Kakeya Conjecture over Finite Fields

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In 1917, Japanese mathematician Sōichi Kakeya proposed the following question:

Kakeya Problem. What is the minimum area of a region $D \subset \mathbb{R}^2$ in which a line segment of unit length can be turned through 360°?

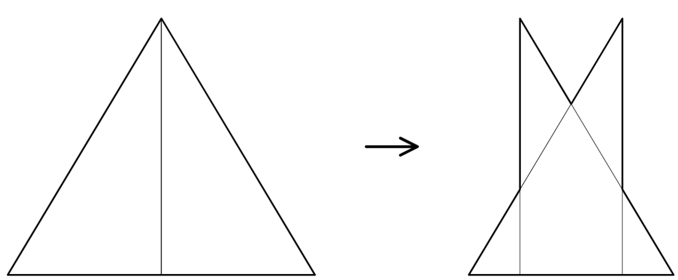
Examples of Kakeya sets are disk of radius 1/2, equilateral triangle of height 1, and deltoid of height 1, which have areas $\pi/4$, $\sqrt{3}/3$, and $\pi/8$ respectively. If we require D to be convex, Kakeya & Fujiwara conjectured that the equilateral triangle of unit height achieves the minimum, which was proved by Pal in 1920. In the non-convex case, Kakeya himself seemed to conjecture that the deltoid is optimal. However, Besicovitch showed in 1928 the following stunning result:

Theorem 1. Given any $\varepsilon > 0$, there exists compact $D \subset \mathbb{R}^2$ in which a unit line segment can be turned through 360°.

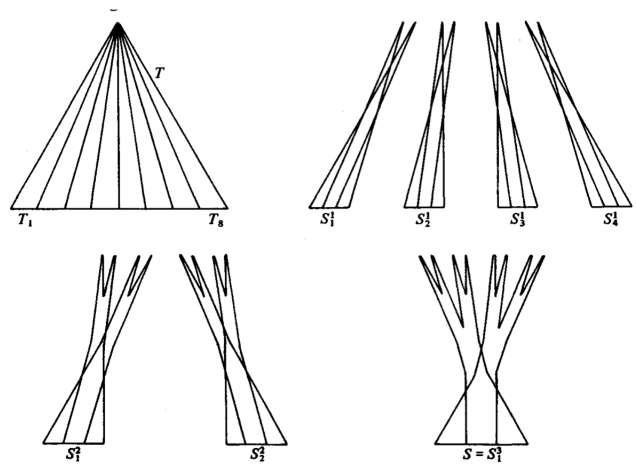
Remark. Besicovitch's construction is based on the following two observations:

1. One can translate a needle using arbitrarily small area, following a path of shape "N".
2. Define a **Kakeya set** in $\mathbb{R}^2$ to be any set which contains a unit line segment in every direction, then there exists Kakeya set of arbitrarily small area.

We illustrate briefly on the second observation, which is not entirely straightforward.



The idea is to preserve the angle that the needle can sweep while reducing the area needed. For instance, the above cut and sliding construction maximally reduces the area while keeping the angle that the needle can sweep through to be 60°.



The main difficulty in justifying the construction actually works lies in the crucial estimate that at each stage, the overlapping annihilates enough amount of area. Those who are interested in a detailed proof are directed to Falconer's *Geometry of Fractal Sets.*

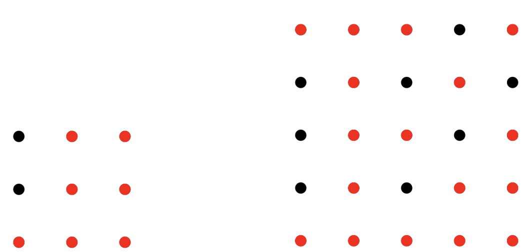
In fact, Davies showed in 1971 that there exist Kakeya sets in the plane of measure zero. At the same time, he also observed that they must have (Hausdorff) dimension 2, which motivates the general Kakeya conjecture.

Kakeya Conjecture. A Kakeya set E in $\mathbb{R}^n$ (i.e. a set that contains a unit line segment in every direction) has dimension n.

The conjecture remains open today since it was proposed in the 70s. It appears that as dimension grows, the problem becomes harder. Except for some partial results, for example $\dim E > 5/2$ when $n = 3$, it looks like the conjecture is far from being solved. However, increasing interests were raised when Wolff proposed the finite field analogue of the Kakeya conjecture as a model problem.

Definition 2. Let F be a finite field, and F^n be the vector space over F. Let $E \subset F^n$ be such that for any $v \in F^n \setminus \{0\}$, there exists $x \in F^n$ such that the line $L_{x,v} = \{x + tv : t \in F\}$ is contained in E, then E is a **Kakeya set** over F.

Examples of Kakeya set when $n = 2$ and $F = \mathbb{Z}_3$ or $F = \mathbb{Z}_5$.



A characterizing feature of a set $E \subset \mathbb{R}^n$ of Hausdorff dimension $s \leq n$ is that $\mathcal{H}^s(rE) = r^s\mathcal{H}^s(E)$, namely the measure of the set is scaled proportionally with respect to the s-dimensional volume when the set is scaled. Hence, the finite field analogue conjecture can be stated as follows:

Kakeya Conjecture over Finite Fields. Let $E \subset F^n$ be a Kakeya set. Then $|E| \geq c_n|F|^n$ for some $c_n > 0$ that only depends on n.

