Microlocal Methods in Hyperbolic Dynamics

September 2023

1 Background

Recall the Riemann zeta function

$$\zeta(s) = \sum \frac{1}{n^s} = \prod_{p \ prime} (1 - p^{-s})^{-1}$$

which converges absolutely for any $\mathfrak{Re}(s) > 1$. Analytic continuation by

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \int_n^{n+1} (n^{-s} - x^{-s}) dx$$

shows that $\zeta(s)$ continues to $\mathfrak{Re}(s) > 0$ except for a simple pole at $s = 1$. Now using the functional equation $\xi(s) = \xi(1-s)$ where

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$$

We see that $\zeta(s)$ continues meromorphically to $\mathbb{C}$.

One pioneering use of the Riemann zeta function is to bundle the collection of all prime numbers into a single analytic function. By studying the poles and zeros of the meromorphic continuation, we would recover some statistical behaviours of primes. For instance, the Prime Number Theorem is equivalent to the fact that $\zeta(s)$ has no zeros on the line $\mathfrak{Re}(s) = 1$. Moreover, the more we are able to push the line of no zero closer to $\mathfrak{Re}(s) = \frac{1}{2}$, the better the error estimate in PNT we will get.

Similar ideas are employed in the study of hyperbolic dynamics, i.e. dynamics that exhibit chaotic behaviours. In these systems, it is rather hopeless to study the trajectory of a single particle. For instance, on a Riemannian manifold with negative sectional curvature, the trajectory of a single particle under the geodesic flow is dense on the manifold.

Instead, it is more realistic to study the statistical behaviour of the trajectories of a group of particles on the manifold. For instance, do these particles eventually equidistribute? If they do, how fast? Thus, it does not seem unnatural to bring back Riemann's idea to bundle a countable set of positive real numbers (related to the dynamics) into an analytic function. By studying the poles and zeros of the meromorphic continuation (if it exists), we hope to discover statistical statements about the dynamics.

One such notion of zeta functions in the study of dynamical systems is the Selberg zeta function:

$$\zeta_S(s) = \prod_{\gamma^\sharp \in \mathcal{G}} \prod_{m=0}^{\infty} (1 - e^{-\ell(\gamma^\sharp)(m+s)})$$

where $\mathcal{G}$ is the set of primitive closed geodesics on a compact negatively curved Riemannian surface (X, g). Since the number of closed geodesics on (X, g) is countable [1] and grow at most exponentially in length [2], we know $\zeta_S(s)$ converges absolutely for $\mathfrak{Re}(s) \gg 1$. By using his trace formula, Selberg showed that $\zeta_S(s)$ continues meromorphically to $\mathbb{C}$ when (X, g) is hyperbolic.

Another notion of zeta function is the Ruelle (dynamical) zeta function, which resembles the Riemann zeta function:

$$\zeta_R(s) = \prod_{\gamma^\sharp} \left(1 - e^{-\ell(\gamma^\sharp)s}\right)$$

It is closely related to ζ_S via the relation

$$\zeta_R(s) = \frac{\zeta_S(s)}{\zeta_S(s+1)}$$

Hence, the Ruelle zeta function on hyperbolic manifolds extends meromorphically to $\mathbb{C}$ via the Selberg trace formula. We refer to Pollicott's beautifully written review for more historical developments and classical results.

On the other hand, in his seminal review paper *Differentiable Dynamical Systems*, Smale conjectured that ζ_R has a meromorphic continuation even for manifolds of variable sectional curvature. He claimed that "a positive answer would be a little shocking" perhaps due to the lack of symmetry in this general setting.

[1] Proof: Closed geodesics are in one-to-one correspondence with conjugacy classes of the fundamental group. Since the fundamental group is finitely generated, as are its conjugacy classes.

[2] Let $N(T)$ be the number of closed geodesics of length no more than T, then $N(T) \leq Ce^{(2n-1)LT}$.

Nevertheless, to his shock, Giulietti-Liverani-Pollicott first established the meromorphic continuation by applying spectral theory methods in 2012. Soon in 2013, Dyatlov-Zworski provided a different proof using microlocal analysis. Eventually, Dyatlov-Guillarmou settled the meromorphic continuation for Axiom A flow, the most general case conjectured by Smale.

2 Meromorphic Continuation

In this section, we will provide a sketch of Dyatlov-Zworski's proof of the meromorphic continuation of the Ruelle zeta function. We'll start by introducing the space we'll be working with, namely Anosov manifolds.

2.1 Dynamical System

Definition. Let X be a compact manifold with C^∞ flow $\varphi_t : X \to X$ and $\varphi_t = \exp tV, V \in C^\infty(X; TX)$. The flow is *Anosov* if the tangent space to X has a continuous decomposition

$$T_x X = E_0(x) \oplus E_s(x) \oplus E_u(x)$$

which is invariant under the flow, namely $d\varphi_t(E_\bullet) = E_\bullet(\varphi_t(x))$ and $E_0(x) = \mathbb{R}V(x)$. Moreover, for some C and $\theta > 0$ [3] fixed

$$\begin{aligned} |d\varphi_t(x)v|_{\varphi_t(x)} &\le Ce^{-\theta|t|}|v|_x, \quad v \in E_u(x), \quad t < 0, \\ |d\varphi_t(x)v|_{\varphi_t(x)} &\le Ce^{-\theta|t|}|v|_x, \quad v \in E_s(x), \quad t > 0, \end{aligned}$$

where $|\bullet|_y$ is a smooth Riemannian metric on X.

Example. Let (M, g) be a compact Riemannian manifold of negative sectional curvature. We define the *geodesic flow* on $X = S^*M$ (the cosphere bundle) as the Hamiltonian flow with

$$p(x, \xi) := |\xi|_g, \quad X := H_p, \quad \varphi_t := \exp(tX) : S^*M \to S^*M.$$

It was shown by Anosov that such geodesic flow is Anosov. See Semyon Dyatlov's notes for a nice proof of the surface case.

[3] Here C depends on the metric but θ does not, so it is intrinsic to the dynamics; see a proof in Smale's review *Differentiable Dynamical Systems*

Let γ be a closed trajectory of φ_t with period T_γ and let $x \in \gamma$. Then the map $\varphi_{-T_\gamma} : X \to X$ maps γ onto γ and x onto x; so $d\varphi_{-T_\gamma}$ is a linear map of T_x onto itself having the tangent space to γ at x as a one-dimensional eigenspace. We call $P_\gamma := d\varphi_{-T_\gamma}|_{E_s \oplus E_u}$ the *(linearized) Poincaré map* of γ.

Remark. Note that although the definition of P_γ depends on the choice of $x \in \gamma$, it is clear that P_γ defined at x is conjugate to P_γ defined at x' for any pair of x and x' on γ. Hence, the notation P_γ makes sense.

