F.



Hence, we may find a disc $B_{\rho_3}(w) \subset B_{\rho_2}(y)$ (see figure above) with no points of F in its interior but with a point v of F_0 on its boundary, and ρ_3 satisfies $\rho_2 \geq \rho_3 \geq c\rho_2$. Let $\boldsymbol{\theta}$ be the normal direction pointing inward to $B_{\rho_3}(w)$ at v, and let ρ_4 be half the length of the chords of $B_{\rho_3}(w)$ through v that makes angles ϕ with $\boldsymbol{\theta}$. It follows from elementary geometry that $\rho_4 = \rho_3 \cos\phi$.

Since the sector $S_{\rho_4}(v, \boldsymbol{\theta}, \phi) \subset B_{\rho_3}(w)$ contains possibly no points of F other than v, then it follows

$$\begin{aligned}\mathcal{H}^s(E \cap S_{\rho_4}(v, \boldsymbol{\theta}, \phi)) &= \mathcal{H}^s((E \setminus F) \cap S_{\rho_4}(v, \boldsymbol{\theta}, \phi)) \\ &\leq \mathcal{H}^s((E \setminus F) \cap B_{\rho_2}(y)) < \varepsilon(2\rho_2)^s\end{aligned}$$

where the second to last inequality follows from (19). However,

$$\varepsilon\rho_2^s \leq \varepsilon(\rho_3/c)^s \leq \varepsilon\left(\frac{\rho_4}{c\cos\phi}\right)^s = \varepsilon c_1\rho_4^s$$

where c_1 is a positive constant that only depends on ϕ, c and s. Hence it follows that for any $\varepsilon > 0$, there is $v \in F_0$ and $\rho_4 < \rho$ and $\boldsymbol{\theta}$ for which

$$\mathcal{H}^s(E \cap S_{\rho_4}(v, \boldsymbol{\theta}, \phi)) < \varepsilon c_1\rho_4^s$$

holds, which clearly contradicts the assumption that $\mathcal{H}^s(F_0) > 0$. Moreover, since α and ρ in the definition of F_0 are arbitrary, the proof is complete. □

It is only a matter of technical details to extend ϕ to the case when $\phi = \pi/2$. Specifically, we may take a sequence of $\{\phi_i\}$ increasing to $\pi/2$. By the above theorem, there exists a corresponding sequence $\{\boldsymbol{\theta}_i\}$ such that $\underline{D}^s(E, x, \boldsymbol{\theta}_i, \phi_i) = 0$ for all i. Since $\{\boldsymbol{\theta}_i\}$ is bounded, by passing to a subsequence, we may assume that $\boldsymbol{\theta}_i \to \boldsymbol{\theta}$.

It follows from the definition of angular density that if $\underline{D}^s(E, x, \boldsymbol{\theta}, \phi) = 0$, then $\underline{D}^s(E, x, \boldsymbol{\theta}, \phi') = 0$ for all $\phi' < \phi$. Hence, we may conclude $\underline{D}^s(E, x, \boldsymbol{\theta}, \phi) = 0$ for all $\phi < \pi/2$. Since for each $\varepsilon > 0$, the circular sector $S_r(x, \boldsymbol{\theta}, \pi/2)$ can be written as

$$S_r(x, \boldsymbol{\theta}, \pi/2) = S_r(x, \boldsymbol{\theta}, \pi/2 - \varepsilon) \cup S_r(x, \boldsymbol{\theta}', \varepsilon/2) \cup S_r(x, \boldsymbol{\theta}'', \varepsilon/2)$$

for some $\boldsymbol{\theta}'$ and $\boldsymbol{\theta}''$. Thus, by the definition of lower angular density, we get from Lemma 5.4 that

$$\begin{aligned}
\underline{D}^s(E, x, \boldsymbol{\theta}, \pi/2) &\leq \underline{D}^s(E, x, \boldsymbol{\theta}, \pi/2 - \varepsilon) + \overline{D}^s(E, x, \boldsymbol{\theta}', \varepsilon/2) + \overline{D}^s(E, x, \boldsymbol{\theta}'', \varepsilon/2) \\
&= \overline{D}^s(E, x, \boldsymbol{\theta}', \varepsilon/2) + \overline{D}^s(E, x, \boldsymbol{\theta}'', \varepsilon/2) \\
&\leq 12 \cdot 7^s(\varepsilon/2)^{s-1}
\end{aligned}$$

for almost all $x \in E$. Letting $\varepsilon \to 0$ shows $\overline{D}^s(E, x, \boldsymbol{\theta}, \pi/2) = 0$.

