Inverse Spectral Problem

Guangqiu Liang

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2022.1.21 Background (Yiran Wang)

Given a compact Riemannian manifold (M, g), consider the Laplace-Beltrami operator $\Delta_M : C^\infty(M) \to C^\infty(M)$ defined as $\Delta f = -\text{div}(\text{grad} f)$. In local coordinates,

$$\Delta := -\frac{1}{\sqrt{g_{ij}}}\partial_i(\sqrt{g_{ij}}g^{ij}\partial_j)$$

Consider Δ acting on $L^2(M, dV)$ where dV is the volume measure induced by the metric. By Green's identity, Δ is positive and symmetric on $C_0^\infty(M)$. By the Friedrich's extension method, Δ is self-adjoint on $\mathcal{D}(\Delta) = \{u \in H_0^1(M) : \Delta u \in L^2(M)\}$ and essentially self-adjoint on $C^\infty(M)$.

A classical spectral theory result gives $\sigma(\Delta) = \sigma_{pt}(\Delta) \subset \mathbb{R}_{\geq 0}$ with $\lim_{k\to\infty} \lambda_k = \infty$. Now the question is what geometric information of (M, g) can be inferred from the eigenvalues $\{\lambda_k\}$. A related question is concerned with the spectrum of the Schrödinger operator $\Delta + V$ where V is a scalar function.

One approach is to consider functions defined in terms of the eigenvalues (i.e. the spectral invariants). For example, the heat trace, also known as the Minakshisundaram–Pleijel zeta function,

$$Z(t) = \text{tr}(e^{-t\Delta_M}) = \sum_{j=1}^{\infty} e^{-t\lambda_j}, \quad t > 0$$

If looking at the behavior of $Z(t)$ as $t \to 0^+$, we get the Pleijel asymptotic expansion,

$$\sum_{j=1}^{\infty} e^{-t\lambda_j} \sim (4\pi t)^{-n/2} \sum_{n=0}^{\infty} a_n t^n \sim \frac{\text{vol}(M)}{2\pi t} - \frac{\ell(\partial M)}{4\sqrt{2\pi t}} + o\left(\frac{1}{\sqrt{t}}\right)$$

By the classical isoperimetric inequality, we have $\ell(\partial M)^2 \geq 4\pi \text{vol}(M)$ with equality holds if and only if M is a disk. Thus, we can "hear" the shape of a disk from its eigenvalues. In fact, disks are the only Lipschitz planar domains that are determined by their spectra. Recently, Hezari and Zelditch proved ellipses of sufficiently small eccentricity are spectrally determined among all smooth domains.

In particular, we ask if two Riemannian metrics g_1 and g_2 are **isospectral** (i.e. $\sigma(\Delta_{M_1}) = \sigma(\Delta_{M_2})$), what information on g_1 and g_2 can we get. One of the most famous counterexamples comes from Sunada's method of constructing pairs of isospectral manifolds. Instead, we study

1. How big is the set of isospectral metrics?

2. Can we conclude anything on a more restricted class of manifolds?

For the first question, there is a decent answer given by Osgood, Phillips, and Sarnak that the set of isospectral metrics is compact in C^∞ topology. Their method uses the notion of the determinant of Laplacian. Roughly speaking, they prove a local uniqueness result so that if $\sigma(\Delta_{g_1}) = \sigma(\Delta_{g_2})$ for g_1 close to g_2, then g_1 is isometric to g_2.

Finally, we look at spectral rigidity & length rigidity problems, which oftentimes involves the study of another spectral invariant, namely the wave trace,

$$W(t) = \operatorname{tr}(e^{it\sqrt{\Delta}}) = \sum_{j=1}^{\infty} e^{-i\sqrt{\lambda_j}t}$$

defined as a distribution $\mathcal{S}(M)$ because the above series does not converge in the usual sense. In terms of the length rigidity problem, we define the length spectrum as the set of lengths of closed geodesics on M. Hence, it is natural to focus on manifolds with an abundance of closed geodesics, such as

1. compact negatively curved manifold

2. Anosov manifold

3. Zoll surface

2022.1.28 V. Guillemin and D. Kazhdan, 1980 (Guangqiu Liang)

Given a compact Riemannian manifold (M, g), we would like to know how much information about the manifold that we can extract from the spectrum of the associated Laplace-Beltrami operator. For $f \in C^\infty(M)$, the Laplacian Δ_g is defined as $\Delta_g(f) = -\text{div}(\text{grad} f)$. In local coordinates,

$$\Delta_g := -\frac{1}{\sqrt{\det g}}\partial_i(g^{ij}\sqrt{\det g}\,\partial_j)$$

By Green's identity,

$$\int_M g(\nabla f, \nabla f)\,dV = \int_M f\Delta_g f\,dV \geq 0,$$

Δ is positive and symmetric on $C_0^\infty(M)$. By the Friedrich's extension method, Δ is self-adjoint on $\mathcal{D}(\Delta) = \{u \in H_0^1(M) : \Delta u \in L^2(M)\}$ and essentially self-adjoint on $C^\infty(M)$. A classical result in functional analysis gives $Spec(\Delta) \subset \mathbb{R}_{\geq 0}$ is discrete with eigenvalues λ_i satisfying $\lim_{k\to\infty} \lambda_k = \infty$.

Here we note that if the Riemannian manifold (M, g) is non-compact, then by Gaffney's theorem, we have that Δ_g is still essential self-adjoint on $C^\infty(M)$. However, $\sigma(\Delta_g)$ will be continuous, and geometric information are much harder to extract. We are forced to work on more restrictive class of manifolds, such as hyperbolic surfaces, and the resonances are studied as an analog of eigenvalues in the compact case. For those who are interested, Borthwick's book *Spectral Theory of Infinite-Area Hyperbolic Surfaces* is an excellent reference for this type of theory.

Mathematically speaking, the *Spec* operator maps a Riemannian metric to a sequence of nonnegative integers in $\mathbb{R}$, and we would like to study given a sequence $\{\lambda_i\}$, what can we say about $Spec^{-1}(\{\lambda_i\})$. Our dream is $Spec^{-1}(\{\lambda_i\})$ is a singleton set, which means the Riemannian manifold is uniquely determined by its spectrum. Such problems are common and important because the spectra are oftentimes the only detectable information in practice. See Kac's beautiful paper "Can one hear the shape of a drum?" for a broader understanding.

From Yiran's introduction, we know this is true for the disk. However, the dream is not true in general, as Sunada (1985) proved there is a systematic way of constructing pairs of isospectral but non-isometric (compact, connected) manifolds.

Instead, we ask a weaker question: Are Riemannian manifolds **spectrally rigid**? Roughly speaking, we would like to know given a smooth perturbation of a Riemannian metric on X, the associated $\sigma(\Delta_g)$ has to be perturbed.

Definition 1. A family of Riemannian metrics $g_t, 0 \leq t \leq 1$, is a **deformation** of g_0 if g_t is smooth in t. If there exists a family of diffeomorphisms ϕ_t such that $\phi_0 = \text{Id}$ and $g_t = \phi_t^* g_0$, then we say the deformation g_t is **trivial**. Moreover, if $Spec(\Delta_{g_t}) = Spec(\Delta_{g_s})$ for all $0 \leq t \leq s \leq 1$, then the deformation is **isospectral**.

Definition 2. A Riemannian manifold (M, g) is **spectrally rigid** if it does not admit non-trivial isospectral deformations.

