Open Mapping Theorem (functional analysis)

Guangqiu Liang

February 2020

Theorem. If X, Y are Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, and if $T : X \to Y$ is a bounded linear surjective map, then T is open.

The following proof is outlined in a homework exercise of UW Math 425 (Fundamentals of Mathematical Analysis). The major weaponry we need are Baire's Category Theorem, the completeness of X and Y, and repeated use of the rescaling argument.

First we show that open mapping theorem can be reduced to its equivalent statement: Let X, Y be Banach spaces with norms $\|\cdot\|_X$, $\|\cdot\|_Y$, and if $T : X \to Y$ is a bounded linear surjective map, then there exists $\epsilon > 0$ such that $B_Y(0, \epsilon) \subset T(B_X(0, 1))$ where $B_X(0, 1)$ is the unit open ball in X:

Suppose the statement holds, we first extend the statement by an arbitrary scalar: For any $y \in B_Y(0, r\epsilon)$ for any $r > 0$, we know that $\tilde{y} = y/r \in B_Y(0, \epsilon)$. By assumption, we know that there exists an $\tilde{x} \in B_x(0, 1)$ such that $T(\tilde{x}) = \tilde{y}$. By the linearity of T, we have $T(r\tilde{x}) = y$. Thus, we have found an $x = r\tilde{x} \in B_X(0, r)$ such that $T(x) = y$, implying $B_Y(0, r\epsilon) \subset T(B_X(0, r))$.

Now we extend the statement by a translation: For any $y \in B_Y(b, \epsilon)$ for any $b \in Y$, we know that $\tilde{y} = y - b \in B_Y(0, \epsilon)$. By assumption, there exists an $\tilde{x} \in B_X(0, 1)$ such that $T(\tilde{x}) = \tilde{y}$. By linearity again and the surjectivity of T, we have $T(a + \tilde{x}) = y$ where $T(a) = b$. Thus, we have found an $x = a + \tilde{x} \in B_X(a, \epsilon)$ such that $T(x) = y$, implying that $B_Y(b, \epsilon) \subset T(B_X(a, 1))$.

Combing the arguments above, we have that for any $x \in X$ and $r > 0$, let $T(x) = y$, then there exists an $\epsilon > 0$ such that $B_Y(y, \epsilon) \subset T(B_X(x, r))$. Now consider an open subset U of X, for any point $y \in T(U)$, by the surjectivity of T, we know there exists an $x \in U$ such that $T(x) = y$. Moreover, by the openness of U, we know there exists an $r > 0$ such that $B_X(x, r) \subset U$. By what we have shown above, we know that there exists an $\epsilon > 0$ such that $B_Y(y, \epsilon) \subset T(B_X(x, r)) \subset T(U)$. Since y is arbitrary in U, we have shown that $T(U)$ is open, implying that T is an open map.

Now we fill up the space X with the balls $B_j = B_X(0, j), j = 1, 2, \ldots$. Clearly, $X = \bigcup_j B_j$. Since T is surjective, we have $T(X) = T(\bigcup_j B_j) \subset \bigcup_j T(B_j)$. Once again, $T(B_j) \subset Y$ for each j, thus $\bigcup_j T(B_j) \subset Y$ and $Y = \bigcup_j T(B_j)$. By Baire's category theorem, we know that there exists some $J \in \mathbb{N}$ such that $\overline{T(B_J)}$ has nonempty interior, which means there exists a $c \in \overline{T(B_J)} \subset Y$ and $r > 0$ such that $B_Y(c, r) \subset \overline{T(B_J)} \subset \overline{T(B_J)}$.

Now we extend the result we get above by rescaling and translation: Let $z \in B_Y(0,1)$, we know that $cz + r \in B_Y(c,r) \subset \overline{T(B_J)}$. Moreover, we know that $cz + r \in T(B_J)$ and $c \in T(B_J)$. Thus, by the surjectivity of T, there exist $d, w \in B_J$ such that $T(d) = c$ and $T(w) = z$.