Moreover, for Anosov flows, all periodic trajectories γ are *nondegenerate*, i.e. $I - P_\gamma$ is invertible. Indeed, if $v \in E_u(x) \oplus E_s(x)$ and $v = d\varphi_{-T_\gamma} v$, then $d\varphi_{-nT_\gamma} v = v$ for all $n \in \mathbb{Z}$, which by the expanding and contracting properties implies that $v = 0$. Hence, $\det(I - P_\gamma)$ is well defined and nonzero for all periodic trajectories of Anosov flow φ_t. Moreover, standard linear algebra [4] shows

$$\det(I - P_\gamma) = \sum_{k=0}^{n-1} (-1)^k \mathrm{tr}\left(\wedge^k P_\gamma\right)$$

and an easy computation from Dyatlov-Zworski pp.6 shows that

$$|\det(I - P_\gamma)| = (-1)^q \det(I - P_\gamma), \quad q = \dim E_s.$$

Note that in general, we can construct Anosov flows with any $\dim E_s$ and $\dim E_u$ by suspending Anosov diffeomorphisms. However, the flow is *contact* if the *Anosov 1-form* $\alpha \in C^0(T^*X)$ defined by

$$\ker(\alpha(x)) = E_u(x) \oplus E_s(x), \quad (\alpha(x))(V(x)) = 1, \quad \forall x \in X.$$

is a contact 1-form, namely $d\alpha|_{E_u \oplus E_s}$ is symplectic, we must have $\dim E_s = \dim E_u$ (Lagrangian subspaces). [5] In this case, $\Omega := \alpha \wedge (d\alpha)^n$ is an invariant smooth volume form on X with $n = \dim E_s$, and $E_u(x) \oplus E_s(x)$ is smooth.

Generally, the last statement is not true: $E_0(x)$ is smooth because the generator V is assumed to be smooth. The distributions $E_u(x), E_s(x), E_u(x) \oplus E_s(x)$ are only Hölder continuous even if the flow φ_t is smooth. For geodesic flow on negatively curved manifolds, the Anosov foliations $E_u(x), E_s(x)$ are C^1; furthermore, if they are C^∞, it automatically implies the manifold has constant negative curvature. See Renato Feres' thesis for more interesting discussions on the regularity of Anosov foliations.

[4] Find reference

[5] Proof: Let $\omega = d\alpha|_{E_u \oplus E_s}$ be the flow invariant symplectic form. If $u_s, v_s \in E_s(x)$, then $\omega(u_s, v_s) = \omega(d\varphi_t(u_s), d\varphi_t(v_s)) \to 0$ as $t \to \infty$. Hence, $\omega|_{E_s} = 0$ and similarly, $\omega|_{E_u} = 0$.

2.2 Guillemin Trace Formula

In this subsection, we collect some useful and interesting results from Guillemin's paper, including some spectral properties of the generator of the geodesic flow and the so-called Atiyah-Bott-Guillemin trace formula.

Let M be a compact Riemannian manifold and let Δ be its Laplace-Beltrami operator. The positive square root $\sqrt{\Delta}$ is a self-adjoint elliptic pseudodifferential operator of order one with symbol $p(x, \xi) = |\xi|_g$. Then p generates a Hamiltonian flow on S^*M which is exactly the geodesic flow on S^*M. Let V be the generator (i.e. the geodesic vector field). Let P be the self-adjoint L^2 extension of the first-order differential operator

$$-i\mathcal{L}_V : C^\infty(S^*X) \to C^\infty(S^*X)$$

Theorem. Either the spectrum of P is the whole real line or all trajectories of the flow are periodic. In particular, if $\varphi_t = \exp(tV)$ is Anosov, $\sigma_{L^2}(P) = \mathbb{R}$.

Remark. First, the proof works for more general operators such as $\nabla^{\mathcal{E}}_V$, where $\nabla^{\mathcal{E}}$ is a unitary connection on a Hermitian vector bundle $\mathcal{E} \to S^*M$. Moreover, this theorem indicates that the L^2-spectrum of the geodesic vector field is useless for spectral theory purposes. Hence, to reveal its inherent spectral data, it is necessary to modify L^2 into a space *(anisotropic Sobolev spaces)* on which P would have a discrete spectrum.

Furthermore, using the above theorem and other results, Guillemin showed that the eigenvalues of $\sqrt{\Delta}$ cluster on $\mathbb{R}$ unless every trajectory of φ_t is periodic. One example for the latter case is the geodesic vector field on $\mathbb{S}^1$ where every trajectory is periodic; a simple computation using Fourier transform shows that $\sigma_{L^2}(V) = \mathbb{Z}$. Next, we turn to Guillemin's main result:

Theorem. If all trajectories of V are non-degenerate, then

$$\operatorname{tr}^\flat e^{-itP} = \sum_\gamma \frac{\ell(\gamma^\sharp)\delta(t - \ell(\gamma))}{|\det(I - P_\gamma)|}, \quad t > 0$$

Here the *flat trace* of an operator $B : C^\infty(M) \to \mathcal{D}'(M)$ satisfying

$$\mathrm{WF}\,(B) \cap \Delta(T^*X) = \emptyset, \quad \Delta(T^*X) = \{(x, \xi, x, \xi) : (x, \xi) \in T^*M\}$$

for a compact manifold X is defined as

$$\operatorname{tr}^\flat B := \int_X (\iota^* K_B)(x)\, dx, \quad \iota : x \to (x, x).$$

where K_B is the Schwartz kernel of B with respect to some density dx on X. The disjoint condition guarantees the distributional pullback $\iota^* K_B$ is well defined (Hörmander I, Thm 8.2.4).

In our case, the flat trace of $\varphi_{-t}^* = e^{-itP}$ is defined similarly by the pullback of $\iota(t,x) = (t,x,x)$ and pushforward of $\pi : (t,x) \to t$, namely

$$\mathrm{tr}^\flat e^{-itP} = \pi_* \iota^* K_{e^{-itP}},$$

where $K_{e^{-itP}}(t,x,y) \in \mathcal{D}'(\mathbb{R}_t \times X \times X)$ is the Schwartz kernel of e^{-itP} and the pullback is well-defined because

$$\mathrm{WF}\left(K_{e^{-itP}}\right) \cap N^*(\mathbb{R}_t \times \Delta(X)) = \emptyset, \quad t > 0$$

where $\Delta(X) \subset X \times X$ is the diagonal and $N^*(\mathbb{R}_t \times \Delta(X)) \subset T^*(\mathbb{R}_t \times X \times X)$ is the conormal bundle.