It is conjectured that most Riemannian manifolds are spectrally rigid. Guillemin and Kazhdan (1980) showed that "most" contains negatively curved Riemannian 2-manifolds. We have the following theorem:

Theorem 3. Let M be a compact 2-manifold. If the curvature of (M, g) is everywhere negative, (M, g) is spectrally rigid.

Before going into the details, we would like to take a closer look at the assumptions of the theorem and see why they are necessary.

1. (Compactness) If the manifolds are non-compact, it is difficult to characterize its geodesics or spectrum. For instance, we know $Spec(\Delta_g)$ will be continuous.

2. (Negative Curvature) Negatively curved manifolds have many closed geodesics. In fact, Anosov showed that if (M, g) is negatively curved, then M has infinitely many distinct closed geodesics. Furthermore, these geodesics are dense in M.

3. (Dimension Two) This is historically the easiest case. Later the authors generalized the results to n-dimensional manifold subject to a pointwise curvature pinching condition. Croke and Sharafutdinov proved a full generality of compact negatively curved Riemannian manifold in 1998.

4. (Implicit Topological Requirements) By Gauss-Bonnet, M has genus strictly greater than 1. The k-th homotopy group $\pi_k(M) = 0$ for all $k > 1$. In fact, we cannot see these surfaces, meaning that they cannot be embedded into $\mathbb{R}^3$. The only way to realize them is through hyperbolic surfaces $\mathcal{H}/\Gamma$ for some properly discontinous isometry subgroup.

Next we state a fairly similar theorem in Guillemin & Kazhdan's paper. Let $q \in C^\infty(M)$, then $\Delta_g + q$ acting on $C^\infty(M)$ is known as the **Schrödinger operator**. Moreover, if the closed geodesics on M are isolated, non-degenerate, and of distinct periods, we say M has **simple length spectrum**, i.e. the set $\mathcal{L}_M = \{\ell(\gamma) : \gamma \text{ closed geodesics}\}$ has multiplicity one for each γ. Note that simple length spectrum is invariant under deformation.

Theorem 4. Let X be a compact negatively curved 2-manifold with simple length spectrum. Let q_1 and q_2 be smooth functions on X. Suppose $Spec(\Delta + q_1) = Spec(\Delta + q_2)$, then $q_1 = q_2$.

The proofs of Theorem 3 and Theorem 4 are very similar, which involve the use of dynamics and harmonic analysis on the cosphere bundle S^*M. Before presenting the main ingredients of the proof, we give a quick introduction to the language of symplectic geometry and dynamical system, which will be used frequently in the rest of the paper.

Definition 5. Let V be a finite-dimensional real vector space and $\omega : V \times V \to \mathbb{R}$ be a bilinear map.

1. ω is **alternating** (anti-symmetric) if $\omega(u, v) = -\omega(v, u)$ for all $u, v \in V$. In this case, w is also called a 2-form.

2. ω is **nondegenerate** if the linear map $\tilde{\omega} : V \to V^*$ defined by $\tilde{\omega}(u)(v) = \omega(u, v)$ is invertible. Equivalently, if $\omega(u, v) = 0$ for all $v \in V$, then $u = 0$.

A nondegenerate 2-form on V is called a **symplectic tensor**. A vector space V endowed with a symplectic tensor is called a **symplectic vector space**. Note that if V is a symplectic vector space, then $\dim V = 2n$. Transporting the idea to a smooth manifold, we say that M is a **symplectic manifold** if there exists a nondegenerate closed 2-form on M.

Example 6. With standard coordinates on $\mathbb{R}^{2n}$ denoted by $(x^1, \ldots, x^n, y^1, \ldots, y^n)$, the 2-form

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy^i$$

is symplectic. This is called the **standard symplectic form** on $\mathbb{R}^{2n}$.

In fact, by Darboux Theorem, the standard symplectic form is the only symplectic form on symplectic manifolds. To facilitate understanding, we make some simple comparisons between Riemannian manifolds and symplectic manifolds.

1. (Existence) All smooth manifolds admit Riemannian structures, but only some of them admit symplectic structures.

2. (Geometry) Riemannian manifolds have rich local geometry (abundant geometric invariants such as curvature), but by Darboux theorem, symplectic manifolds do not.

3. (Relation) The cotangent bundle of every Riemannian manifold has a symplectic structure. Geodesics on Riemannian manifolds lift to geodesic flows on their cotangent bundles.

2022.2.4 Guillemin & Kazhdan continued (Guangqiu Liang)

We elaborate a little more on the last remark: In general, we can define a tautological 1-form on T^*M by the bundle isomorphism induced by the symplectic form. Namely, given a symplectic manifold (M, ω), the map $\hat{\omega} : TM \to T^*M$ defined by $\hat{\omega}(X)(Y) = \omega(X, Y)$ for any $X, Y \in \mathfrak{X}(M)$ is an isomorphism.

Let (x^i, ζ_i) be a local coordinate of $(q, \varphi) \in T^*M$ where $q = (x^1, x^2, \ldots, x^n) \in M$ and $\varphi = \zeta_i dx^i \in T_q^*M$. Consider the natural projection map $\pi : T^*M \to M$ such that $\pi(q, \varphi) = q$, which induces the differential $d\pi : T(T^*M) \to TM$ and the pointwise pullback $d\pi^*_{(q,\varphi)} : T_q^*M \to T^*_{(q,\varphi)}(T^*M)$. We define the **tautological 1-form** on T^*M to be $\alpha_{(q,\varphi)} = d\pi^*_{(q,\varphi)}(\varphi)$. Note that α is defined intrinsically in terms of the structure of T^*M.

In local coordinates, we can compute for any $v \in T_{(x,\zeta)}(T^*M)$,

$$\begin{aligned} d\pi^*(dx^i)(v) &= dx^i(d\pi(v)) \quad \text{(definition of } d\pi^*) \\ &= d\pi(v)(x^i) \quad \text{(definition of } dx^i) \\ &= v(x^i \circ \pi) \quad \text{(definition of } d\pi) \\ &= d(x^i \circ \pi)(v) \quad \text{(definition of } d(x^i \circ \pi)(v) \end{aligned}$$

Therefore, $d\pi^*(dx^i) = d(x^i \circ \pi)$. Since the context is clear, we simplify the notation by stating $d\pi^*(dx^i) = dx^i$. Therefore, $\alpha_{(x,\zeta)} = d\pi^*_{(x,\zeta)}(\zeta_i dx^i) = \zeta_i dx^i$. It is clear that α is a smooth 1-form on T^*M.

Now let $\omega = -d\alpha$. Immediately, we have that ω is a closed 2-form on T^*M. Moreover, in local coordinates,

$$\omega = -d\alpha = -d(\zeta_i dx^i) = \sum_{i=1}^{n} dx^i \wedge d\zeta_i.$$

which is clearly nondegenerate. Thus, ω is a symplectic form on T^*M.

