Weierstrass ℘ Function

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Motivation: We have the result from classical complex analysis that

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n \neq 0} \left(\frac{1}{z-n} + \frac{1}{n} \right)$$

which is a periodic meromorphic function that has a simple pole at every integer. Now we wish to extend the result to obtain a "nice" doubly periodic function over $\mathbb{C}$. That is for $w_1, w_2 \in \mathbb{C} \setminus \{0\}$ with $w_1/w_2 \notin \mathbb{R}$, we have $f(z+w_1) = f(z+w_1) = f(z)$ for all z.

Sadly, there is no non-constant entire function satisfying the above property, by Liouville's theorem. We take a step back to look for meromorphic function perhaps having poles at the lattice points $mw_1 + nw_2$ where $m, n \in \mathbb{Z}$. One naive try would be simply copying down what we have above, namely

$$G(z) = \frac{1}{z} + \sum_{(m,n) \neq 0} \left(\frac{1}{z - (mw_1 + nw_2)} + \frac{1}{mw_1 + nw_2} \right)$$

However, this doesn't quite work because the series on the right hand side does not converge. Moreover, a doubly periodic function has to be even, while the function $G(z)$ we came up with is odd. A cheap fix would be defining

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left(\frac{1}{(z - mw_1 - nw_2)^2} - \frac{1}{(mw_1 + nw_2)^2} \right) \tag{1}$$

This is called the Weierstrass $\wp$ function. Now we show that the function is indeed well defined, or equivalently the series on the right hand side converges.

Proof. For simplicity, we let $\zeta_{m,n} = mw_1 + nw_2$. We apply a standard trick: For $|z| < R$, we split the sum in (1) into a finite sum of terms with $|\zeta_{m,n}| \leq 2R$ and the sum of terms with $|\zeta_{m,n}| > 2R$. That is

$$\wp(z) = \frac{1}{z^2} + \sum_{|\zeta_{m,n}| \leq 2R} \left(\frac{1}{(z - \zeta_{m,n})^2} - \frac{1}{\zeta_{m,n}^2} \right) + \sum_{|\zeta_{m,n}| > 2R} \left(\frac{1}{(z - \zeta_{m,n})^2} - \frac{1}{\zeta_{m,n}^2} \right) \tag{2}$$

Now if $|z| \leq R$ and $|\zeta| > 2R$,

$$\left| \frac{1}{(z-\zeta)^2} - \frac{1}{\zeta^2} \right| = \left| \frac{2z\zeta - z^2}{\zeta^2 (z-\zeta)^2} \right| \leq \frac{R(2|\zeta| + R)}{|\zeta|^2 |\zeta/2|^2} < \frac{10R}{|\zeta|^3}$$

where the first inequality above follows from reverse triangle inequality that $|z-\zeta| \geq |\zeta| - |z| \geq |\zeta/2|$. Now by Weierstrass M-test it suffices to show that

$$\sum_{(m,n)\neq(0,0)} \frac{10R}{|\zeta|^3} < \infty \tag{3}$$

To estimate the doubly indexed sum, we first find a lower bound for $|\zeta_{m,n}|$. Indeed, if $m, n \neq 0$, then

$$|mw_1 + nw_2| = |w_2||m(w_1/w_2) + n| \geq |w_2||\mathfrak{Im}(w_1/w_2)|$$

If $m = 0$ and $n \neq 0$, then $|mw_1 + nw_2| \geq |w_2|$. Let $\delta > 0$ be the minimum of the above lower bounds. We have shown that no two points on the lattice formed by $\{\zeta_{m,n}\}$ is closer than δ. In another word, if we place a disk of radius $\delta/2$ centered at each point of the lattice, then the disks are disjoint.

Now we sum (3) over sequence of annulus $k \leq |\zeta| \leq k+1$ with common center: Clearly, the area of the annulus $k \leq |\zeta| \leq k+1$ is $(2k+1)\pi$, and the number of lattice points $\zeta_{m,n}$ in this annulus is at most $4(2k+1)/\delta^2$, which is increasing with respect to k at a linear rate. Thus,

$$\sum_{|\zeta_{m,n}|>2R} \left| \frac{1}{(z-\zeta_{m,n})^2} - \frac{1}{\zeta_{m,n}^2} \right| = \sum_{k=2R}^{\infty} \sum_{k<|\zeta_{m,n}|\leq k+1} \left| \frac{1}{(z-\zeta_{m,n})^2} - \frac{1}{\zeta_{m,n}^2} \right|$$
$$\leq \sum_{k=2R}^{\infty} Ck\frac{10R}{k^3} < \infty$$

By Weierstrass theorem on convergent sequence of analytic functions, we know that the second sum in (2) converges uniformly on $\{|z| \leq R\}$ to an analytic function, while the first sum is meromorphic with singular parts $S_{m,n} = 1/(z-\zeta_{m,n})^2$ provided $|\zeta_{m,n}| \leq 2R$. Since R is arbitrary, we have shown that the $\wp$ function is a well defined meromoprhic function that has double poles at all $\zeta_{m,n}$.

Next we show that $\wp$ is doubly periodic: We avoid cumbersome computation by looking at $\wp'$. Indeed, by Weierstrass again,

$$\wp'(z) = -\frac{2}{z^3} - \sum_{(m,n)\neq(0,0)} \frac{2}{(z-\zeta_{m,n})^3} = -\sum_{(m,n)\in\mathbb{Z}^2} \frac{2}{(z-\zeta_{m,n})^3}$$

By the same estimate above, this series converges uniformly and absolutely so that we can rearrange the terms, which implies $\wp'(z+w_1) = \wp'(z)$. Hence, $\wp(z+w_1) - \wp(z)$ is a constant. Moreover, by construction $\wp$ is even, so $\wp(z+w_1) = \wp(z)$ when $z = -w_1/2$. Thus, $\wp(z+w_1) = \wp(z)$ for all z. By the same argument, $\wp(z+w_2) = \wp(z)$.

Remark: This above proof is outlined in standard complex analysis textbooks such as Marshall and Gamelin. One more thing that I want to point out is that a doubly periodic meromorphic function in $\mathbb{C}$ is also called an elliptic function. The Weierstrass $\wp$ function is in some sense the simplest elliptic function we can write down because applying the residue theorem over an elliptic function cannot only have simple poles at its lattice points:

Consider integrating an elliptic function along the boundary of the **fundamental domain** $\mathbb{P} = \{tw_1 + sw_2 : t, s \in [0, 1)\}$. The double periodicity tells that the integral has to be 0, which implies the residue of any elliptic function within any copy of $\mathbb{P}$ is 0. This forces elliptic functions to have either non-simple poles at the lattice points or to have simple poles but the corresponding residues cancel.

The simplest example in the former case is the $\wp$ function, while the examples in the latter case give rise to a type of elliptic functions called Jacobian elliptic function.