Formula for $\zeta(2n)$

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Since Euler computed the famous infinite sum (i.e. the Bessel Problem) in 1737,

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

numerous similar results have been established. For example, math enthusiasts might well remember

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

These sums are now known as special values of the Riemann zeta function, which is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

An easy check shows that $\zeta(s)$ converges when $\mathfrak{Re}(s) > 1$. In this write-up, we presented a formula for even-integer values of the Riemann zeta function, which generalizes all the above results. The proof of the formula assumes several classical results from complex analysis whose proofs will not be given here.

Theorem.

$$\zeta(2m) = \frac{1}{2}(-1)^{m+1} \frac{B_{2m}}{(2m)!}(2\pi)^{2m}$$

where B_n's are the Bernoulli numbers defined as

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n. \tag{1}$$

Proof. Recall a well-known result from complex analysis (e.g. page 174, Marshall):

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right) = \frac{1}{z} + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) = \frac{1}{z} + 2\sum_{n=1}^{\infty} \frac{z}{z^2 - n^2}$$

Notice from above we have

$$(\pi z) \cot \pi z = 1 + 2\sum_{n=1}^{\infty} \frac{z^2}{z^2 - n^2} = 1 - 2\sum_{n=1}^{\infty} \frac{(z/n)^2}{1-(z/n)^2} = 1 - 2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \left(\frac{z}{n}\right)^{2m}$$

where the geometric series converges absolutely in $\{|z| < n\}$ and uniformly in any compact subsets for each n. Thus, the series above converges wherever the function $\pi z \cot \pi z$ is defined. Now we can interchange the order of summation and get

$$\pi z \cot \pi z = 1 - 2 \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} \frac{1}{n^{2m}} \right) z^{2m} = 1 - 2 \sum_{m=1}^{\infty} \zeta(2m) z^{2m} \tag{2}$$

We now expand $\pi z \cot \pi z$ in terms of Bernoulli numbers:

$$\pi z \cot \pi z = \pi i z \left(\frac{e^{2\pi i z} + 1}{e^{2\pi i z} - 1} \right) = \pi i z \left(1 + \frac{2}{e^{2\pi i z} - 1} \right)$$

with definition (1), we have

$$\pi z \cot \pi z = \pi i z + \sum_{n=0}^{\infty} \frac{B_n}{n!} (2\pi i z)^n = 1 + \sum_{n=2}^{\infty} \frac{B_n}{n!} (2\pi i)^n z^n \tag{3}$$

where the last equality above follows from the fact that $B_0 = 1$ and $B_1 = -1/2$. Now with the fact that $B_n = 0$ for n odd and $n > 1$, we can compare the coefficients of equations (2) and (3), which gives

$$\zeta(2m) = \frac{1}{2} (-1)^{m+1} \frac{B_{2m}}{(2m)!} (2\pi)^{2m}$$

as desired. □

Remarks: This complex analysis based proof is vastly different from the computation that Euler gave by manipulating the infinite sum directly. It also worth mentioning that although all even-integer values of ζ are obtained, the odd integer value case still remains a mystery. Euler himself tried to compute $\zeta(3)$ but failed.