Next we discuss the relation between symplectic geometry and dynamical system, as the language will be useful in understanding the proof. Suppose (M, ω) is a symplectic manifold. For any smooth function $f \in C^\infty(M)$, we define the **Hamiltonian vector field of** f to be the smooth vector field X_f defined by

$$X_f = \hat{\omega}^{-1}(df), \quad i.e.\ X_f \lrcorner\, \omega = df.$$

where $\hat{\omega}$ is the natural bundle isomorphism induced by ω. Equivalently, X_f is the unique vector field that satisfies

$$\omega(X_f, Y) = df(Y) = Yf, \quad \text{for any } Y \in \mathfrak{X}(M).$$

In local (Darboux) coordinates (x^i, y^i), suppose

$$X_f = \sum_{i=1}^{n} \left(a^i \frac{\partial}{\partial x^i} + b^i \frac{\partial}{\partial y^i} \right) \quad \text{for some coefficients } a^i \text{ and } b^i.$$

Computing $X_f \lrcorner\, \omega$ gives

$$X_f \lrcorner\, \omega = \sum_{j=1}^{n} \left(a^j \frac{\partial}{\partial x^j} + b^j \frac{\partial}{\partial y^j} \right) \lrcorner \sum_{i=1}^{n} dx^i \wedge dy^i = \sum_{i=1}^{n} (a^i dy^i - b^i dx^i)$$

On the other hand,

$$df = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x^i} dx^i + \frac{\partial f}{\partial y^i} dy^i \right)$$

Comparing the coefficients gives

$$X_f = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial y^i} \frac{\partial}{\partial x^i} - \frac{\partial f}{\partial x^i} \frac{\partial}{\partial y^i} \right). \tag{1}$$

Now we are ready to see the connection between symplectic geometry and dynamics. A symplectic manifold (M, ω) together with a smooth function $H \in C^\infty(M)$ is called a **Hamiltonian system**. The function H is called the **Hamiltonian** of the system. The flow of the Hamiltonian vector field X_H is called its **Hamiltonian flow**, and the integral curves of X_H are called the **orbits** of the system.

In Darboux coordinates, we know from equation (1) that the orbits are those curves $\gamma(t) = (x^i(t), y^i(t))$ satisfying (By definition of integral curves $\dot{\gamma}(t) = X_H(\gamma)$),

$$\dot{x}^i(t) = \frac{\partial H}{\partial y^i}(x(t), y(t)),$$

$$\dot{y}^i(t) = -\frac{\partial H}{\partial x^i}(x(t), y(t)).$$

These are called **Hamilton's equations**. Moreover, since $X_H H = dH(X_H) = \omega(X_H, X_H) = 0$, we have that H is constant along each integral curve of X_H. This is also known as the conservation of energy in physics where H is the energy function of the Hamiltonian system.

In terms of Theorem 2.1 in Guillemin & Kazhdan, which says:

Theorem 7. Suppose one has a family of Hamiltonians p_t, homogeneous of degree one on $T^*M - 0$ and a corresponding family of closed orbits γ_t, depending smoothly on t such that γ_t lies on the energy surface $p_t = 1$. Let $\gamma = \gamma_0$ and $\dot{p} = (dp/dt)_{t=0}$. Then if the γ_t are all of the same length, $\int_\gamma \dot{p} = 0$.

Remark. In the context of this paper, the family of Hamiltonians p_t are exactly the family of Riemannian metrics $g_t, 0 \leq t \leq 1$, which can be regarded as a C^∞-function on T^*M (via the bundle isomorphism). The Hamiltonian vector field X_{p_t} is the corresponding geodesic spray, and the flow of X_{p_t} is the geodesic flow. The family of closed orbits γ_t corresponds to the closed geodesics on M [1] (such curves are abundant because our surface is negatively curved). Moreover, the energy surface $p_t = 1$ corresponds exactly to the cosphere bundle S^*M.

On the other hand, the assumption that γ_t are all of the same length may seem absurd at first, the following theorem gurantees its connection with the original problem.

Theorem 8. Let M be a compact negatively curved Riemannian manifold. Then the spectrum of the Laplace-Beltrami operator determines the lengths of the periodic geodesics on M.

[1] A proof of the statement is given in the appendix (to be included).

This theorem shows under an isospectral deformation of g_0, the closed geodesics on M all have the same length. Finally, $\dot{p} = (dp/dt)_{t=0}$ can be understood as the derivative of the deformation g_t at $t = 0$, and if we define $\dot{L}_{\gamma_t} = (dL_{\gamma_t}/dt)_{t=0}$, then $\dot{L}_{\gamma_t} = \int_\gamma \dot{g}$. [2] Therefore, the theorem gives a variational result on the length of the geodesics.

Proof. (Theorem 7) Consider the band B in T^*M, the union of the images of the closed geodesics γ_t. Then it follows from the Stokes' Theorem (note T^*M is orientable because ω^n is a non-vanishing $2n$-form) that $\int_B \omega = \int_{\partial B} \alpha = 0$ since $L(\gamma_0) = L(\gamma_1)$.

Along each γ_t, the Hamiltonian vector field X_{p_t} induced by p_t is by definition $X_{p_t} \lrcorner\, \omega = dp_t$. On the other hand, by the definition of $\hat{p}$, we have

$$dp_t = dp_0 + td\dot{p} + O(t^2) = dp_0 + d(t\dot{p}) - \dot{p}dt + O(t^2). \tag{2}$$

Moreover, since the orbits γ_t lies in the energy surface $p_t = 1$ (equivalently, the cosphere bundle S^*M), we have the following identity

$$1 = p_t = p_0 + t\hat{p} + O(t^2) \quad \Rightarrow \quad d(p_0 + t\dot{p}) = O(t)dt + O(t^2).$$

Applying the above identity to equation (2) gives

$$X_{p_t} \lrcorner\, \omega = -\dot{p}dt + O(t)dt + O(t^2). \tag{3}$$

on γ_t. Let L be the common length of the curves γ_t and let B be parametrized by $\phi(s,x), 0 \leq s \leq t, 0 \leq x \leq L$, which maps the curves, with t being constant, onto the image of γ_t. Then by construction $\phi_*(\partial/\partial x) = X_{p_t}$ and the pullback of t is still t. Now since

$$\phi^*(X_{p_t} \lrcorner\, \omega)(\partial/\partial t) = \omega(X_{p_t}, \phi_*(\partial/\partial t)) = \omega\left(\phi_*\left(\frac{\partial}{\partial x}\right), \phi_*\left(\frac{\partial}{\partial t}\right)\right) = \phi^*(\omega)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right)$$

by equation (3), we have

$$\phi^*(\omega)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) dx\, dt = -\phi^*(\dot{p})(x,t)\, dx\, dt + O(t)\, dx\, dt.$$

Integrating both sides with respect to x and t gives

$$0 = \int_B \omega = \int_0^t \int_0^L \phi^*(\omega)\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}\right) dx\, dt = -\int_0^t \int_0^L \phi^*(\dot{p})\, dx\, dt + O(t^2)$$

Differentiating both sides with respect to t gives

$$\int_0^L \phi^*(\dot{p})(x,t)dx = O(t)$$

Setting $t = 0$ gives

$$\int_\gamma \dot{p} = \int_0^L \phi^*(\dot{p})(x,0)dx = 0.$$

□

[2] A proof is given in the appendix (to be included).

Appendix.

Theorem 1. Given a Riemannian manifold (M, g), the cotangent bundle T^*M naturally admits a symplectic structure ω. Let (T^*M, ω, g) be a Hamiltonian system, then the integral curves generated by the Hamiltonian vector field X_g correspond exactly to the geodesics on M.

