Weyl's Lemma

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Definition 1. A continuous function f is **weakly-analytic** on a region Ω if

$$\int_\Omega f \frac{\partial \varphi}{\partial \overline{z}} dA = 0$$

for all compactly supported continuously differentiable functions φ on Ω. Here dA denotes the area measure.

Definition 2. A continuous function $f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ is **weakly-harmonic** on a region Ω if

$$\int_\Omega f \Delta\varphi \, dA = 0$$

for all compactly supported twice continuously differentiable function φ on Ω.

The proofs for the following two versions of Weyl's lemma are mainly outlined in Exercise 7.13, Complex Analysis by Marshall. We use the notation

$$f_z \equiv \frac{\partial f}{\partial z} \equiv \frac{1}{2}(f_x - i f_y), \quad f_{\overline{z}} \equiv \frac{\partial f}{\partial \overline{z}} \equiv \frac{1}{2}(f_x + i f_y)$$

for the first-order derivatives where f is holomorphic and $z = x + iy$. Note that Cauchy-Riemann equation can be written as $f_{\overline{z}} = 0$ under this notation.

Theorem 1 (analytic version). A continuous function is weakly-analytic if and only if it is analytic.

Proof. First we show an analytic function is weakly-analytic: If R is a rectangle and φ is continuously differentiable on $\overline{R}$ then by Green's theorem

$$\int_R \psi_{\overline{z}} dA = \frac{1}{2}\int_R (\psi_x + i\psi_y) dA = \frac{1}{2}\int_{\partial R} -i\psi dx + \psi dy = \frac{-i}{2}\int_{\partial R} \psi dz$$

Now applying the above result to $\psi = f\varphi$ on a union of squares $\Gamma = \bigcup R_\alpha \subset \Omega$ gives

$$\int_\Gamma (f\varphi)_{\overline{z}} \, dA = \int_\Gamma (f_{\overline{z}}\varphi + f\varphi_{\overline{z}}) dA = -\frac{i}{2}\int_{\partial\Gamma} f\varphi dz \tag{1}$$

Since f is holomorphic and φ is continuous on compacts and thereby bounded, the integral on the right hand side is zero by the Cauchy's theorem. Moreover, since f is analytic and thereby satisfying the Cauchy-Riemann equation, namely $f_{\overline{z}} = 0$, we have

$$\int_\Gamma f\varphi_{\overline{z}} \, dA = 0$$

Notice the above result holds for all compactly supported continuously differentiable φ and for all $\Gamma \subset \Omega$. Therefore, taking refinements of $\bigcup R_\alpha$ gives that f is weakly-analytic.

Now we show that a weakly-analytic function is analytic: Set $\varphi(z) = \frac{3}{\pi}(1 - |z|^2)^2$ on $|z| < 1$ and 0 elsewhere. In practice, any radially symmetric, compactly supported and continuously differentiable φ will work.

Indeed, $\varphi(z) = \frac{3}{\pi}(1-|z|^2)^2 = \frac{3}{\pi}(1-(x^2+y^2))^2$ is composition of continuously differentiable functions and thereby continuously differentiable. Moreover, we want to show that φ is a probability density function, namely

$$\int_{\mathbb{C}} \varphi dA = \int_{\mathbb{D}} \varphi \, dA = \frac{3}{\pi} \int_0^{2\pi} \int_0^1 r(1-r^2)^2 dr d\theta = \frac{3}{\pi} \int_0^{2\pi} \frac{1}{6} \, d\theta = 1$$

We perform a standard convolution argument by letting $\varphi_\delta(z) = \frac{1}{\delta^2}\varphi(\frac{z}{\delta})$ and

$$f_\delta(z) = \int_{\mathbb{C}} f(w)\varphi_\delta(z-w) \, dA(w)$$

Now suppose f is weakly-analytic on region Ω, then for rectangle $R \subset \overline{R} \subset \Omega$, from equation (1) we have

$$\int_{\partial R} f_\delta(z) dz = 2i \int_R (f_\delta(z))_{\overline{z}} \, dz = 2i \int_R \frac{\partial}{\partial \overline{z}} \left(\int_{\mathbb{C}} f(w)\varphi_\delta(z-w) dA(w) \right) dz$$

Since φ_δ is continuously differentiable and compactly supported, we can differentiate under the integral sign, which gives

$$\int_{\partial R} f_\delta(z) dz = 2i \int_R \left(\int_{\mathbb{C}} f(w) \frac{\partial \varphi_\delta}{\partial \overline{z}}(z-w) dA(w) \right) dz \tag{2}$$

