Theorem. A non-constant analytic function $f$ defined on a region $\Omega$ is an open map.

Proof. Given any open set $U \subset \Omega$, for any $z_0 \in U$, there exists $R > 0$ such that $f$ has a power series expansion that converges on $\{|z - z_0| < R\} \subset U$. Now pick $r < R$ and let 

$$
\delta = \inf_{|z-z_0|=r} |f(z) - f(z_0)|
$$

Since we assume $f$ is non-constant, thus the zeros of $f - f(z_0)$ are isolated by the identity theorem, which enables us to find an $r$ so that $\delta > 0$.

Now we show $\{|f(z) - f(z_0)| < \delta/2\}$ is a neighborhood of $f(z_0)$ that is contained in the image of $\{|z - z_0| < r\}$ under $f$, thereby implying $f$ is an open map. We proceed by contradiction: Suppose there exists $w$ such that $|w - f(z_0)| < \delta/2$ but $f(z) \neq w$ for all $z \in \{|z - z_0| \leq r\}$, then $1/(f(z) - w)$ is analytic in $\{|z - z_0| \leq r\}$, and 

$$
\left| \frac{1}{f(z) - w} \right| \leq \frac{1}{|f(z) - f(z_0)| - |w - f(z_0)|} < \frac{1}{\delta - \delta/2} = \frac{2}{\delta}
$$

on $\{|z - z_0| = r\}$, where the first inequality follows from the reverse triangle inequality that

$$
|f(z) - w| = |f(z) - f(z_0) + f(z_0) - w| \geq |f(z) - f(z_0)| - |w - f(z_0)|
$$

and the second from the assumptions that $|f(z) - f(z_0)| \geq \delta$ and $|w - f(z_0)| < \delta/2$.

Now applying maximum principle to $1/(f(z) - w)$, we have $|1/(f(z) - w)| < 2/\delta$ for all $|z - z_0| < r$, which implies $|f(z_0) - w| > \delta/2$, a contradiction. This shows the image of an open set contains a neighborhood of each of its points, and thus $f$ is open.

Remark: The above little ingenious proof is given in Complex Analysis by Marshall, which my teacher Steffen Rohde describes it as the most elegant one among the standard complex analysis textbooks.