Proof. Let $\pi_m^* : C^\infty(M; S^m T^*M) \to C^\infty(TM)$ be the natural pullback map by evaluation (defined in PSU or Guillarmou's paper). Then the Riemannian metric $g \in C^\infty(M; S^2_+ T^*M)$ can be regarded as a C^∞-function $\tilde{g}$ on T^*M, namely

$$\tilde{g}(x, v) = \frac{1}{2}(\pi_2^* g)(x, v^\#) = \frac{1}{2} g_{ij} dx^i(v^\#) dx^j(v^\#) = \frac{1}{2} g_{ij} v^i v^j$$

Then the Hamiltonian vector field X_g is given by

$$(X_g)_{(x,v)} = \sum_{k=1}^{n} \left(\frac{\partial g}{\partial v^k} \frac{\partial}{\partial x^k} - \frac{\partial g}{\partial x^k} \frac{\partial}{\partial v^k} \right) = \sum_{k=1}^{n} \left(v^k \frac{\partial}{\partial x^k} - v^i v^j \Gamma_{ij}^k \frac{\partial}{\partial v^k} \right)$$

It follows from the characterization of Hamiltonian orbits that the integral curves $\gamma(t) = (x^k(t), v^k(t))$ of X_g are those satisfying

$$\dot{x}^k(t) = v^k \frac{\partial}{\partial x^k}, \quad \dot{v}^k(t) = -v^i v^j \Gamma_{ij}^k \frac{\partial}{\partial v^k}.$$

Identifying $(x, v) \in T^*M$ with $(x, \dot{x})$ for $x \in M$, we see that the integral curves of X_g on T^*M descend exactly to the geodesics on M satisfying the geodesic equation,

$$\ddot{x}^k + \dot{x}^i \dot{x}^j \Gamma_{ij}^k = 0.$$

□

Theorem 2. Given a Riemannian manifold (M, g), let g_t be a smooth isospectral deformation of $g_0 = g$. Let γ_t be the corresponding closed geodesics, $\gamma = \gamma_0$, $\dot{g} = (dg_t/dt)_{t=0}$, and $\dot{L}_\gamma = (dL(\gamma_t)/dt)_{t=0}$. Then $\dot{L}_\gamma = \int_\gamma \dot{g}$.

Proof. This follows from a straightforward computation: Assuming $\gamma_t(s) : [0, 1] \to M$ for each t, then by definition,

$$\frac{dL_{\gamma_t}}{dt} = \frac{d}{dt} \left(\int_0^1 g(\dot{\gamma}_t(s), \dot{\gamma}_t(s))^{1/2} \, ds \right) = \int_0^1 \frac{\dot{g}(\dot{\gamma}_t, \dot{\gamma}_t)}{2\sqrt{g(\dot{\gamma}_t, \dot{\gamma}_t)}} \, ds$$

restricting to $t = 0$ gives

$$\dot{L}_\gamma = \frac{1}{2\sqrt{g(\dot{\gamma}, \dot{\gamma})}} \int_0^1 \dot{g}(\dot{(\gamma)}, \dot{(\gamma)}) \, ds$$

as desired. □

Question 3. In the proof of Theorem 7, only the first order Taylor expansion is used. Can we extract more information using a higher order estimate?

2022.2.17 Guillemin & Kazhdan Continued (Guangqiu Liang)

From Max's talk last week, we learned that there exists three vector fields $\xi_1, \xi_2, \partial/\partial\theta \in \mathfrak{X}(S^*M)$ such that the following equations hold

$$[\partial/\partial\theta, \xi_1] = \xi_2, \quad [\partial/\partial\theta, \xi_2] = -\xi_1, \quad [\xi_1, \xi_2] = K\partial/\partial\theta$$

Note that for our purpose, ξ_1 is the vector field that generates the geodesic flow on M, and $\partial/\partial\theta$ is the vector field on $\mathbb{S}^1$. Here we remark that the above identities can be computed via isothermal coordinates explicitly, or computed via the *connection form* (dual basis $\omega_1, \omega_2, \varphi$) construction in Singer & Thorpe using the following useful identity.

$$d\tau(X, Y) = X\tau(Y) - Y\tau(X) - \tau([X, Y]).$$

Moreover, since the Lie derivatives of the volume form $\Omega = \omega_1 \wedge \omega_2 \wedge \varphi$ on S^*M with respect to ξ_1 and with respect to $\partial/\partial\theta$ are both zero, $-i\partial/\partial\theta$ and $-i\xi_1$ extend to a densely defined self-adjoint operator on $L^2(S^*M)$. Finally, recall the vector fields η^+ and η^- defined as

$$\eta^+ = \frac{\xi_1 - i\xi_2}{2}, \quad \eta^- = \frac{\xi_1 + i\xi_2}{2}$$

with the commutator equations rewritten as

$$[-i\partial/\partial\theta, \eta^+] = \eta^+, \quad [-i\partial/\partial\theta, \eta^-] = -\eta^-, \quad [\eta^+, \eta^-] = (-K/2)(-i\partial/\partial\theta).$$

Now our task is to study the properties of the operators η^+ and η^- (elaborate more of these vector fields from PSU & Guillarmou). First we note that S^*M is not a product space globally but is indeed a local product space. Therefore, we may assume from now on that

$$L^2(S^*M) = \sum L^2(S^*M_i) = \sum L^2(\mathbb{S}^1) \bigotimes L^2(X_i) = \sum H_n$$

where each H_n is the space of polynomials of degree n, and the vector field $-i\partial/\partial\theta$ acts like n times the identity operator. Next we examine how η^+ and η^- act on each subspace H_n. As the name suggests, η^+ takes a function from H_n to H_{n+1} and η^- takes a function from H_n to H_{n-1}.

Indeed, if $f \in H_n$, then by the commutator identity involving η^+,

$$\begin{aligned}(-i\partial/\partial\theta)\eta^+ f &= \eta^+(-i\partial/\partial\theta f) + [-i\partial/\partial\theta, \eta^+]f \\ &= n\eta^+ f + \eta^+ f \\ &= (n+1)\eta^+ f\end{aligned}$$

which implies $\eta^+ f$ is a function in H_{n+1}. By a similar argument using the commutator identity involving η^-, we may show that η^- extends to an operator from H_n to H_{n-1} for all n. Moreover, note that since $-i\xi_1$ is self-adjoint, $\xi_1 = \eta^+ + \eta^-$ is skew-adjoint, i.e.

$$-(\eta_+^t + \eta_-^t) = \eta_+ + \eta_-$$

Considering the domain of these operators, we get $\eta_+^t = -\eta_-$ and $\eta_-^t = -\eta_+$. The next lemma is where the negative curvature condition comes in. It shows roughly speaking that both η^+ and η^- are stable when pushing functions upwards or downwards.

Lemma 1. Let $a_0 = \min(-K/2)$ and $a_1 = \max(-K/2)$ where K is the scalar curvature function. Then for all $f \in H_n \cap \mathcal{D}(\eta^+) \cap \mathcal{D}(\eta^-)$ and $n \geq 0$,

$$\left\|\eta^- f\right\|^2 + a_0 n \left\|f\right\|^2 \leq \left\|\eta^+ f\right\|^2 \leq \left\|\eta^- f\right\|^2 + a_1 n \left\|f\right\|^2$$

Proof. By the commutator relation again,

$$[\eta^+, \eta^-] = \eta^+\eta^- - \eta^-\eta^+ = (\eta_+)^t \eta^+ - (\eta_-)^t \eta_- = (-K/2)(-i\partial/\partial\theta)$$

Thus,

$$\left\|\eta^+ f\right\|^2 = \langle (\eta_+)^t \eta^+ f, f\rangle \geq \langle (\eta_-)^t \eta_- f, f\rangle + a_0 n \left\|f\right\|^2$$

with a similar inequality the other way. □

At this point, it is imperative to summarize what we have accomplished so far: To prove that compact 2-manifolds of negative curvature are spectrally rigid, we assume that there exists an isospectral deformation g_t of (M, g) and would like to show that it is trivial, i.e. there exists a family of diffeomorphisms ϕ_t such that $\phi_t^* g = g_t$.