But notice

$$\begin{aligned}
\frac{\partial}{\partial \overline{w}} \varphi_\delta(z-w) &= \frac{1}{2}\left(\frac{\partial}{\partial x_w}\varphi_\delta(z-w) + i\frac{\partial}{\partial y_w}\varphi_\delta(z-w) \right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x_z}\varphi_\delta(w-z) + i\frac{\partial}{\partial y_z}\varphi_\delta(w-z) \right) \\
&= -\frac{1}{2}\left(\frac{\partial}{\partial x_z}\varphi_\delta(z-w) + i\frac{\partial}{\partial y_z}\varphi_\delta(z-w) \right) \\
&= -\frac{\partial}{\partial \overline{z}}\varphi_\delta(z-w)
\end{aligned}$$

where the second equality holds by symmetry. Also with the assumption that f is weakly-analytic, we have from (2) that

$$\int_{\partial R} f_\delta(z)dz = -2i \int_R \left(\int_{\mathbb{C}} f(w) \frac{\partial \varphi_\delta}{\partial \overline{w}}(z-w)dA(w) \right) dz = 0$$

Therefore, since f_δ is by definition continuous, by Morera's theorem, f_δ is analytic.

Now we want to show the limiting function of f_δ is f and thereby showing f is analytic: Indeed, by direct estimate,

$$\begin{aligned}
|f_\delta(z) - f(z)| &= \left| \int_{\mathbb{C}} f(w)\varphi_\delta(z-w)dA(w) - f(z) \right| \\
&= \left| \int_{\mathbb{C}} (f(w) - f(z))\varphi_\delta(z-w)dA(w) \right| \\
&\leq \int_{\mathbb{C}} |f(w) - f(z)||\varphi_\delta(z-w)|dA(w)
\end{aligned}$$

Since f is continuous, we can always pick $|z-w| < \delta$ for δ small enough so that $|f(w) - f(z)| < \epsilon$ for any given $\epsilon > 0$. Moreover, since $\varphi_\delta(z-w)$ by definition is positive and compactly supported, we have for δ small enough,

$$|f_\delta(z) - f(z)| \leq \epsilon \int_\Omega \varphi_\delta(z-w)dA(w)$$

which shows that f_δ converges pointwise. Now since f_δ is uniformly continuous for each $\delta > 0$ on $\overline{R}$, we know from triangle inequality that f_δ converges uniformly to f on compact subsets of Ω. Thus, the limiting function f is analytic on Ω, by Weierstrass theorem.

Theorem 2. A continuous function is harmonic if and only if it is weakly-harmonic.

Proof. (Harmonic implies weakly-harmonic) Suppose f is harmonic, then by definition $\Delta f = 0$. Now for any φ satisfying the assumption and any closed rectangle $R \subset \Omega$, integration by parts gives

$$\begin{aligned}
\int_R f\Delta\varphi \, dA &= \int_R \nabla f \cdot \nabla\varphi \, dA - \left(\int_R f\varphi_x dxdy + \int_R f\varphi_y dydx \right) \\
&= \int_R \Delta f\varphi \, dA - \left(\int_R \varphi f_x dxdy + \int_R \varphi f_y dydx \right) \\
&= \int_R \Delta f\varphi \, dA = 0
\end{aligned}$$

where the sum of integrals in the brackets is zero by Fubini's theorem. Since the integral on the left hand side is zero for any R and thereby zero for any union of rectangles in Ω. The integral over Ω can be approximated up to $\epsilon > 0$, therefore $\int_\Omega f\Delta\varphi \, dA = 0$, implying f is weakly-harmonic.

(Weakly-harmonic implies harmonic) Suppose f is weakly harmonic, then follow the same procedure as in the proof that weak analyticity implies analyticity: Define

$$f_\delta(z) = \int_{\mathbb{C}} f(w)\varphi_\delta(z-w)\, dA(w)$$

where $\varphi_\delta(z) = \frac{1}{\delta^2}\varphi(\frac{z}{\delta})$. Moreover, by the same argument in part (c), we have $\Delta_z \varphi_\delta(z - w) = -\Delta_w \varphi_\delta(z-w)$ where Δ_z denotes the laplacian operator with respect to the variable z. Now take a rectangle $R \subset \overline{R} \subset \Omega$,

$$\begin{aligned}
\Delta_z f_\delta(z) &= \Delta_z \left(\int_{\mathbb{C}} f(w)\varphi_\delta(z-w)\, dA(w) \right) \\
&= \int_{\mathbb{C}} f(w)\Delta_z \varphi_\delta(z-w)\, dA(w) \\
&= -\int_{\mathbb{C}} f(w)\Delta_w \varphi_\delta(z-w)\, dA(w) = 0
\end{aligned}$$

where the second equality follows from the fact that since φ_δ is continuously differentiable and compactly supported, thus we can differentiate under the integral sign. The last equality follows from the assumption that f is weakly harmonic.

Now we know that f_δ is harmonic for all $\delta > 0$. Furthermore, by the same estimate in the analytic case, we know that f_δ is a sequence of harmonic function that converges uniformly to f on compact subsets of Ω. Finally by Harnack's principle, the limiting function f is harmonic.