The Weierstrass Function

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We know from elementary calculus that a differentiable function is continuous but the converse is not necessarily true. We may, starting from the very first day we learn about this fact, wonder how badly the converse fails to be true. Weierstrass, the grandmaster in analysis, answered in 1860s that it can be super bad.

Theorem. If $a \geq 3$ is an odd integer and if $b \in (0,1)$ such that $ab > 1 + 3\pi/2$, then the function

$$f(x) = \sum_{k=0}^{\infty} b^k \cos\left(\pi a^k x\right)$$

is continuous everywhere but nowhere differentiable.

Lemma. If $B > 0$, then

$$\left| \frac{\cos\left(A\pi + B\pi\right) - \cos\left(A\pi\right)}{B} \right| \leq \pi$$

The proof of this lemma can serve as a good elementary calculus exercise.

Proof. First we show the function f is continuous. Indeed, since $|b^k \cos \pi a^k x| \leq b^k$ and $b \in (0,1)$, we know the series defining f converges absolutely and uniformly on $\mathbb{R}$ by Weierstrass M-test. Moreover, since $b^k \cos(\pi a^k x)$ is continuous on $\mathbb{R}$ for each k, the limiting function f is continuous on $\mathbb{R}$.

Now we show f is nowhere differentiable: For any $r \in \mathbb{R}$, we want to show that $f'(r)$ does not exist. For each $m = 1, 2, 3, \ldots$, since $a^m r$ is a real number, there exists an integer α_m such that

$$\alpha_m - \frac{1}{2} < a^m r \leq \alpha_m + \frac{1}{2}$$

Let $\epsilon_m = 21^m r - \alpha_m$ be the corresponding gap for each m, we see that

$$\alpha_m + \epsilon_m = a^m r \tag{1}$$

Moreover, since $-1/2 < \epsilon_m \leq 1/2$, it follows that

$$0 < \frac{1/2}{a^m} \leq \frac{1 - \epsilon_m}{a^m} < \frac{3/2}{a^m}$$

Let $h_m = (1-\epsilon_m)/a^m$. Then (4) guarantees that $h_m \to 0$ as $m \to \infty$. Moreover,

$$a^m h_m = 1 - \epsilon_m, \quad \frac{1}{h_m} > \frac{a^m}{3/2} \tag{2}$$

Now fix the integer m and consider the difference quotient

$$\begin{aligned}\frac{f(r+h_m)-f(r)}{h_m} &= \frac{\sum_{k=0}^{\infty} b^k \cos(a^k\pi(r+h_m)) - \sum_{k=0}^{\infty} b^k \cos(a^k\pi r)}{h_m} \\ &= \sum_{k=0}^{m-1} \frac{\cos(a^k\pi(r+h_m)) - \cos(a^k\pi r)}{h_m/b^k} \\ &\quad + \sum_{k=m}^{\infty} \frac{\cos(a^k\pi(r+h_m)) - \cos(a^k\pi r)}{h_m/b^k}\end{aligned}$$

Now we estimate the finite sum first: Applying the lemma to each summand with $A = a^k r$ and $B = a^k h_m$ gives

$$\left|\frac{\cos(a^k\pi(r+h_m)) - \cos(a^k\pi r)}{h_m/b^k}\right| \le (ab)^k \pi$$

Thus by triangle inequality

$$\left|\sum_{k=0}^{m-1} \frac{\cos(a^k\pi(r+h_m)) - \cos(a^k\pi r)}{h_m/b^k}\right| \le \sum_{k=0}^{m-1} \left|\frac{\cos(a^k\pi(r+h_m)) - \cos(a^k\pi r)}{h_m/b^k}\right| \tag{3}$$

$$\le \sum_{k=0}^{m-1} (ab)^k \pi = \pi \frac{a^m b^m - 1}{ab-1} < \frac{2}{3}(ab)^m \tag{4}$$

where the last inequality follows from the assumption on ab. Moreover, we can extend the result that the finite sum above is less than $(2/3-\epsilon)(ab)^m$ for some $\epsilon > 0$. To estimate the infinite sum, we first observe