Since the spectrum of the Laplace-Beltrami operator determines the lengths of the periodic geodesics on X, we know that the lengths of the closed geodesics under an isospectral deformation are of the same length. Then, by Proposition 2.1 we have that the geodesic X-ray transform of the infinitesimal deformation $\dot{p} = dp_t/dt$ is zero. By a theorem in Livsic cohomology problem, we know that there exists a smooth function q such that $\xi q = \dot{p}$. We will use the fact that $\dot{p}$ is a real quadratic form on T^*M (i.e. $\dot{p} \in H_{-2} \oplus H_0 \oplus H_2$) to argue such a smooth q (belonging to $H_1 \oplus H_{-1}$) cannot exist unless the deformation is trivial. Now we present the ingredient that there exists a smooth q on S^*M such that $\xi q = p$.

Lemma 2. Let p be a smooth function on S^*M of the form

$$p = \sum_{|i| \leq N} p_i, \quad p_i \in H_i.$$

Suppose the integral of p over every periodic integral curve of ξ is zero. Then there exists a smooth function q of the form

$$q = \sum_{|i| \leq N-1} q_i, \quad q_i \in H_i$$

such that $\xi q = p$.

Proof. By Livsic, given such smooth function p in the hypothesis, there exists a C^1 function q on S^*M such that $\xi q = p$. Suppose $q = \sum q_i$ with $q_i \in H_i$, then the equation $\xi q = p$ is equivalent to the system

$$\eta^- q_{i+1} + \eta^+ q_{i-1} = p_i, \quad i = 0, \pm 1, \pm 2, \ldots$$

Since $p_i = 0$ for all $i > N$, we have that $\eta^+ q_{i-1} + \eta^- q_{i+1} = 0$ for all $i > N$. Therefore, by Lemma 1,

$$\left\|\eta^+ q_{i-1}\right\| = \left\|\eta^- q_{i+1}\right\| \leq \left\|\eta^+ q_{i+1}\right\|$$

for all $i > N$. Since q is C^1, the differential $\eta^+ q_i$ converges to zero in the L^2 topology. Hence, the above estimate implies $\eta^+ q_i = 0$ for all $i \geq N$. By Lemma 1 again, we have $q_i = 0$ for all $i \geq N$.

By a similar argument, we can show that $q_i = 0$ for $i \leq -N$. Finally, to show the required smoothness, write the system as $\eta^+ q_{i-1} = p_i - \eta^- q_{i+1}$. It is clear that the right hand side is smooth for all $i \geq N$, seeing η^+ as a first order elliptic differential operator, we may conclude that q_{N-1} is smooth. By induction on i, we have that q_i is smooth for all i. □

So far we have obtained all the ingredients we need. Applying to the problem where $\| \, \|_t$ is an isospectral deformation of $\| \, \|_0$ on M, let $p_t : T^*M \to R$ be the function $p(x, \xi) = \|\xi\|_t^2$ and let $\dot{p} = dt/dt|_{t=0}$ restricted to S^*X.

Then since $\dot{p}$ is a real valued quadratic form on T_p^*M for each $p \in M$, we have that $\dot{p}$ as a linear combination of the basis $(dz)^2, (d\overline{z})^2$ and $\| \, \|_p^2$ must satisfy $\overline{p}_2 = p_{-2}$ and $\overline{p}_0 = p_0$.

Similarly, for the corresponding solution q such that $\xi q = \dot{p}$, we have that $q \in H_{-1} \oplus H_1$: Indeed, by Lemma 2, there exists smooth $q = q_1 + q_0 + q_{-1}$ such that $\xi q = \dot{p}$. But since $p_1 = p_{-1} = 0$, we have that $\eta^+ q_0 = \eta^- q_0 = 0$. Moreover, since $q_0 \in H_0$, it follows $(\partial/\partial)\theta q_0 = 0$. However, $\{\eta^+, \eta^-, \partial/\partial\theta\}$ is a basis of $T(S^*M)$ at each point, so q_0 must be a constant. Without loss of generality, we may assume $q_0 = 0$, and thus $q \in H_1 \oplus H_{-1}$. (missing the end of the proof)

2022.3.18 Osgood, Phillips & Sarnak, 1988 (Borthwick)

Let (M, g) be a compact Riemannian surface and $-\Delta_g$ be the associated Laplacian, then the set of eigenvalues $\sigma(-\Delta_g)$ is an invariant of the isometry class $\hat{g}$. OPS proved in their 1987 papers, *Isospectral Sets of Surfaces* and *Extremals of Determinants of Laplacian*, that

Theorem 1. An isospectral family $\{\hat{g}_n\}$ of isometry classes of metrics on M is compact in the C^∞ topology.

Remark. Here compactness in the C^∞ topology means sequentially compactness, i.e. for any sequence $\{\hat{g}_n\}$, there exist representatives $g_n \in \hat{g_n}$ such that a subsequence $\{g_{n_k}\}$ converges in the C^∞ topology.

It is conjectured that in the case of Riemannian surfaces (Riemannian manifolds of dimension 2), the set of isospectral isometry classes $\{\hat{g_n}\}$ is also discrete. If the conjecture is true, together with the compactness result, we may conclude that the set $\{\hat{g_n}\}$ is in fact finite. However, the conjecture still remains open.

Now we present the tools used in OPS:

1. Conformal uniformization
2. Determinant of $-\Delta$
3. Heat asymptotics
$$\sum_{j=0}^{\infty} e^{-t\lambda_j} \sim \frac{1}{4\pi t} \sum_{j=0}^{\infty} a_j t^j$$

Conformal uniformization. In dimension 2, if g and g_0 are conformal metrics, i.e. $g = e^{2\varphi} g_0$, then their corresponding Laplacians have a simple relation, namely $\Delta_g = e^{-2\varphi}\Delta_{g_0}$. Up to conformal diffeomorphisms, we would like to reduce the problem to a background metric with constant curvature. In fact, the curvature $k = \mathrm{sgn}(\chi)$ where χ is the Euler characteristic of the Riemannian surface.

Here we sketch a proof of the reduction process: Given $g = e^{2\varphi} g_0$, the scalar curvature of g is given by (recall conformal transformation of the curvature),

$$S_g = e^{-2\varphi}(S_{g_0} + 2(n-1)\Delta_{g_0}\varphi - (n-1)(n-2)\left|\mathrm{grad} f\right|_g^2)$$

In the case when $n = 2$, we have that

$$K_g = e^{-2\varphi}(K_{g_0} + \Delta_{g_0}\varphi)$$

or equivalently,

$$K_{g_0} = e^{2\varphi}(-\Delta_g \varphi + K_g)$$

Then the problem reduces to solving φ for which K_g is constant, which gives rise to the pde $-\Delta_g \varphi = K_g + e^{-2\varphi}$ subject to the condition obtained by integrating the equation on both sides and applying Gauss-Bonnet Theorem, i.e.

$$\int_M e^{-2\varphi}\, dV_g = -2\pi\chi(M).$$

The strategy to solve this equation resembles the proof of the Riemann Mapping Theorem, in which the pde is converted into a minimization problem. In this case, let

$$f(\varphi) = \int_M \left(\frac{1}{2}|\nabla \varphi|^2 + K_g \varphi \right) dV_g, \quad \varphi \in H^1(M),$$

subject to

$$\mathcal{W} := \{\varphi \in H^1 : \int_M e^{-2\varphi} dV_g = -2\pi\chi(M)\}$$

We claim that $f(\varphi)$ achieves a minimum at some function $\varphi \in C^\infty(M)$ and φ satisfies $-\Delta_g \varphi = K_g + e^{-2\varphi}$. Indeed, we may show that $f(\varphi)$ is bounded below on $\mathcal{W}$. Taking a sequence in $\mathcal{W}$ that converges to the minimum and passing to a subsequence by compactness give the desired solution φ.