Generally, suppose $\mathcal{E} \to S^*X$ is a vector bundle, and suppose the one-parameter group of diffeomorphisms $\exp(tV) : C^\infty(S^*X) \to C^\infty(S^*X)$ lifts to a one-parameter group of vector bundle automorphisms

$$\left(\widetilde{\exp(tV)}\right) : \mathcal{E} \to \mathcal{E},$$

then there is an induced one parameter group of linear mappings $e^{-it\mathbf{P}} : C^\infty(X;\mathcal{E}) \to C^\infty(X;\mathcal{E})$. Let γ be a periodic trajectory of period $\ell(\gamma)$. For each $x \in \gamma$, we get a map

$$\left(\widetilde{\exp(tV)}\right)_x : \mathcal{E}_x \to \mathcal{E}_x$$

Let $\chi_{\mathcal{E}}(\gamma)$ be the trace of this map. Then Guillemin's trace formula generalizes to

$$\mathrm{tr}^\flat U(t) = \sum_\gamma \frac{\ell(\gamma^\sharp)\delta(t - \ell(\gamma))}{|\det(I - P_\gamma)|} \chi_{\mathcal{E}}(\gamma)$$

In particular, when $\mathcal{E} = \mathcal{E}_0^k$ the smooth invariant vector bundle given by all differential k-forms $\mathbf{u}$ satisfying $\iota_V \mathbf{u} = 0$, then $\chi_{\mathcal{E}}(\gamma) = \mathrm{tr}\left(\wedge^k P_\gamma\right)$. [6]

[6] check this

2.3 Reduction

Now we are ready to start on the meromorphic continuation of $\zeta_R(s)$:

$$\begin{aligned}
\log \zeta_R(s) &= \sum_{\gamma^\sharp} \log(1 - e^{-\ell(\gamma^\sharp)s}) \\
&= \sum_{\gamma^\sharp} \sum_{k=1}^{\infty} \frac{1}{k} e^{-\ell(\gamma^\sharp)ks} \qquad \big(\ell(\gamma) = k\ell(\gamma^\sharp)\big) \\
&= \sum_{\gamma} \frac{\ell(\gamma^\sharp)}{\ell(\gamma)} e^{-\ell(\gamma)s} \\
&= -\sum_{\gamma} \sum_{k=0}^{n-1} (-1)^{k+q} \frac{\ell(\gamma^\sharp)\, e^{-\ell(\gamma)s}\, \mathrm{tr}\left(\wedge^k P_\gamma\right)}{\ell(\gamma)\, |\det(I - P_\gamma)|} \\
&= \sum_{k=0}^{n-1} (-1)^{k+q} \log \zeta_k(s)
\end{aligned}$$

where

$$\log \zeta_k(s) = -\sum_{\gamma} \frac{\ell(\gamma^\sharp)\, e^{-\ell(\gamma)s}\, \mathrm{tr}\left(\wedge^k P_\gamma\right)}{\ell(\gamma)\, |\det(I - P_\gamma)|}.$$

Differentiating $\log \zeta_k$ gives

$$\frac{d}{ds} \log \zeta_k(s) = \frac{\zeta_k'(s)}{\zeta_k(s)} = \sum_{\gamma} \frac{\ell(\gamma^\sharp)\, \mathrm{tr}\left(\wedge^k P_\gamma\right)\, e^{-\ell(\gamma)s}}{|\det(I - P_\gamma)|}$$

Recall from complex analysis that: When h is a meromorphic function on a simply connected domain $D \subset \mathbb{C}$ with simple poles and integral residues, then there exists g meromorphic on D such that $h = g'/g$. Hence, it suffices to continue $\xi_k'(s)/\xi_k(s)$.

Now Guillemin's trace formula says that

$$\mathrm{tr}^\flat(e^{-it\mathbf{P}})|_{C^\infty(X;\mathcal{E}_0^k)} = \sum_{\gamma} \frac{\ell(\gamma^\sharp)\, \mathrm{tr}\left(\wedge^k P_\gamma\right)\, \delta(t - \ell(\gamma))}{|\det(I - P_\gamma)|}$$

Hence, we have

$$\frac{d}{ds} \log \zeta_k(s) = \sum_{\gamma} \frac{\ell(\gamma^\sharp)\, \mathrm{tr}\left(\wedge^k P_\gamma\right)\, e^{-\ell(\gamma)s}}{|\det(I - P_\gamma)|} = \int_0^\infty e^{-ts}\, \mathrm{tr}^\flat(e^{-it\mathbf{P}})|_{C^\infty(X;\mathcal{E}_0^k)}\, dt$$

Then it is enough to show that the right-hand side has a meromorphic continuation to $\mathbb{C}$ with simple poles and integral residues. Furthermore, it suffices to take $t_0 > 0$ smaller than $\ell(\gamma)$ for all γ and consider a continuation of

$$\int_{t_0}^{\infty} e^{-ts} \operatorname{tr}^\flat(e^{-it\mathbf{P}})|_{C^\infty(X;\mathcal{E}_0^k)}\, dt = e^{-t_0 s} \int_0^{\infty} e^{-ts} \operatorname{tr}^\flat(\varphi_{-t_0}^* e^{-it\mathbf{P}})|_{C^\infty(X;\mathcal{E}_0^k)}\, dt$$

Now by spectral theorem

$$\frac{1}{i}\int_0^{\infty} e^{-ts}\, \varphi_{-t_0}^* e^{-it\mathbf{P}}\, dt = \varphi_{-t_0}^*(\mathbf{P} - is)^{-1}, \quad \text{for } \mathfrak{Re}(s) \gg 1$$

We see that it is sufficient to continue

$$\operatorname{tr}^\flat\left(\varphi_{-t_0}^*(\mathbf{P} - \lambda)^{-1}\right), \quad \text{for } \mathfrak{Im}(\lambda) \gg 1.$$

to $\mathbb{C}$ with simple poles and integral residues. Recall that it was shown in Faure-Sjöstrand that $(P - \lambda)^{-1}$ continues meromorphically on suitably chosen anisotropic Sobolev spaces. Hence, we only need to check the disjoint wave front set condition for the distributional kernel of $\varphi_{-t_0}^*(P - \lambda)^{-1}$, namely that this wave front set does not intersect $N^*(\Delta(X))$.

In the following, we review Dyatlov-Zworski's proof on the meromorphic continuation of $(P-\lambda)^{-1}$, which is slightly different from Faure-Sjöstrand's proof and relies on the propagation of singularities and radial points estimates.

2.4 Anisotropic Sobolev Spaces

In section 2.2, we briefly mentioned Helton's result that the L^2-spectrum of the generator $P = -i\mathcal{L}_V$ consists of the entire real line, which is not helpful from a spectral theory perspective. It turns out that we are simply looking at the operator on the wrong space ($L^2(S^*X)$ in this case). Consider the following examples:

Example. The first order differential operator $-i\frac{d}{dx} : C^\infty(0,1) \to C^\infty(0,1)$ is not closed and unbounded with respect to the sup norm. On the other hand, $-i\frac{d}{dx}$: $H_0^1(0,1) \to L^2(0,1)$ is a self-adjoint closed bounded operator with finite-dimensional kernel and cokernel, in which case $-i\frac{d}{dx}$ has discrete eigenvalues.