Now we are ready to prove the following crucial lemma that leads to our desired conclusion on the density property. The lemma says at each regular point of an s-set $(1 < s < 2)$ a weak tangent exists, and proof is an ingenious argument of Marstrand which shows that, informally speaking, if such a set is sparse to one side of a line through one of its points, then it must also be sparse on the other side, allowing a suitably defined linear approximation to exist.

Lemma 5.6. *Let E be an s-set in $\mathbb{R}^2$ with $1 < s < 2$. Let x be a regular point of E at which the upper convex density equals 1, and suppose that $\underline{D}^s(E, x, -\boldsymbol{\theta}, \pi/2) = 0$ for some $\boldsymbol{\theta}$. Then E has a weak tangent at x perpendicular to $\boldsymbol{\theta}$.*

Proof. Since $D^s(E, x) = \overline{D}_c^s(E, x) = 1$ and $\underline{D}^s(E, x, -\boldsymbol{\theta}, \pi/2) = 0$, by definitions given any $\eta > 0$, there exists a $\rho > 0$ such that

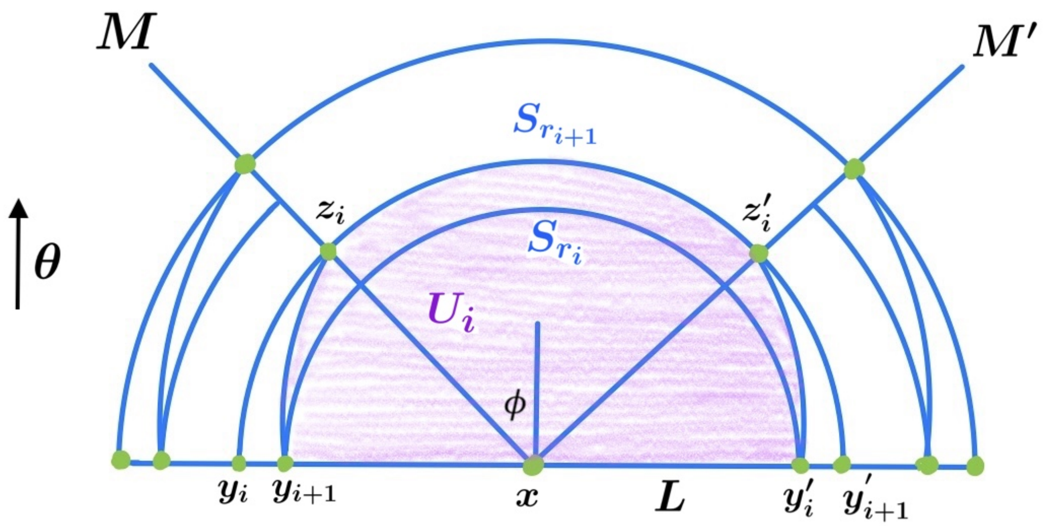
$$\mathcal{H}^s(E \cap B_r(x)) > (1 - \eta)(2r)^s \quad \text{for all } r \leq \rho, \tag{21}$$

$$\mathcal{H}^s(E \cap U) < (1 + \eta)|U|^s \quad \text{if } x \in U \text{ and } 0 < |U| \leq 2\rho \tag{22}$$

and

$$\mathcal{H}^s(E \cap S_\rho(x, -\boldsymbol{\theta}, \pi/2)) < \eta(2\rho)^s. \tag{23}$$

Now for any $0 < \phi < \pi/2$, let L be the line through x and perpendicular to $\boldsymbol{\theta}$, and let M and M' be the two rays of the cone $S_\infty(x, \boldsymbol{\theta}, \phi)$ (See the figure from Falconer [Fal85] below).



For a fixed positive integer m, we inductively construct a sequence of $m+1$ semicircles $\{S_{r_i}\}_{i=1}^{m+1}$ of strictly decreasing radii r_i, all of which are centered at x and based on L. Specifically, for each i the semicircle S_{r_i} is perpendicular to L at y_i and y_i' and meets M and M' at z_i and z_i' respectively.

Suppose S_{r_i} has been constructed for all $1 \le i \le m$, then $S_{r_{i+1}}$ is obtained by taking y_{i+1} to be the point on the line segment $[x, y_i]$ such that

$$|y_{i+1} - y_i'| = |y_{i+1} - y_i| + |y_{i+1} - z_i'|.$$

As y_{i+1} ranges from x to y_i, the left hand side above increases continuously from r_i to $2r_i$, and the right hand side decreases from $2r_i$ to a value strictly less than $2r_i$, we know from Intermediate Value Theorem that such a y_{i+1} exists.