Determinant of the Laplacian

It is known from standard spectral theory on compact Riemannian manifold that the eigenvalues of the Laplacian are discrete and they tend to infinity. Formally, we would like to borrow the traditional definition and define the determinant of the Laplacian as $\det'(-\Delta_g) = \prod_{k=1}^\infty \lambda_k$. However, a regularization process is needed to make sense of the infinite product. For this purpose, we will use the zeta function

$$Z(s) = \sum_{k=1}^\infty \frac{1}{\lambda_k^s}, \quad s \in \mathbb{C}$$

It follows from the Weyl's law that

$$\lambda_k \sim (2\pi)^2 \left(\frac{k}{\omega_n \mathrm{vol}(M)} \right)^{2/n}$$

Thus, $Z(s)$ converges absolutely for all $\mathfrak{Re}(s) > 1$. Morally speaking,

$$Z'(s) = \sum_{k=1}^\infty -\frac{1}{\lambda_k^s} \log \lambda_k \quad \Rightarrow \quad Z'(0) = -\sum_{k=1}^\infty \log \lambda_k$$

and we can formally define

$$\det'(-\Delta_g) = \prod_{k=1}^\infty \lambda_k = e^{-Z'(0)}$$

To make sense of the value of $Z'(s)$ at $s = 0$, we claim that $Z(s)$ admits an analytic continuation to a neighborhood of $s = 0$ with $Z(0) = \frac{\chi(M)}{6} - 1$. Indeed, using Riemann's trick, we can write

$$\frac{1}{\lambda_k^s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-\lambda_k t}\, dt$$

By the Dominated Convergence Theorem,

$$\begin{aligned} Z(s) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\sum_{k=1}^\infty e^{-\lambda_k t} \right) dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left(\frac{A}{4\pi t} + \left(\frac{\chi(M)}{6} - 1 \right) + O\left(\min(t, e^{-\lambda_1 t})\right) \right) dt \\ &= \frac{1}{\Gamma(s)} \left(\frac{A}{4\pi} \frac{1}{s-1} + \left(\frac{\chi(M)}{6} - 1 \right) \frac{1}{s} + h(s) \right) \end{aligned}$$

where $h(s)$ is analytic for $\mathfrak{Re}(s) > -1$. Near $s = 0$, $Z(s)$ has no pole at $s = 0$. Hence, the notion of the determinant of the Laplacian is well-defined. (elaborate more on conformal uniformization & determinant of the Laplacian)

Heat Asymptotics

Using the following Tauberian theorem, we are able to study the behavior of the heat operator on Riemannian manifold, which yields fruitful results on the geometry of the given manifold. Here we give a relatively detailed exposition because of the fundamental importance of the heat asymptotics in geometric inverse problems. First, we state the Karamata's Tauberian Theorem without proving:

Theorem 1. (Karamata's Theorem) Let μ be a measure on $[0, \infty)$, such that e^{-tx} is integrable with respect to $d\mu(x)$ for each $t > 0$. Suppose that for $\alpha > 0$,

$$\int_0^\infty e^{-tx}\, d\mu(x) \sim At^{-\alpha}$$

as $t \to 0^+$ (or $t \to \infty$). Then

$$\mu[0, s] \sim \frac{A}{\Gamma(\alpha + 1)} s^\alpha,$$

as $s \to \infty$ (or $s \to 0^+$, respectively).

Let (M, g) be a compact Riemannian manifold, and let Δ be the Laplace-Beltrami operator on M, namely

$$\Delta := -\frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j)$$

We know from basic functional analysis that Δ is a self-adjoint and positive operator on $L^2(M)$, it follows from the spectral theorem that Δ has nonnegative discrete eigenvalues $\{\lambda_k\}_{k=0}^\infty$ accumulating at infinity.

Consider the heat equation $\partial_t u = \Delta u$ under the initial condition $u|_{t=0} = f$. Formally, the heat equation has the solution $u = e^{t\Delta} f$. Indeed, the heat operator $e^{t\Delta}$ that solves the heat equation is well defined as a bounded operator by functional calculus.

The interplay between geometry and heat propagation is manifested into the initial evolution of the heat diffusion process, i.e. how the heat operator behaves when $t \to 0^+$? The question is answered by studying the kernel of the heat operator, namely the heat kernel. The following fundamental theorem is given by Minakshisundaram and Pleijel in 1949:

Theorem 2. Let M be a compact Riemannian manifold. There exists a function $H \in C^\infty(\mathbb{R}_+ \times M \times M)$ such that

$$e^{t\Delta} f(x) = \int_M H(t; x, y) f(y)\, dV_g(y)$$

for $f \in L^2(M)$. If f is continuous, then

$$\lim_{t \to 0^+} \int_M H(t; x, y) f(y)\, dV_g(y) = f(x).$$

For each $x \in M$, there is a uniform asymptotic expansion as $t \to 0^+$,

$$H(t, x, x) \sim (4\pi t)^{-n/2} \sum_{j=0}^{\infty} \alpha_j(x) t^j.$$

where $\alpha_0 = 1$ and α_j depends only on the metric g and its derivatives.

The proof is rather involved, which we will not cover here. However, the idea is simple: Consider the heat equation on the Sobolev space $H^2(\mathbb{R}^n)$. The heat operator $e^{t\Delta}$ is again well defined by the functional calculus and can be written explicitly using Fourier transform. In particular,

$$e^{t\Delta} f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\xi - t|\xi|^2} f(y)\, d^n y\, d^n \xi.$$

which implies the heat kernel is given explicitly by

$$\Psi(t, x, y) = (4\pi t)^{-n/2} e^{-|x-y|^2/4t}$$

We sketch the construction of the general heat kernel, which is based on the Euclidean heat kernel. The strategy is to first construct an approximate solution (a parametrix) that captures the essential features of the true heat kernel and then modify it to get the general kernel. Set

$$H_k(t, x, y) = \sum_{j=0}^{k} t^j u_j(x, y) \Psi(t, x, y)$$

Up to choosing the coefficients u_j and choosing a cutoff function, the parametrix is an approximate solution to the heat equation, i.e.

$$(\partial_t - \Delta) F_k(t, x, y) = R_k \sim O(t^{k-n/2})$$

as $t \to 0$, uniformly for $(x, y) \in M \times M$. Moreover, for $f \in C^\infty(M)$,

$$\lim_{t \to 0^+} F_k(t, x, \cdot) = \delta_x$$

uniformly for $x \in M$. Finally, choosing $k > n/2$ and setting $H = Q_k - H_k$ where Q_k is the solution of the inhomogeneous heat equation (using Duhamel's principle),

$$(\partial_t - \Delta) Q_k = R_k.$$

which gives our desired heat kernel H.