Example. The Laplacian operator on $\mathbb{R}^n$ serves as a more interesting example. Even after isotropically modifying the domain, namely adding a weight $\langle\xi\rangle^2$ to the domain, the operator $\Delta : H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is not closed and remains true for all weighted Sobolev spaces $\langle x \rangle^r H^s(\mathbb{R}^n)$. However, if M is a compact manifold, then $\Delta : H^2(M) \to L^2(M)$ is self-adjoint and bounded with finite-dimensional kernel and cokernel, in which case Δ has discrete eigenvalues.

In general, our goal is to find the appropriate spaces on which $P - \lambda$ is Fredholm. Since it is invertible for some $\mathfrak{Im}(\lambda) \gg 1$, it follows from the *analytic Fredholm theorem* that $(P - \lambda)^{-1}$ is meromorphic.

Such a scheme is in fact quite common in geometric scattering theory, where given an operator $\mathcal{P}$, we would like to find function spaces $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{P} : \mathcal{X} \to \mathcal{Y}$ is Fredholm, i.e.

1. $\mathcal{P}$ has a finite-dimensional kernel (uniqueness up to finite-dimensional error).
2. $\mathcal{P}^*$ has a finite-dimensional kernel (solvability up to finite-dimensional error).
3. $\mathcal{P}$ has closed range (solution $\mathcal{P}u = f$ can be controlled as $\|u\|_{\mathcal{X}} \leq C \|f\|_{\mathcal{Y}}$).

We will accomplish this goal by applying microlocal analysis to construct suitable Sobolev spaces $\mathcal{H}^s$ adapted to the flow so that the following type of estimate holds: For any $N \in \mathbb{Z}$,

1. $\|u\|_{\mathcal{H}^s} \lesssim \|\mathcal{P}u\|_{\mathcal{H}^r} + \|u\|_{H^{-N}}$
2. $\|u\|_{\mathcal{H}^r} \lesssim \|\mathcal{P}^*u\|_{\mathcal{H}^s} + \|u\|_{H^{-N}}$

Granted such estimates, we see that $\mathcal{P}u = 0$ implies $u \in \mathcal{H}^{-N} \to u \in H^s$ is bounded. Choosing N sufficiently negative, we have by *Rellich compact embedding theorem* that the composition $\mathcal{H}^{-N} \to H^s \hookrightarrow \mathcal{H}^{-N}$ is compact, which is impossible unless u belongs to a finite-dimensional subspace of $\mathcal{H}^s$.

Now to set up the stage, we introduce some notations: Let $\mathcal{E} = \bigoplus_{j=0}^n \Lambda^j(T^*X)$ be the vector bundle of differential forms of all orders on X and consider the first-order differential operator

$$\mathbf{P} : C^\infty(X; \mathcal{E}) \to C^\infty(X; \mathcal{E}), \quad \mathbf{P} := -i\mathcal{L}_V$$

where V is the generator of the Anosov flow φ_t and $\mathcal{L}$ the Lie derivative with principal symbol $\sigma(\mathbf{P})(x, \xi) = \xi(V(x)) \in S^1(X; \mathbb{R})$ being diagonal and homogeneous of degree 1. Define the dual decomposition

$$T_x^*X = E_0^*(x) \oplus E_s^*(x) \oplus E_u^*(x)$$

where E_0^*, E_s^* and E_u^* are dual to E_0, E_u and E_s. We are now ready to describe the anisotropic Sobolev spaces that grant $\mathbf{P} - \lambda$ nice spectral properties. Let $m_G \in C^\infty(T^*X \setminus o; [-1,1])$ be an escape function homogeneous of degree 0 satisfying the following properties:

$$m_G(x,\xi) = \begin{cases} 1 & \text{near } E_s^* \\ -1 & \text{near } E_u^* \end{cases}$$

and $H_p m_G \leq 0$ everywhere, where H_p is the Hamiltonian vector field generated by the principal symbol p. We refer to Dyatlov-Zworski's appendix for the construction of such an escape function and to Faure-Sjöstrand for a finer version. Now with m_G in hand, we choose a pseudodifferential operator $G \in \Psi^{0+}(X)$ satisfying

$$\sigma(G)(x,\xi) = m_G(x,\xi)\log|\xi|,$$

where $|\bullet|$ is any smooth norm on the fibers of T^*X. Then $e^{\pm sG} \in \Psi^{s+}(X)$ [7] for any $s > 0$ and the anisotropic Sobolev spaces are defined using this weight

$$H_{sG} := e^{-sG}(L^2(X)), \quad \|\mathbf{u}\|_{H_{sG}} := \left\|e^{sG}\mathbf{u}\right\|_{L^2}$$

The analogy here would be that standard Sobolev space can be phrased as $H^s := \Lambda^{-s}(L^2)$ where Λ is a pseudodifferential operator with principal symbol $\sigma(\Lambda) = \langle\xi\rangle$, and $\|u\|_{H^s} = \|\Lambda^s u\|_{L^2} = \left\|\langle\xi\rangle^s \hat{f}(\xi)\right\|_{L^2}$.

Moreover, since the symbol of $e^{\pm sG}$ belongs to $S^s_{1-\varepsilon,\varepsilon}$ for each $\varepsilon > 0$, we have that $H^s \subset H_{sG} \subset H^{-s}$. More specifically, in these anisotropic Sobolev spaces H_{sG}, functions are smoother in the stable direction and rougher in the unstable direction with

$$H_{sG} = \begin{cases} H^s & \text{near } E_s^* \\ H^{-s} & \text{near } E_u^* \end{cases}$$

Now since the flow moves from stable to unstable, we "move" smoother functions into spaces of rougher functions, which results in the Fredholm properties. We define the domain $D_{sG} := \{\mathbf{u} \in H_{sG} : \mathbf{P}\mathbf{u} \in H_{sG}\}$ and the Hilbert space norm on D_{sG} by $\|\mathbf{u}\|_{D_{sG}} := \|\mathbf{u}\|_{H_{sG}} + \|\mathbf{P}\mathbf{u}\|_{H_{sG}}$.

2.5 Propagation of Regularities

Once the correct spaces are set up, we are ready to analyze the operator $\mathbf{P} - \lambda$. First let's consider its principal symbol $p(x,\xi) = \xi(V(x))$, which is a linear function on

[7] We refer to the appendix of Faure-Roy-Sjöstrand for a precise description of microlocal theorems related to symbols of variable orders.

T^*X. This implies the *characteristic set* (i.e. zero set of the symbol) is a hyperplane in T^*X and hence noncompact. Now outside this hyperplane, the principal symbol is nonzero and thereby invertible. Elliptic regularity from microlocal analysis takes care of the local invertibility of the operator $\mathbf{P}$ for us.

