Riemann Mapping Theorem

Guangqiu Liang

March 2020

Theorem. Suppose $\Omega \subset \mathbb{C}$ is simply-connected and $\Omega \neq \mathbb{C}$. Then there exists a ono-to-one analytic function f of Ω onto $\mathbb{D} = \{z : |z| < 1\}$. If $z_0 \in \Omega$ then there is a unique such map with $f(z_0) = 0$ and $f'(z_0) > 0$.

The main recipe of the following proof is the use of normal family to solve an extremal problem, and the proof is outlined in Complex Analysis, Marshall. Two lemmas needed throughout the proof will be provided at the end.

Proof. We present the proof in three step: First we show that for fixed $z_0 \in \Omega$, the family

$$\mathcal{F} = \{f : f\text{is one-to-one, analytic, } |f| < 1 \text{ on } \Omega, f(z_0) = 0, f'(z_0) > 0\}$$

is nonempty and normal. Then we show that there exists a function $f \in \mathcal{F}$ that maximizes the set $\mathcal{F}' = \{f'(z_0) : f \in \mathcal{F}\}$. Finally, we show that the function f we found with the optimal derivative is indeed to Riemann map we are looking for.

By the assumption that $\Omega \neq \mathbb{C}$, then there exists $z_1 \in \mathbb{C} \setminus \Omega$. Since Ω is simply-connected and $z - z_1 \neq 0$ in Ω, the logarithm $\log(z - z_1)$ is well-defined and analytic on Ω. Let $g_1(z) = \sqrt{z - z_1}$. Now we show that g_1 is one-to-one: Suppose $g_1(z) = g_1(w)$, squaring both sides gives $z - z_1 = w - z_1$ and implies $z = w$. Moreover, if $z \neq w$, we also have $g_1(z) \neq -g_1(w)$ by the same argument above. Thus, $g_1(z)$ is univalent.

Choose $z_2 \in \Omega$ such that $g_1(z_2) \neq 0$, then there exists $r > 0$ such that $B(g(z_2), r) \subset g(\Omega)$ by the open mapping theorem. Moreover, $B(-g(z_2), r') \subset g(\Omega)^C$ for some r'. Thus, $|g_1(z) + g(z_2)| \geq r'$ and hence $g_2(z) = r'/(g_1(z) + g(z_2))$ maps Ω conformally onto a subset of $\mathbb{D}$. Moreover, we can choose an automorphism T to get $f = T \circ g_2(z)$ such that $f(z_0) = 0$. Moreover, we have the freedom to choose the rotational constant to make $f'(z_0) > 0$. Thus, $\mathcal{F} \neq \emptyset$.

Now since $\mathcal{F}$ is a family of globally bounded analytic functions, by Lemma 1 (an analytic version of Arzela-Ascoli theorem), $\mathcal{F}$ is normal. Moreover, choose a sequence $\{f_n\} \subset \mathcal{F}$ such that

$$\lim_{n\to\infty} f_n'(z_0) = M = \sup\{f'(z_0) : f \in \mathcal{F}\}$$

By normality, we can relabel $\{f_n\}$ with a subsequence that converges uniformly on compact subsets of Ω. Now by Weierstrass theorem, the limit function f is analytic in Ω, $|f| \leq 1$

and $\{f_n'\}$ converges uniformly to f'. Thus $f'(z_0) = M$. In particular, this implies M is bounded and $M \neq 0$. Moreover, $f(z_0) = \lim_{n\to\infty} f_n(z_0) = 0$ and f is one-to-one by Hurwitz theorem. Thus, $f \in \mathcal{F}$.