The existence of a smooth heat kernel H implies that the heat operator $e^{t\Delta}$ is Hilbert-Schmidt and

$$\mathrm{tr}(e^{t\Delta}) = \int_M H(t,x,x)\, dV_g(x) \sim (4\pi t)^{-n/2} \sum_{j=0}^{\infty} a_j t^j$$

where the coefficients $a_j = \int_M \alpha_j \, dV_g$ are called the heat invariants of M and the last asymptotic relation holds since the asymptotics of $H(t,x,x)$ is uniform. On the other hand, writing the trace as the sum over the eigenvalues gives

$$\mathrm{tr}(e^{t\Delta}) = \sum_{k=0}^{\infty} e^{-\lambda_k t} \sim (4\pi t)^{-n/2} \sum_{j=0}^{\infty} a_j t^j$$

In particular, we can compute the first few a_j, namely

$$a_0 = \int_M \alpha_0(x)\, dV_g = \int_M dV_g = \mathrm{Vol}(M).$$

$$a_1 = \int_M \alpha_1(x)\, dV_g = \frac{1}{6}\int_M S\, dV_g = \frac{2\chi(M)}{3}$$

$$a_2 = \frac{1}{360}\int_M \left(2|R|^2 - 2|\mathrm{Ric}|^2 + 5\mathrm{Scal}_g^2\right) dV_g$$

The higher order coefficients are much more difficult to compute due to the complex combinations of the metric g and its derivatives. Note that these heat invariants computation implies one can hear the volume, dimension, and number of holes of a compact Riemannian manifold from its spectrum. See Marc Kac's famous expository paper "Can one hear the shape of a drum?" if interested.

Finally, we are ready to present an asymptotic formula for the eigenvalues of a compact Riemannian manifold, also known as the Weyl's law. If we let $N_M(x) := \#\{\lambda_k \leq x\}$, then

$$N_M(x) \sim (2\pi)^{-n} \omega_n \mathrm{Vol}(M) x^{n/2}$$

as $t \to \infty$, where ω_n is the volume of the unit ball in $\mathbb{R}^n$. Equivalently, if the eigenvalues $\{\lambda_k\}$ are arranged in increasing order,

$$\lambda_k \sim (2\pi)^2 \left(\frac{k}{\omega_n \mathrm{Vol}(M)}\right)^{2/n}$$

as $k \to \infty$. Both results follow immediately from Karamata's theorem by letting $\mu = N_M$. Add the notes of Paternain-Salo-Uhlmann's 2014 paper

2022.6.7. Guillarmou, Invariant Distribution Anosov, 2017 (Guangqiu)

For the first summer section of the reading seminar, we continue with our study of geometric inverse problems. In particular, we are interested in generalizing the spectral rigidity results initiated by Guillemin & Kazhdan, who proved that negatively curved compact Riemannian surfaces are spectrally rigid. There are mainly two directions of generalizations: Dimension and type of manifold.

Dimension-wise, Croke & Sharafutdinov generalized GK's result to negatively curved compact manifolds of arbitrary dimension. In terms of generalizing to a broader class of manifolds, attention has been drawn to Anosov manifolds, a type of manifolds that possess good dynamical properties to deal with. So far, Paternain, Salo & Uhlmann showed that closed Anosov surfaces are spectrally rigid by proving the geodesic ray transform I_2 is injective. What Guillarmou accomplished in this paper is an innovative proof of the injectivity of I_m for any $m \geq 0$, which has been an important open problem in integral geometry. We first provide some definitions that will be helpful in later discussion:

Definition 1. A Riemannian manifold (M, g_0) is *spectrally rigid* if given a smooth deformation g_t of g_0 such that $Spec(-\Delta_{g_t}) = Spec(-\Delta_{g_0})$ for all $-\varepsilon < t < \varepsilon$, then there exists a family of diffeomorphisms φ_t such that $g_t = \varphi_t^* g_0$.

Definition 2. A smooth vector field X on a compact manifold M without boundary is *Anosov* if its flow φ_t has the following property: there exists a continuous flow-invariant splitting (meaning $d\varphi_t(E_s) = E_s$ and $d\varphi_t(E_u) = E_u$),

$$TM = E_0 \oplus E_s \oplus E_u$$

such that $E_0 = \mathbb{R}X$ is the direction of the flow andE_s and E_u are the stable and unstable bundles, which are defined as follows: there exists $C > 0, \nu > 0$ such that

$$\xi \in E_s(y), y \in M \quad \iff \quad \|d\varphi_t(y)(\xi)\| \leq Ce^{-\nu t}\|\xi\| \quad \text{for } t \geq 0,$$

$$\xi \in E_u(y), y \in M \quad \iff \quad \|d\varphi_t(y)(\xi)\| \leq Ce^{-\nu|t|}\|\xi\| \quad \text{for } t \leq 0.$$

where $\|\cdot\|$ is the norm induced by any fixed metric on M. Moreover, if the geodesic flow of a closed Riemannian manifold (M, g) is Anosov, then we say M is an *Anosov manifold*.

Remark. Anosov manifolds are natural generalization of negatively curved manifolds from a dynamical system point of view: All negatively curved compact manifolds are Anosov, but there are examples of Anosov manifolds with small positive curvature part. Anosov manifolds also have the following properties:

1. The orbit of a point $p \in M$ is dense in M for almost all p.
2. The set of closed geodesics is dense in M.
3. The geodesic flow is mixing (stronger than ergodicity).

Definition 3. The geodesic flow on a closed manifold M is *ergodic* if the only invariant $L^2(SM)$ functions are the constants.

Definition 4. A flow φ_t is mixing with respect to an invariant probability measure $d\mu$ if for all $u, v \in L^2(M)$

$$C_t(u,v) = \int_M u(\varphi_t(y))v(y)\,d\mu(y) - \int_M u(y)\,d\mu(y)\int_M v(y)\,d\mu(y)$$

approaches zero as $t \to \infty$.

Definition 5. Given a symmetric m-tensor field $f = f_{i_1\cdots i_m}dx^{i_1}\otimes\cdots\otimes dx^{i_m}$ on M, we define the corresponding function on SM by

$$f(x,v) = f_{i_1\cdots i_m}v^{i_1}\cdots v^{i_m}.$$

In our case, suppose (M,g) be a closed Anosov surface and $\mathcal{G}$ be the set of closed geodesics on (M,g) parametrized by arc length, then the *geodesic ray transform* of a symmetric m-tensor field f on M is defined by

$$I_m f(\gamma) = \int_0^T f(\gamma(t),\dot\gamma(t))\,dt, \quad \gamma\in\mathcal{G} \text{ has period } T.$$

Remark. Note that if h is a symmetric $(m-1)$-tensor field, its inner (symmetrized) covariant derivative dh is a symmetric m-tensor field defined by $dh = \sigma\nabla h$ where σ denotes the symmetrization and ∇ is the Levi-Civita connection. We see that

$$dh(x,v) = Xh(x,v)$$

where X is the geodesic vector field associated with ϕ_t. This shows that $I_m(f)(\gamma) = 0$ for all $\gamma\in\mathcal{G}$ if $f = dh$ for some symmetric $(m-1)$-tensor field. In particular, we say I_m is s-injective if these are the only elements in $\ker I_m$.

Remark. The terminology "s-injective" comes from the fact that any symmetric m-tensor field f can be decomposed into $f = f^s + dh$ where f^s is a symmetric m-tensor field with zero divergence and h is an $(m-1)$-tensor. The decomposition f^s and dh are called respectively the *solenoidal* and *potential* parts of f. Thus, that I_m is s-injective means exactly I_m is injective on the set of solenoidal tensors.

The general idea in both Paternain-Salo-Uhlmann and Guillarmou originates from GK. Recall that Roughly speaking, we are looking at the linearization,

$$g_t = \varphi_t^* g_0 + t\dot g + O(t^2).$$

where $\dot g$ is defined as $\dot g = \frac{dg}{dt}|_{t=0}$. To show rigidity, it suffices to show $\dot g$ is "trivial" for any isospectral deformation g_t. It is hard to conclude anything from the definition of $\dot g$ directly; instead, we consider the geodesic ray transform $I_2(\dot g)$, which $\dot g$ is considered as a C^∞ function over the cosphere bundle SM, i.e.

$$I_2(\dot g)(\gamma) = \int_0^T \dot g(\gamma,\dot\gamma)\,dt, \quad \gamma\in\mathcal{G} \text{ has period } T$$

It is relatively easy to show that $I_2(\dot{g}) = 0$ using the fact that the lengths of closed geodesics are a spectral invariant, after which showing spectral rigidity is reduced to showing the s-injectivity of the geodesic ray transform I_2. Here the notion of a trivial $\dot{g}$ is clear; that is, $\dot{g} = dh$ for some 1-tensor field h.