Thus, our major enemy lives on the hyperplane $\mathcal{O}_V := \{(x,\xi) \in T^*X : \xi(V(x)) = 0\}$ normal to the flow direction. This generally won't cause a problem because another powerful tool from microlocal analysis called *propagation of singularities* allows us to obtain microlocal estimates in the characteristic set, if an a priori control of regularity is known. However, since we would like to set up a global Fredholm problem, global estimates (particularly estimates when $\xi \to \infty$) are necessary. Hence, we start by compactifying the cotangent bundle T^*X into $\overline{T}^*X$, the *fiber-radially compactified* cotangent bundle.

Definition. Let $B^*X = \{(x,\xi) : |\xi| \leq 1\}$ denote the coball bundle. Embed T^*X into B^*X by

$$(x,\xi) \to \left(x, \frac{\xi}{1 + \langle \xi \rangle}\right)$$

with fiber infinity $\partial B^*X = S^*X$ identified via the map $\kappa : T^*X \setminus o \to S^*X$ by $\kappa(x,\xi) = (x, \xi/|\xi|)$. Such B^*X is a fiber-radially compactified cotangent bundle and denoted as $\overline{T}^*X$. In particular, for each $(x,\xi) \in T^*X \setminus o$, the ray $(x, s\xi)$ converges to $\kappa(x,\xi)$ in $\overline{T}^*X$ as $s \to \infty$.

Note that the Hamiltonian flow H_p extends to a smooth vector field on $\overline{T}^*X$ which is tangent to $\partial\overline{T}^*X$ [8]. On the other hand, the drawback of this compactification is also clear: We have artificially created fiber infinity points on $\overline{T}^*X$ that are radial: Points $\alpha \in \partial\overline{T}^*X$ such that the Hamiltonian vector field $H_p|_\alpha$ is parallel to $\xi\partial_\xi$.

More precisely, these radial points are classified into sources and sinks. Let $e^{tH_p} : T^*X \to T^*X$ denote the canonical lift of the flow $\varphi : X \to X$ to the cotangent bundle, namely $e^{tH_p}(x,\xi) = (\varphi_t(x), (d\varphi_{-t}(x))^T\xi)$ for $(x,\xi) \in T^*X$. Assume that $L \subset T^*X \setminus o$ is a closed conic set invariant under the flow $e^{tH_p}(x,\xi)$ and there exists an open conic neighbourhood U of L with the following properties for some $\theta > 0$:

$$d(\kappa(e^{-tH_p}(U)), \kappa(L)) \to 0 \quad \text{as } t \to \infty;$$
$$(x,\xi) \in U \quad \Rightarrow \quad |e^{-tH_p}(x,\xi)| \geq C^{-1}e^{\theta t}|\xi|$$

[8] Proposition E.5. from Dyatlov-Zworski's book: Let $p \in S^k(T^*X)$. Then $\langle\xi\rangle^{1-k}H_p$ extends to a smooth vector field on $\overline{T}^*X$ which is tangent to $\partial\overline{T}^*X$. Hence, the above statement is true thanks to the fact that $p(x,\xi) \in S^1(T^*X)$.

for any norm on the fibers. We call L a *radial source*, and *radial sink* is defined analogously by reversing the direction of the flow.

Note that in our setting, we have chosen E_u^* and E_s^* so that the following holds:

$$\begin{aligned} \left|d\varphi_{-t}(x)^T\xi\right| &\leq Ce^{-\theta|t|}|\xi|, \quad \xi \in E_u^*, \quad t<0 \\ \left|d\varphi_{-t}(x)^T\xi\right| &\leq Ce^{-\theta|t|}|\xi|, \quad \xi \in E_s^*, \quad t>0 \end{aligned}$$

This implies for any $\xi \notin E_0^*(x) \oplus E_s^*(x)$, applying e^{tH_p} for positive time flow eventually kills any E_s^* component and thus $d(\kappa(e^{tH_p}(x,\xi), \kappa(E_u^*)) \to 0$ as $t \to \infty$. Similarly, for any $\xi \notin E_0^*(x) \oplus E_u^*(x)$, we have $d(\kappa(e^{tH_p}(x,\xi)), \kappa(E_s^*)) \to 0$ as $t \to -\infty$. Hence, E_u^* is a radial sink and E_s^* a radial source.

At these radial points, the Hamiltonian vector fields are simply dilations of the fibers. Since propagation of singularities is by default conic in the fibers, it is trivial near these points. In summary, we are faced with the following two obstacles:

1. We need an a priori control on regularity to start propagating regularity.

2. We need to find a way to deal with regularity at radial points.

Miraculously, these two problems are solved simultaneously by the work of Melrose and Vasy on radial points estimate. What these estimates accomplish is that they provide unconditional control of regularity at the radial points, assuming a certain regularity threshold is achieved. This is possible due to exactly the degeneracy of $\mathbf{P}$ at these points.

Provided with all the tools mentioned above, we are ready to establish a global estimate of $\mathbf{P}$ on $\overline{T}^*X$. Instead of working directly with $\mathbf{P}$, we consider an artificial semiclassical adaption, namely $\mathbf{P}_\delta(z) = h\mathbf{P} - i\mathbf{Q}_\delta - z$ where $z = h\lambda$ and $\mathbf{Q}$ is a compact operator supported near the zero section of T^*M, which will eliminate trapping and guarantee invertibility. The use of semiclassical pseudodifferential operators will help us keep track of the wavefront set in the compactified cotangent bundle, but eventually, we will let $h > 0$ be small but fixed.

More precisely, we choose $\mathrm{WF}_h(\mathbf{Q}_\delta) \subset \{|\xi| < \delta\}$ and $\sigma_h(\mathbf{Q}_\delta) > 0$ on $\{|\xi| \leq \delta/2\}$ with $\sigma_h(\mathbf{Q}_\delta) \geq 0$ everywhere. And we modify the anisotropic Sobolev spaces accordingly: Choose a semiclassical pseudodifferential operator $G(h) \in \Psi_h^{0+}$ such that

$$\sigma_h(G(h))(x,\xi) = (1-\chi(x,\xi))m_G(x,\xi)\log|\xi|,$$

where $\chi \in C_0^\infty(T^*X)$ is equal to 1 near the zero section and $\mathrm{WF}_h(G(h))$ does not intersect the zero section. Define the space $H_{sG(h)} = e^{-sG(h)}(L^2(X))$ and similarly

$D_{sG(h)}$ with the norm. Note that since $\sigma(G(h)-G)$ is bounded as $|\xi| \to \infty$, the spaces $H_{sG(h)}$ and H_{sG}, $D_{sG(h)}$ and D_{sG} have equivalent norms with the constant depending on h. [9]

Recall our goal is to show $\mathbf{P}-\lambda : C^\infty \to \mathcal{D}'$ is Fredholm, which we do strip by strip: For any constant $C_0 > 0$, there exists s large enough so that $\mathbf{P}-\lambda : D_{sG} \to H_{sG}$ is Fredholm on $\{\mathfrak{Im}(\lambda) > -C_0\}$.