Finally, we want to show that $\mathbb{D} \subset f(\Omega)$ by contradiction: Suppose there is $w \in \mathbb{D} \setminus f(\Omega)$, then the function

$$h_1(z) = \frac{f(z) - w}{1 - \overline{w} f(z)} \equiv T_1 \circ g_1(z)$$

is nonzero analytic function on the simply-connected domain Ω. Thus, we can define the logarithm function and thereby an analytic square root $h_2(z) = \sqrt{h_1(z)}$ which is also one-to-one. Composing h_2 with a LFT gives

$$h(z) = \frac{h_2(z) - h_2(z_0)}{1 - \overline{h_2(z_0)} h_2(z)} \equiv T_2 \circ g_2(z)$$

which fixes $h(z_0) = 0$. We may also choose a rotational constant $\lambda = |h'(z_0)|/h'(z_0)$ so that $\lambda h \in \mathcal{F}$. Now if we look at the composition

$$\varphi = T_1^{-1} \circ S \circ T_2^{-1}, \quad S(z) = z^2$$

which is an automorphism of the disk and which fixes the origin $\varphi(0) = 0$. Moreover, from the other direction, $f(z) = \varphi \circ h(z)$, equivalently $\varphi(z) = f \circ h^{-1}(z)$. By Schwarz's lemma, we have $|\varphi'(0)| = |(f \circ h^{-1})'(0)| < 1$, which implies $|f'(z_0)| < |h'(z_0)|$ by the inverse function theorem. Now we have found another function in $\mathcal{F}$, namely λh, that has a derivative larger than the optimal value of $f'(z_0)$, which is a contradiction. Thus, $\mathbb{D} \subset f(\Omega)$ and therefore $f(\Omega) = \mathbb{D}$. Moreover, the uniqueness of the conformal map f follows from Lemma 2.

Lemma 1. The following are equivalent for a family $\mathcal{F}$ of analytic functions on a region Ω: (i) $\mathcal{F}$ is normal on Ω; (ii) $\mathcal{F}$ is locally bounded on Ω; (iii) $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ is locally bounded on Ω and there is a $z_0 \in \Omega$ so that $\{f(z_0) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$.

Proof. Suppose $\mathcal{F}$ is normal. By the proof of Arzela-Ascoli theorem or Exercise 10.2, we know that $\mathcal{F}$ is globally bounded and thereby locally bounded. Now suppose $\mathcal{F}$ is locally bounded, that is $|f| \leq M$ on a closed disk $\overline{B(z_1, r)} \subset \Omega$ for all $f \in \mathcal{F}$ and all z in the disk.

Now by Cauchy's estimate, we have

$$|f'(z)| \leq \frac{\sup |f|}{r/2} = \frac{2M}{r}$$

which implies $\mathcal{F}'$ is locally bounded, and (iii) holds. Finally, if (iii) holds and $z_1 \in \Omega$, then $|f'(z)| \leq L < \infty$ for all z in a disk $B(z_1, r)$. Integrating f' along a line segment from z_1 to z gives the estimate $|f(z) - f(z_1)| \leq L|z - z_1|$ for all $f \in \mathcal{F}$. By definition $\mathcal{F}$ is equicontinuous at z_1. Since z_1 is arbitrarily chosen, we know $\mathcal{F}$ is equicontinuous on Ω. By Arzela-Ascoli, $\mathcal{F}$ is normal.

Lemma 2. If there exists a conformal map of a region Ω onto $\mathbb{D}$, then given any $z_0 \in \Omega$, there exists a unique conformal map f of Ω onto $\mathbb{D}$ such that $f(z_0) = 0$ and $f'(z_0) > 0$.

Proof. If g is a conformal map of Ω onto $\mathbb{D}$, then

$$h(z) = c\frac{z - g(z_0)}{1 - \overline{g(z_0)}z}$$

is an automorphism of $\mathbb{D}$ for $|c| = 1$, and $f = h \circ g$ maps Ω onto $\mathbb{D}$ with $f(z_0) = 0$. Then, a direct computation shows $f'(z_0) = cg'(z_0)/(1 - |a|^2)$, and we have the freedom to choose the rotational constant c so that $f'(z_0) > 0$.

Now suppose k also maps Ω onto $\mathbb{D}$ with $k(z_0) = 0$ and $k'(z_0) > 0$, then $H = k \circ f^{-1}$ is an automorphism of the disk with $H(0) = 0$. By Schwarz's lemma, $|H(z)| \leq |z|$ and similarly $|H^{-1}(z)| \leq |z|$ so that $|H(z)| = |z|$ and $H(z) = cz$, with $|c| = 1$. Since $H'(0) = k'(z_0)/f'(z_0) > 0$ by assumption, we have $c = 1$ and therefore $k = f$.