$$a^k\pi r + a^k\pi h_m = a^{k-m}\pi(a^m r + a^m h_m) = a^{k-m}\pi(\alpha_m + 1) \tag{5}$$

where the last equality follows from (1) and (2). Since $k \ge m$ and α_m is an integer, it follows $\cos(a^k\pi r + a^k\pi h_m) = (-1)^{\alpha_m+1}$. Moreover, we make another observation that

$$\begin{aligned}\cos(a^k\pi r) &= \cos(a^{k-m}\pi\alpha_m + a^{k-m}\pi\epsilon_m) \\ &= \cos(a^{k-m}\pi\alpha_m)\cdot\cos(a^{k-m}\pi\epsilon_m) - \sin(a^{k-m}\pi\alpha_m)\cdot\sin(a^{k-m}\pi\epsilon_m) \\ &= (-1)^{\alpha_m}\cdot\cos(a^{k-m}\pi\epsilon)\end{aligned}$$

Now from (5) and the above result, the infinite sum becomes

$$\sum_{k=m}^{\infty} \frac{\cos(a^k\pi(r+h_m)) - \cos(a^k\pi r)}{h_m/b^k} = \frac{(-1)^{\alpha_m+1}}{h_m} \sum_{k=m}^{\infty} b^k(1 + \cos(a^{k-m}\pi\epsilon_m))$$

Thus,

$$\left|\sum_{k=m}^{\infty} \frac{\cos(a^k\pi(r+h_m)) - \cos(a^k\pi r)}{h_m/b^k}\right| = \frac{1}{h_m}\sum_{k=m}^{\infty} b^k(1+\cos(a^{k-m}\pi\epsilon_m))$$

where the summand on the right hand side above is nonnegative. Therefore, the infinite sum is greater than its first term (where $k = m$), which gives

$$\left|\sum_{k=m}^{\infty} \frac{\cos(a^k\pi(r+h_m)) - \cos(a^k\pi r)}{h_m/b^k}\right| \geq \frac{b^m(1+\cos(\pi\epsilon_m))}{h_m} \geq \frac{b^m}{h_m} > \frac{2}{3}(ab)^m$$

where the second to the last inequality follows because $-1/2 < \epsilon_m \leq 1/2$ and the last inequality follows from (2). Finally, we are able to appreciate Weierstrass' main performance: From above,

$$\begin{aligned}
\frac{2}{3}(ab)^m &< \left|\sum_{k=m}^{\infty} \frac{\cos(a^k\pi(r+h_m)) - \cos(a^k\pi r)}{h_m/b^k}\right| \\
&= \left|\frac{f(r+h_m)-f(r)}{h_m} - \sum_{k=0}^{m-1} \frac{\cos(a^k\pi(r+h_m)) - \cos(a^k\pi r)}{h_m/b^k}\right| \\
&\leq \left|\frac{f(r+h_m)-f(r)}{h_m}\right| + \left|\sum_{k=0}^{m-1} \frac{\cos(a^k\pi(r+h_m)) - \cos(a^k\pi r)}{h_m/b^k}\right| \\
&< \left|\frac{f(r+h_m)-f(r)}{h_m}\right| + \left(\frac{2}{3}-\epsilon\right)(ab)^m
\end{aligned}$$

where the last inequality follows from the extension of (4). This implies

$$\left|\frac{f(r+h_m)-f(r)}{h_m}\right| > \epsilon(ab)^m \to \infty \quad \text{as} \quad m \to \infty$$

Since $h_m \to 0$, we know

$$f'(r) = \lim_{h\to 0} \frac{f(r+h)-f(r)}{h}$$

is also unbounded.

Remark: A proof for a specific choice of a and b is presented in The Calculus Gallery by Dunham. Here we show the general case, but the main tastes are exactly the same. This proof demonstrates the "Weierstrassian rigor" to an extreme and puts the last piece of intuitive and geometric foundation of calculus into the coffin.