Since $\mathbf{Q}$ is a compact perturbation and $z = h\lambda$, it suffices to show $\mathbf{P}_\delta(z) : D_{sG(h)} \to H_{sG(h)}$ is invertible on $\{\mathfrak{Im}(z) > -hC_0\}$ provided s is chosen to be sufficiently large and h sufficiently small but fixed. Injectivity is achieved by the estimate

$$\|u\|_{H_{sG(h)}} \leq Ch^{-1} \|\mathbf{P}_\delta(z)u\|_{H_{sG(h)}}$$

and surjectivity is similar by an estimate on $\mathbf{P}_\delta^*$. By a microlocal partition of unity, we are able to prove the estimate above in pieces of the form:

$$\|Au\|_{H_h^s} \leq C\|Bu\|_{H_h^s} + Ch^{-1}\|B_1 f\|_{H_h^s} + \mathcal{O}(h^\infty)$$

where $A \in \Psi_h^0$ falls into the following categories:

1. When $\mathrm{WF}_h(A) \cap \{p=0\} \cap \{|\xi| \geq \delta/2\} = \emptyset$, then $\mathbf{P}_\delta$ is elliptic on $\mathrm{WF}_h(A)$. Hence, by standard elliptic estimate, we can take $B=0$ and $B_1 = I$.

$$\|Au\|_{H_{sG(h)}} \leq C\|B_1 f\|_{H_{sG(h)}} + \mathcal{O}(h^\infty)$$

2. When $\mathrm{WF}_h(A)$ is contained in a small neighborhood of $\kappa(E_s^*)$, namely the radial source, we choose s large enough so that $H_{sG(h)}$, microlocally equivalent to H_h^s near $\kappa(E_s^*)$, has regularity beyond the critical threshold in Melrose/Vasy's radial points estimate, which gives the estimate with $B=0$ and $\mathrm{WF}_h(B_1)$ in a neighborhood of $\kappa(E_s^*)$.

$$\|Au\|_{H_{sG(h)}} \leq Ch^{-1}\|B_1 f\|_{H_{sG(h)}} + \mathcal{O}(h^\infty)$$

3. When $\mathrm{WF}_h(A)$ is contained in a small neighborhood of $(x_0,\xi_0) \in \{p=0\} \setminus \overline{E}_u^*$, namely everything happens inside the characteristic set of $\mathbf{P}_\delta$, hence standard propagation of singularity applies. In this case, we can take $\mathrm{WF}_h(B)$ contained

[9] check Zworski's Semiclassical Analysis

in a neighborhood of $\kappa(E_s^*)$ and $\mathrm{WF}_h(B_1)$ contained a small neighborhood of $e^{tH_p}(\mathrm{WF}_h(A)) \subset ell(B)$ for $t \in [-T, 0]$.

$$\|Au\|_{H_{sG(h)}} \leq C \|Bu\|_{H_{sG(h)}} + Ch^{-1} \|B_1 f\|_{H_{sG(h)}} + \mathcal{O}(h^\infty)$$

Applying step 2 again gives

$$\|Au\|_{H_{sG(h)}} \leq C \|B_2 f\|_{H_{sG(h)}} + Ch^{-1} \|B_1 f\|_{H_{sG(h)}} + \mathcal{O}(h^\infty)$$

for some $B_2 \in \Psi_h^0$ with $\mathrm{WF}_h(B_2)$ in a neighborhood of $\kappa(E_s^*)$.

4. When $\mathrm{WF}_h(A)$ is contained in a small neighborhood of some $(x_0, \xi_0) \in E_u^*$, then $e^{tH_p}(x_0, \xi_0)$ converges to the zero section as $t \to -\infty$. Since the zero section is damped by $\mathbf{Q}$, the estimate follows similarly to the previous case.

5. Finally, when $\mathrm{WF}_h(A)$ is contained in a small neighborhood of $\kappa(E_u^*)$, the radial sink, hence radial points estimate applies again, which gives the estimate with $B, B_1 \in \Psi_h^0$ with $\mathrm{WF}_h(B), \mathrm{WF}_h(B_1)$ contained in a small neighborhood of $\kappa(E_u^*)$ and $\mathrm{WF}_h(B) \cap \kappa(E_u^*) = \emptyset$.

$$\|Au\|_{H_{sG(h)}} \leq C \|Bu\|_{H_{sG(h)}} + Ch^{-1} \|B_1 f\|_{H_{sG(h)}} + \mathcal{O}(h^\infty)$$

Applying the previous steps allows us to bound $\|Bu\|$ by terms $\|B_1 f\|, \|B_2 f\|$.

Altogether, we have been able to bound any pieces of $\|u\|_{H_{sG(h)}}$ by pieces of $\|f\|_{H_{sG(h)}}$. Hence, the bound $\|u\|_{H_{sG(h)}} \leq Ch^{-1} \|\mathbf{P}_\delta u\|_{H_{sG(h)}}$ is global. Reversing the direction of the flow gives the corresponding estimate for $\mathbf{P}_\delta^*$, i.e.

$$\|v\|_{H_{-sG(h)}} \leq Ch^{-1} \|\mathbf{P}_\delta^* v\|_{H_{-sG(h)}}, \quad v \in D_{-sG(h)}$$

These two estimates together give the invertibility of $\mathbf{P}_\delta(z)$ and hence the Fredholm property of $h(\mathbf{P} - \lambda)$. Since h overall is a small but fixed constant and H_{sG} is topologically equivalent to $H_{sG(h)}$, we have that $\mathbf{P} - \lambda$ is Fredholm as well. Moreover, since

$$(\mathbf{P} - \lambda)^{-1} = \frac{1}{i} \int_0^\infty e^{i\lambda t} e^{it\mathbf{P}} \, dt$$

is well-defined and holomorphic for $\mathfrak{Im}(\lambda) \gg 1$, it follows from analytic Fredholm theory that $(\mathbf{P} - \lambda)^{-1} : H_{sG} \to H_{sG}$ is meromorphic with poles of finite rank.

Note that the Schwarz kernel of $(\mathbf{P}-\lambda)^{-1}$ is characterized by its action on $C^\infty(X)$. Hence, the poles are independent of the choice of s and the weight G and are also known as the *Pollicott-Ruelle resonance* of $\mathbf{P}$.