Let y_{i+1}' denote the intersection point between L and the arc centered at y_{i+1} through z_i'. By symmetry, y_{i+1} is the intersection point between L and the arc centered at y_{i+1}' through z_i. Denote the shaded convex region as U_i, then we know that by construction $|U_i| = 2r_{i+1}$. Now we estimate the measure of E contained in the intersection of annulus $A(x, r_i, r_{i+1})$ and sector $S_\infty(x, \boldsymbol{\theta}, \phi)$; that is

$$\begin{aligned} S_{r_i}(x, \boldsymbol{\theta}, \phi) \setminus S_{r_{i+1}}(x, \boldsymbol{\theta}, \phi) &\subset U_i \setminus S_{r_{i+1}}(x, \boldsymbol{\theta}, \phi) \\ &\subset U_i \cup S_\rho(x, -\boldsymbol{\theta}, \pi/2) \setminus B_{r_{i+1}}(x), \end{aligned}$$

Hence,

$$\begin{aligned} &\mathcal{H}^s(E \cap S_{r_i}(x, \boldsymbol{\theta}, \phi)) - \mathcal{H}^s(E \cap S_{r_{i+1}}(x, \boldsymbol{\theta}, \phi)) \\ &\quad\le \mathcal{H}^s(E \cap U_i) + \mathcal{H}^s(E \cap S_\rho(x, -\boldsymbol{\theta}, \pi/2)) - \mathcal{H}^s(E \cap B_{r_{i+1}}(x)) \\ &\quad\le (1+\eta)|U_i|^s + \eta(2\rho)^s - (1-\eta)(2r_{i+1})^s \\ &\quad= 2^{s+1}\eta r_{i+1}^s + 2^s \eta \rho^s \\ &\quad\le 2^{s+2}\eta\rho^s \end{aligned}$$

where the second inequality follows from (21), (22) and (23). Now summing over the m annular sectors gives

$$\begin{aligned} \mathcal{H}^s(E \cap S_\rho(x, \boldsymbol{\theta}, \phi)) &\le m(2^{s+2}\eta\rho^s) + \mathcal{H}^s(E \cap S_{r_m}(x, \boldsymbol{\theta}, \phi)) \\ &\le m(2^{s+2}\eta\rho^s) + (1+\eta)(2r_m)^s \end{aligned}$$

where the last inequality follows from applying (22) to the convex region $S_{r_m}(x, \boldsymbol{\theta}, \phi)$. This shows

$$\frac{\mathcal{H}^s(E \cap S_\rho(x, \boldsymbol{\theta}, \phi))}{(2\rho)^s} \le 4\eta m + (1+\eta)\left(\frac{r_m}{\rho}\right)^s$$

For any fixed $\eta > 0$, we may choose m large enough, independent of η, so that (r_m/ρ) the second term on the right hand side above is small, and then choose η sufficiently small so that the first term is small. Thus, $\underline{D}^s(E, x, \boldsymbol{\theta}, \phi) = 0$ for all $0 < \phi < \pi/2$. Together with the assumption that $\underline{D}^s(E, x, -\boldsymbol{\theta}, \pi/2) = 0$, we have

$$\underline{D}^s(E, x, \boldsymbol{\theta}, \phi) + \underline{D}^s(E, x, -\boldsymbol{\theta}, \phi) = 0$$

for all $0 < \phi < \pi/2$, which implies the existence of a weak tangent at x in the direction perpendicular to $\boldsymbol{\theta}$. □

Corollary 5.7. *Let E be an s-set in $\mathbb{R}^2$ with $1 < s < 2$. Then E is irregular.*

Proof. Suppose for the sake of contradiction that there exists a regular subset $F \subset E$ of positive measure, then by the remark at the end of the Definition 3.6 and Theorem 5.5, we know that for any $x \in F$, $\overline{D}_c^s(E, x) = 1$ and $\underline{D}^s(E, x, \boldsymbol{\theta}, \phi) = 0$ for some $\boldsymbol{\theta}$ and for all $0 < \phi < \pi/2$. Moreover, since $D^s(E, x) = D^s(F, x) = 1$, it follows from Lemma 5.6 that E has a weak tangent at x. However, this is impossible according to Theorem 5.3, and the proof is complete. □

Remark. Marstrand's work went much deeper. He first showed in [Mar55] that the density of s-set fails to exist at almost all of its points when $1 < s < 2$. The generalization to arbitrary $n \in \mathbb{N}$ and s was eventually proved by Marstrand in 1964 [Mar64].

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