We present a beautiful proof of this conjecture by Dvir in 2008, using polynomial methods from combinatorics. In particular, we would like to control the size of a set E by studying polynomials vanishing on E. We have the following basic facts from high school:

Theorem 3. (Factor Theorem) Let F be a field and $d \geq 0$ be an integer.

1. If $p(x) \in F[x]$ nontrivial and $\deg p(x) \leq d$, then p has at most d roots.
2. If $E \subset F$ and $|E| \leq d$, there exists a nontrivial $p \in F[x]$ such that $\deg p(x) \leq d$ and $p(x) = 0$ on E.

From 2, we see that to lower bound the set E, it suffices to show that the only small degree polynomials vanishing on E is the zero polynomial. We will achieve this through the following two higher-dimensional results:

Theorem 4. Let $E \subset F^n$ with $|E| < \binom{n+d}{n}$ for some d, then there exists a nontrivial $p \in F[x_1, \ldots, x_n]$ such that $\deg p \leq d$ and p vanishes on E.

Proof. The proof is a simple dimension counting argument. Let V be the vector space of polynomials in $F[x_1, \ldots, x_n]$ of degree less than or equal to d, then by balls-and-urns formula, $\dim V = \binom{n+d}{n}$. On the other hand, since F is a field, the set of functions $W = \{f : E \to F\}$ has a vector space structure. In particular, $\dim W = |E|$, as the basis is given by the characteristic function on each element of E.

Thus, the evaluation homomorphism $ev_E : V \to W$ defined as $ev_E(p) = (p(x))_{x \in E}$ has nontrivial kernel. Thus, there exists nontrivial $p \in V$ such that $p \neq 0$ and $p(x) = 0$ on E. $\square$

Theorem 5. Let $P \in F[x_1, x_2, \ldots, x_n]$ be a polynomial of degree at most $|F| - 1$ and P vanishes on a Kakeya set E, then P is the zero polynomial.

Proof. Suppose for the sake of contradiction that P is nonzero, and let $P = \sum_{i=1}^d P_i$ be the corresponding homogeneous decomposition, where $0 \leq d \leq |F| - 1$. Then we must have $P_d \neq 0$.

Now for any $v \in F^n \setminus \{0\}$, we know that there exists $x \in E$ such that the line $\{x + tv : t \in F\} \subset E$ and by hypothesis $P(x + tv) = 0$ for all $t \in F$. Thinking of $P(x + tv)$ as a polynomial of degree d in t, we have that the polynomial $P(x + vt)$ vanishes at $|F|$ points. However, since $d \leq |F| - 1$, so $P(x + tv)$, as a polynomial in t, is of degree at most $|F| - 1$. By the factor theorem, we know that the polynomial $P(x + tv)$ in t vanishes identically, which implies the coefficient $P_d(v)$ of t^d vanishes for any nonzero v.

Since P_d is homogeneous of degree $d > 0$, we have P_d vanishes on all of F^n. But since $d < |F|$, then applying the factor theorem repeatedly to each variable x_i shows $P_d = 0$, a contradiction. □

Corollary 6. Every Kakeya set in F^n has cardinality at least $\binom{n+|F|-1}{n}$.

In particular, the corollary implies the Kakeya conjecture over finite fields:

$$|E| \geq \binom{n+|F|-1}{n} = \frac{(n+|F|-1)(n-1+|F|-1)\cdots(|F|)}{n!} \geq \frac{1}{n!}|F|^n$$

We end our discussion by some interesting remarks on the current progress of the Kakeya conjecture and its application in other areas of math:

1. The original Kakeya conjecture ($F = \mathbb{R}$) still remains far from reach; the currently best estimate we have is about $\dim_{\mathcal{H}} E > \frac{n+2}{2}$ (1995, Wolff/Katz/Tao). However, there has been interesting developments in the case when F is a ring in the past two years.

   In Nov 2020, Dvir & Dhar proved that the case when $F = \mathbb{Z}/n\mathbb{Z}$ where n is square-free.

   In Aug 2021, Arsovski proved the case when $F = \mathbb{Q}_p$, the p-adic numbers.

   In Oct 2021, Dhar proved the case when $F = \mathbb{Z}/n\mathbb{Z}$ for any $n \in \mathbb{N}$.

   In Feb 2022, Salvatore generalized the result to local fields of positive characteristic.

2. The existence of Kakeya set in $\mathbb{R}^n$, which can be obtained by $E \times [0,1]^{n-2}$ for any Kakeya set $E \subset \mathbb{R}^2$, is surprisingly useful in the study of harmonic analysis. In particular, it is used by Fefferman in 1971 to prove the following striking result related to the L^p convergence of Fourier inversion formula:

   Theorem 7. If $n > 1$, then S_R is unbounded for any $p \neq 2$.

   where the operator S_R is defined as the following:

   $$S_R f(x) = \int_{|x| \leq R} \hat{f}(\xi) e^{2\pi i \xi \cdot x} \, d\xi$$

   where $\hat{f}(\xi)$ is the Fourier transform of f. Upon invoking the uniform boundedness principle, Fefferman's theorem implies that $S_R f \nrightarrow f$ unless $p = 2$.

3. There are fascinating connections among the Kakeya conjecture, arithmetic combinatorics (in particular the sum-product phenomenon), and dispersive linear PDE. Terence Tao's survey article "From rotating needles to stability of waves: emerging connections between combinatorics, analysis, and PDE." explains these connections in details.