Given the microlocal estimates on $\mathbf{P}_\delta(z)u = f$, we also obtain [10] the wavefront set description of $\mathbf{R}_\delta(z)$, namely

$$\mathrm{WF}'_h(\mathbf{R}_\delta(z)) \cap T^*(X \times X) \subset \Delta(T^*X) \cup \Omega_+$$

where $\Delta(T^*X)$ is the diagonal and Ω_+ is the *positive flow-out* of e^{tH_p} on $\{p = 0\}$:

$$\Omega_+ = \{(e^{tH_p}(x,\xi), x, \xi) : t \geq 0, p(x,\xi) = 0\}$$

2.6 Wavefront Set Description

So far the results mentioned above can also be found in Faure-Sjöstrand where in replacement of a propagation of singularity argument, they constructed a more sophisticated escape function adapted to the flow which gives the meromorphic continuation.

The novel addition to Dyatlov-Zworski which allows them to obtain the meromorphic continuation of the Ruelle zeta function ζ_R is a precise microlocal description of $\mathbf{R}(\lambda) = (\mathbf{P} - \lambda)^{-1}$, namely for λ near λ_0,

$$\mathbf{R}(\lambda) = \mathbf{R}_H(\lambda) - \sum_{j=1}^{J(\lambda_0)} \frac{(\mathbf{P} - \lambda_0)^{j-1} \prod}{(\lambda - \lambda_0)^j}$$

where $\mathbf{R}_H(\lambda)$ is holomorphic near λ_0 and $\prod : H_{sG} \to H_{sG}$ is the commuting projection [11] onto the kernel $(\mathbf{P} - \lambda_0)^{J(\lambda_0)}$ and

$$\mathrm{WF}'(\mathbf{R}_H(\lambda)) \subset \Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*).$$

Effectively, this microlocal description allows us to describe the wavefront set of $\varphi^*_{-t_0}(\mathbf{P} - \lambda)^{-1}$ and in the end justify the validity of taking the flat trace.

Since $\mathbf{R}(\lambda)$ is meromorphic with poles of finite rank, it has the following Laurent expansion near a pole $\lambda = \lambda_0$

$$\mathbf{R}(\lambda) = \mathbf{R}_H(\lambda) + \sum_{j=1}^{J(\lambda_0)} \frac{A_j}{(\lambda - \lambda_0)^j}$$

[10] Review definition of semiclassical operator wavefront set and go through this argument

[11] See Dyatlov-Zworski's scattering resonance book Appendix C Theorem 3.9

where $J(\lambda_0)$ is the order of the pole λ_0 and A_j's are finite rank operators. By residue theorem, we have

$$-\prod := A_1 = \frac{1}{2\pi i}\oint_{\lambda_0} \mathbf{R}(\lambda)\, d\lambda$$

Moreover, we claim that $A_{j+1} = (\mathbf{P} - \lambda_0)A_j$ for $j > 1$. Since the computation is elementary but not entirely trivial, we include it here: From the Laurent series expansion,

$$A_j = \frac{1}{2\pi i}\oint_{\lambda_0} \frac{\mathbf{R}(\lambda)}{(\lambda - \lambda_0)^{-(j+1)}}\, d\lambda$$

Then

$$(\mathbf{P} - \lambda_0)A_j = \frac{1}{2\pi i}\oint_{\lambda_0} \frac{(\mathbf{P} - \lambda_0)\mathbf{R}(\lambda)}{(\lambda - \lambda_0)^{-(j+1)}}\, d\lambda = \frac{1}{2\pi i}\oint_{\lambda_0} \frac{\mathbf{P}\mathbf{R}(\lambda)}{(\lambda - \lambda_0)^{-(j+1)}}\, d\lambda - \lambda_0 A_j$$

Note the resolvent identity

$$P(P - \lambda)^{-1} = ((P - \lambda) + \lambda)(P - \lambda)^{-1} = I - \lambda(P - \lambda)^{-1}$$

gives

$$\frac{1}{2\pi i}\oint_{\lambda_0} \frac{\mathbf{P}\mathbf{R}(\lambda)}{(\lambda - \lambda_0)^{-(j+1)}}\, d\lambda = \frac{1}{2\pi i}\oint_{\lambda_0} \frac{I}{(\lambda - \lambda_0)^{-(j+1)}}\, d\lambda - \frac{1}{2\pi i}\oint_{\lambda_0} \frac{\lambda\mathbf{R}(\lambda)}{(\lambda - \lambda_0)^{-(j+1)}}\, d\lambda$$

Since $j > 1$, the first integral on the right hand side vanishes. Writing $\lambda\mathbf{R}(\lambda) = (\lambda - \lambda_0)\mathbf{R}(\lambda) + \lambda_0\mathbf{R}(\lambda)$ for the second integral gives the desired result. Hence, $A_j = -(\mathbf{P} - \lambda_0)^{j-1}\prod$ and $(\mathbf{P} - \lambda_0)^{J(\lambda_0)}\prod = 0$.

Finally, we present the wavefront set description of $\mathbf{R}_H(\lambda)$. Recall that $\mathbf{R}(\lambda) = (\mathbf{P} - \lambda)^{-1}$ and $\mathbf{R}_\delta(z) = (h(\mathbf{P} - \lambda) - iQ_\delta)^{-1}$ with $z = h\lambda$, then the following identity holds with some computation

$$\mathbf{R}(\lambda) = h(\mathbf{R}_\delta(z) - i\mathbf{R}_\delta(z)Q_\delta\mathbf{R}_\delta(z)) - \mathbf{R}_\delta(z)Q_\delta\mathbf{R}(\lambda)Q_\delta\mathbf{R}_\delta(z)$$

Since Q_δ is a pseudodifferential operator, we have

$$\mathrm{WF}'_h(\mathbf{R}_\delta(z) - i\mathbf{R}_\delta(z)Q_\delta\mathbf{R}_\delta(z)) \cap T^*(X \times X) \subset \Delta(T^*X) \cup \Omega_+$$

To deal with the second term, first note that $Q_\delta\mathbf{R}(\lambda)Q_\delta$ is microlocally supported in $\{|\xi| < 2\delta\}$. Hence,

$$\mathrm{WF}'_h(\mathbf{R}_\delta(z)Q_\delta\mathbf{R}(\lambda)Q_\delta\mathbf{R}_\delta(z)) \cap T^*(X \times X) \subset \Upsilon_\delta$$

where $\Upsilon_\delta = \{(\rho', \rho) \,|\, \exists t, s \geq 0 : e^{tH_p}(\rho) \in \mathrm{WF}'_h(Q_\delta), e^{-sH_p}(\rho') \in \mathrm{WF}'_h(Q_\delta)\}$. Note that $\mathbf{R}(\lambda)$ depends neither on h or δ; hence,

$$\mathrm{WF}(\mathbf{R}(\lambda)) \subset \Delta(T^*X) \cup \Omega_+ \cup \bigcap_{\delta>0} \Upsilon_\delta = \Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*)$$

as desired.