Now in the space X, by triangle inequality we have

$$\|rw\|_X = \|rw + d - d\|_X \leq \|rw + d\|_X + \|d\|_X \leq 2J$$

Thus, $rw \in B_X(0, 2J)$ and $rz = T(rw) \in T(B_X(0,2J)) \subset \overline{T(B_X(0,2J))}$. Again by linearity, it implies

$$B_Y(0,1) \subset \overline{T(B_X(0,2J/r))}$$

Let $M = 2J/r$ and $z \in B_Y(0,1)$, by the surjectivity of T we know there exists a $w \in B_X(0,M)$ such that $T(w) = z$. Moreover, since T is a linear bounded map between Banach spaces, we know that T is also continuous. Thus, given $\epsilon = 1/2$ there exists a $\delta > 0$ such that

$$\|z - Tx\|_Y < 1/2 \quad \text{whenever} \quad \|w - x\|_X < \delta$$

Since X is a Banach space, we know $(B_X(w,\delta) \bigcap B_X(0,M)) \setminus \{w\} \neq \emptyset$. Thus we know such $x \in B_X(0,M)$ and $x \neq w$ exists.

Now we show that for any $y \in B_Y(0, 2^{-n+1})$, there exists $x \in B(0, 2^{-n+1}M)$ such that $\|y - Tx\|_Y \leq 2^{-n}$ by rescaling: Let $\tilde{y} = (2^{n-1})y \in B_Y(0,1)$, by the previous part we know that there exists an $\tilde{x} \in B_X(0,M)$ such that $\|\tilde{y} - T(\tilde{x})\|_Y \leq 1/2$, which by induction implies $\|y - T(2^{-n+1}\tilde{x})\|_Y \leq 2^{-n}$. Since $2^{-n+1}\tilde{x} \in B_X(0, 2^{-n+1}M)$, we have found the $x = (2^{-n+1})\tilde{x}$ that satisfies $\|y - Tx\|_Y \leq 2^{-n}$.

Now we construct a sequence $\{x_n\}$ by picking $y_n \in B_Y(0, 2^{-n+1})$ for each $n \in \mathbb{N}$ where $y_n = y_{n-1} - T(x_{n-1})$ and $y_0 = y, x_0 = 0$, the above result shows that there exists a corresponding $x_n \in B_X(0, 2^{-n+1}M)$ such that $\|y_n - Tx_n\|_Y \leq 2^{-n}$ holds.

Since $x_n \in B_X(0, 2^{-n+1}M)$ for each n, we know $\|x_n\|_X \leq 2^{-n+1}M$. Moreover, when $n = 1$, we have $\|y - T(x)\|_Y \leq 1/2$ following from above. Suppose the inequality holds for n, then by construction

$$\left\| y - \sum_{i=1}^{n+1} T(x_i) \right\|_Y = \|y_{n+1} - T(x_{n+1})\|_Y \leq 2^{-(n+1)}$$

Since M is finite and the norm $\|\cdot\|_X$ is continuous, we know that

$$\|x\|_X = \lim_{I \to \infty} \left\| \sum_{i=1}^{I} x_n \right\|_X \leq \lim_{I \to \infty} \sum_{i=1}^{I} \|x_n\|_X \leq M \lim_{I \to \infty} \sum_{i=1}^{I} 2^{-i+1} = 2M$$

and for any $\epsilon > 0$, taking N large enough gives

$$\left\| y - T\left(\sum_{i=1}^{N} x_i \right) \right\|_Y \leq 2^{-N} < \epsilon$$

Since ϵ is arbitrary, we see that $\|y - Tx\|_Y = 0$ in the limit, which implies $y = Tx$.

Let $y \in B_Y(0, 1/2M)$, consider $\tilde{y} = 2My \in B_Y(0, 1)$. We know that there exists an $\tilde{x} \in B_X(0, 2M)$ such that $T(\tilde{x}) = \tilde{y}$, which by linearity implies $T(\tilde{x}/2M) = y$. Thus we have found an $x = \tilde{x}/2M \in B_X(0, 1)$ such that $T(x) = y$, which implies $y \in T(B_X(0, 1))$. Since y is an arbitrary point in $B_Y(0, 1/2M)$, we know that $B_Y(0, 1/2M) \subset T(B_X(0, 1))$. Therefore, we have found an $\epsilon = 1/2M$ that shows the reduced statement, and T is thereby an open map.