A Cute but Brutal Geometry Problem

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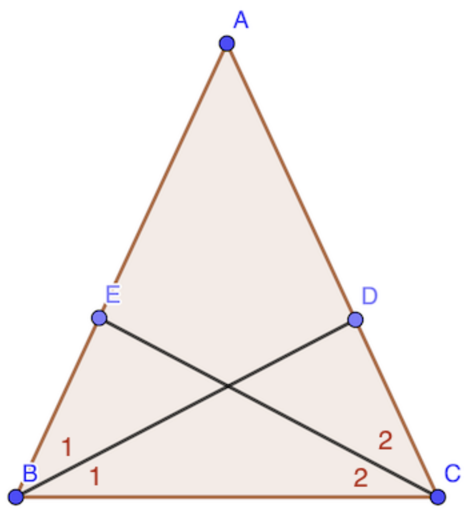
Theorem. Let BD and CE be the angular bisectors of $\angle ABC$ and $\angle ACB$ respectively. If $BD = CE$, then $AB = AC$.



Proof. Suppose for the sake of contradiction that $AB \neq AC$, then without loss of generality we may assume $AB > AC$, which implies $\angle ABC < \angle ACB$ and thus $\angle 1 < \angle 2$. Since $BD = CE$, law of cosines implies $BE > CD$. By law of sines,

$$\frac{CE}{\sin \angle A} = \frac{AE}{\sin \angle 2} \quad \Rightarrow \quad CE = \frac{\sin \angle A}{\sin \angle 2} AE.$$

$$\frac{BD}{\sin \angle A} = \frac{AD}{\sin \angle 1} \quad \Rightarrow \quad BD = \frac{\sin \angle A}{\sin \angle 1} AD.$$

Since $\sin \angle 1 < \sin \angle 2$ and $BD = CE$, we have $AE > AD$. Now by *angle bisector theorem*,

$$AB \cdot AD = BC \cdot CD, \quad AC \cdot AE = BC \cdot BE$$

which implies

$$\frac{AB \cdot AD}{CD} = \frac{AC \cdot AE}{BE}$$

With $BE > CD$, we have $AB \cdot AD < AC \cdot AE$. By law of cosine again,

$$\cos \angle A = \frac{AB^2 + AD^2 - BD^2}{2AB \cdot AD} = \frac{AC^2 + AE^2 - EC^2}{2AC \cdot AE}$$

which implies

$$AB^2 + AD^2 - BD^2 < AC^2 + AE^2 - EC^2$$

and thus $AB^2 + AD^2 < AC^2 + AE^2$. Notice

$$(AB + AD)^2 = AB^2 + AD^2 + 2AB \cdot AD < AC^2 + AE^2 + 2AC \cdot AE = (AC + AE)^2$$

implying $AB + AD < AC + AE$. However,

$$AB + AC = AB + AD + CD < AC + AE + BE = AC + AB$$

a contradiction. Thus, we have shown that $AB \leq AC$. By exactly the same argument, we can also show that $AC \leq AB$. Thus, $AB = AC$. □

Remark. This problem was given, by my friend Pubo Huang, to me who initially went gentle into the amusing delusion that it is cute but trivial. Three weeks later, I decided the title above would serve as a better indication of how much fun I had with this problem.