2.7 Summary

Since we've covered the bulk of the proof, we'll forgive ourselves for not providing the rest of the argument for the meromorphic continuation of $\zeta_R(\lambda)$. Instead, we only mention that:

1. Given the wavefront set description of $\mathbf{R}(\lambda)$ and the choice of t_0 satisfying $0 < t_0 < \ell(\gamma)$ for all γ, the wavefront set of $\varphi^*_{-t_0}(\mathbf{P} - \lambda)^{-1}$ does satisfy the condition on taking the flat trace (even though $\mathrm{WF}(\mathbf{R}(\lambda))$ in fact does not).

2. By standard Fredholm theory, $\mathbf{R}(\lambda)$ behaves like the resolvent of a matrix near its pole. Together with the fact that traces of nilpotent operators [12] are 0, we obtain that $f_k(\lambda) = -e^{i\lambda t_0}\mathrm{tr}^\flat\left(\varphi^*_{-t_0}\mathbf{R}_k(\lambda)\right)$ has simple poles and integral residues. Hence, its meromorphic continuation follows.

A few years after Dyatlov-Zworski's paper, Dyatlov-Guillarmou was able to show the meromorphic continuation of the dynamical zeta function ζ_R for Axiom A flow over general vector bundles using similar but technically more sophisticated methods.

This has opened up new classes of models on which the meromorphic continuation of its dynamical zeta function can be studied. We briefly discuss one particular branch in the next section.

3 Order of Vanishing at Zero

As pointed out in the opening section, the dynamical zeta function carries statistical information about the underlying dynamical system. For instance, given the meromorphic continuation of the zeta function for (topologically weakly mixing) Anosov flow, the first pole h_ϕ is simple and $\zeta_R(s)$ has no zero on the line $\mathfrak{Re}(s) = h_\phi$, where h_ϕ is also known as the topological entropy of the flow ϕ and defined as

$$h_\phi = \lim_{T\to\infty} \frac{1}{T} \log N(T) > 0$$

[12] find a reference (maybe linear algebra via exterior forms)

$N(T)$ denotes the counting function for primitive orbits of length less than T. In particular, we have the *prime orbit theorem* analogous to the prime number theorem:

$$N(T) \sim \frac{e^{h_\phi}T}{h_\phi T} \quad \text{asymptotically as } T \to \infty$$

For geodesic flows on compact negatively curved surfaces, Dolgopyat was able to show in 1998 that there exists $\varepsilon > 0$ such that $\zeta_R(s)$ has an analytic zero-free extension to $\mathfrak{Re}(s) > h_\phi - \varepsilon$, hence improving the error term in the prime orbit theorem:

$$N(T) = \int_0^{e^{h_\phi T}} \frac{1}{\log u}\, du + O(e^{(h_\phi - \varepsilon)T})$$

For higher dimensional compact negatively curved manifolds with a $\frac{1}{9}$-pinching condition, similar conclusions were obtained by Giulietti-Liverani-Pollicott. Yet, we don't know if the pinching condition is necessary.

On the other hand, if we have a noncompact manifold with interesting dynamics, we are able to deduce the decay of correlation information from the meromorphic continuation. These cases include hyperbolic manifolds or asymptotically hyperbolic manifolds with strictly negative variable sectional curvature.

In particular, if (M, g) is one such manifold and $\varphi_t : SM \to SM$ is the corresponding geodesic flow, then we can define a correlation function as follows: Given $f, g \in C_0^\infty(SM)$,

$$\rho_{f,g}(t) = \int_{SM} (f \circ \varphi_{-t}) g \, d\mu$$

where $d\mu$ is the invariant Liouville measure. Then the long-term behavior of $\rho_{f,g}$ depends on how much we are able to meromorphically continue its Laplace transform

$$\hat{\rho}_{f,g}(\lambda) = \int_0^\infty e^{-\lambda t} \rho_{f,g}(t)\, dt = \langle R(\lambda) f, g \rangle_{L^2}$$

or effectively the resolvent operator $R(\lambda)$. Now existence of a simple pole $\lambda_0 < 0$ and no other poles/zeros on the line $\mathfrak{Re}(\lambda) = \lambda_0$ implies

$$\hat{\rho}_{f,g}(\lambda) = e^{\lambda_0 t} \left(\int_{SM} f \, d\mu \int_{SM} g \, d\tilde{\mu} + o(1) \right)$$

and the existence of a spectral gap gives rise to an exponential error term in the above relation. Nevertheless, we are instead interested in a region deep into the meromorphic continuation, i.e. meromorphic continuation of the Ruelle zeta function at zero. It is shown that topological information of the dynamical system can still be recovered.

Theorem 1. (Dyatlov-Zworski 17) Suppose M is a compact negatively curved surface, then near $s = 0$:

$$\zeta_R(s) = Cs^{-\chi(M)}(1 + \mathcal{O}(s))$$

Theorem 2. (Hadfield 18) Suppose M is a compact negatively curved surface with boundary, then near $s = 0$:

$$\zeta_R(s) = Cs^{1-\chi(M)}(1 + \mathcal{O}(s))$$

Theorem 3. (Borthwick-Judge-Perry 05) Suppose M is a geometrically finite hyperbolic surface, then

$$\zeta_R(s) = Cs^{1-\chi(M)}(1 + \mathcal{O}(s))$$

Theorem 4. Suppose M is a negatively curved asymptotically hyperbolic surface, then near $s = 0$:

$$\zeta_R(s) = Cs^{1-\chi(M)}(1 + \mathcal{O}(s))$$

Just like the analogy between L-function and Riemann zeta function, we can also consider a twisted version of the Ruelle zeta function

$$\zeta_\alpha(s) = \prod_{\gamma^\sharp} \det\left(1 - \alpha(\gamma^\sharp)e^{-\ell(\gamma^\sharp)s}\right)$$

for an arbitrary (not necessarily unitary) finitely dimensional complex representation $\alpha : \pi_1(S^*X) \to GL(N, \mathbb{C})$. Its meromorphic continuation follows exactly as before and we have:

Theorem 5. (Cekić-Paternain 20) Suppose M is a compact negatively curved surface and α is a unitary representation. Let $N(\alpha)$ denote dimension of the representation α, then near $s = 0$:

$$\zeta_\alpha(s) = Cs^{-\chi(M)N(\alpha)}(1 + O(s))$$

Theorem 6. (Frahm-Spilioti 23) Suppose M is a hyperbolic surface, then near $s = 0$:

$$\zeta_\alpha(s) = Cs^{-\chi(M)N(\alpha)}(1 + O(s))$$

However, the interesting question of the order of vanishing for surfaces of variable curvature with non-unitary twists remains open, and we refrain from mentioning a closely related but overwhelmingly vast study of Fried's conjecture.