The first step to prove the s-injectivity of I_m is to consider a transport equation (or cohomological equation). By Livsic-type theorems, given $I_m(f)(\gamma) = 0$ for all closed geodesics γ, there exists a smooth function $u : SM \to \mathbb{R}$ such that $Xu = f$. Thus, to show f is trivial, it suffices to show the projection $VXu = 0$ where V is the vertical vector field on the cosphere bundle SM. GK and PSU showed this by essentially obtaining estimates from the Pestov identity,

$$\|XVu\|^2 - (KVu, Vu) + \|Xu\|^2 - \|VXu\|^2 = 0.$$

where K is the Gaussian curvature of the surface. For Guillemin-Kazhdan, the estimate is easily obtained from the negative curvature assumption. As for the Anosov case, PSU work directly with the operator $P = XV$ and show the surjectivity of I_1^*, which allows them to derive injectivity conclusion on I_m.

On the other hand, the innovative part of Guillarmou's paper is, instead of working directly with the geodesic ray transform, to introduce an well-behaved intermediate operator Π that approximates I_m (resembling the proof of the inverse function theorem). At the end, the easier-to-analyze properties of Π are carried to I_m using tools from the theory of anisotropic Sobolev spaces. Such a roundabout allows Guillarmou to obtain the following important result:

Theorem 7. On a Riemannian surface with Anosov geodesic flow, for all $m \geq 0$ we have $\ker I_m \cap \ker D^* = 0$ and $\ker \Pi\pi_m^* \cap \ker D^* = 0$.

The operator $\Pi : H^s(M) \to H^r(M), \forall s > 0, \forall r < 0$ can be defined as follows: Given a smooth invariant probability measure $d\mu$ and assume the geodesic flow is mixing with respect to this measure, then Π is defined in a distributional sense as a weak limit of damped correlations

$$\langle \Pi f, \psi \rangle = \lim_{\lambda \to 0^+} \int_{\mathbb{R}} e^{-\lambda |t|} \langle f \circ \varphi_t, \psi \rangle_{L^2} \, dt, \quad f, \psi \in C^\infty(M)$$

if $\langle f, 1 \rangle = 0$. Here the correlation function $\langle f \circ \varphi_t, \psi \rangle_{L^2}$ converges to equilibrium exponentially for geodesic flow φ_t of negative curvature, so the definition of the pairing makes sense. In fact, the motivation for Π comes from the attempt to define integral operator along all geodesics (not just closed geodesics, which are discrete, in the definition of I_m), formally as

$$\Pi f(x) = \int_{\mathbb{R}} f \circ \varphi_t(x) \, dt$$

Nevertheless, the operator Π has a bunch of desired properties:

1. range(Π) is of infinite dimension and is dense in the space of invariant distributions, $\mathcal{I} := \{w \in C^{-\infty}(M), Xw = 0\}$.

2. Π is self-adjoint as a map from $H^s(M)$ to $H^{-s}(M)$ for any $s > 0$ and satisfies

$$X\Pi f = 0, \ \forall f \in H^s(M), \quad \text{and } \Pi X f = 0, \ \forall f \in H^{s+1}(M).$$

Appendix

Remark. David pointed out that the operator Π is roughly speaking a reconstruction of the spectral measure of $-iX$ by the Stone's formula, justifying the use of the notation Π:

$$\frac{1}{2}\left(P_{[\alpha,\beta]} + P_{(\alpha,\beta)}\right) = \int_a^b d\Pi(\lambda)$$

where $d\Pi$ is the operator valued measure defined as

$$d\Pi(\lambda) = \frac{1}{2\pi i}\lim_{\varepsilon\to 0}\left((A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1}\right) d\lambda.$$

and P_I is the spectral projector associated to the self-adjoint operator A and is defined as $P_I = \chi_I(A)$ by functional calculus for self-adjoint operators.

Yiran also mentioned that the goal of considering the resolvent $R_\pm(\lambda) = (-X \pm \lambda)^{-1}$ is to, as we will see later, examine what happens when $\lambda \to 0^+$, which is roughly the inverse X^{-1}.

It is also pointed out by Guillarmou that the intermediate operator Π is an analog of the normal operator $I_0^* I_0$ in the study of boundary rigidity problems on manifolds with boundary. Here I_0 denotes the geodesic X-ray transform over $C^\infty(SM)$.

It is an interesting phenomenon that normal operator of the form A^*A is used extensively in reconstruction/inverse problems. We include a brief survey of inverting the Radon transform on $\mathbb{R}^3$ to illustrate this common feature. For the sake of brevity, we assume all functions below belong to the Schwartz space $\mathcal{S}(\mathbb{R}^3)$.

Definition 1. The **Radon Transform** of a function $f \in \mathcal{S}(\mathbb{R}^3)$ is defined by

$$\mathcal{R}(f)(t,\gamma) = \int_{P_{t,\gamma}} f$$

where the plane $P_{t,\gamma}$ for some unit vector $\gamma \in \mathbb{S}^2$ and $t \in \mathbb{R}$ is defined as,

$$P_{t,\gamma} = \{x \in \mathbb{R}^3 : x \cdot \gamma = t\}.$$

Given unit vectors e_1, e_2 so that $\{\gamma, e_1, e_2\}$ is an orthonormal coordinate of $\mathbb{R}^3$, then the integral over $P_{t,\gamma}$ is realized (independent of coordinates) as

$$\int_{P_{t,\gamma}} f = \int_{\mathbb{R}^2} f(t\gamma + u_1 e_1 + u_2 e_2)\, du_1\, du_2$$

In particular,

$$\int_{\mathbb{R}} \left(\int_{P_{t,\gamma}} f\right) dt = \int_{\mathbb{R}^3} f(x)\, dx.$$

The question is can we reconstruct f if we know everything about $\mathcal{R}(f)$, namely its integral value over all planes $P_{t,\gamma} \subset \mathbb{R}^3$? The answer is affirmative and takes a particularly elegant form in $\mathbb{R}^3$.

Theorem 2. The Radon transform $\mathcal{R}$ is injective on $\mathcal{S}(\mathbb{R}^3)$. Moreover, for any $f \in \mathcal{S}(\mathbb{R}^3)$,

$$f = -\frac{\Delta(\mathcal{R}^*\mathcal{R}(f))}{8\pi^2}.$$

where the **dual Radon transform** is defined as

$$\mathcal{R}^*(F)(x) = \int_{\mathbb{S}^2} F(x\cdot\gamma, \gamma)\, d\sigma(\gamma).$$

Exercise: Show that $\langle \mathcal{R}f, F\rangle_{L^2(\mathbb{R}\times\mathbb{S}^2)} = \langle f, \mathcal{R}^*F\rangle_{L^2(\mathbb{R}^3)}$ for $F \in \mathcal{S}(\mathbb{R}\times\mathbb{S}^2)$, $f \in \mathcal{S}(\mathbb{R}^3)$.

Proof. We first show the injectivity of $\mathcal{R}$ by examining the relation between Radon and Fourier transforms. In particular, we claim that $(\mathcal{R}f)(t,\gamma) \in \mathcal{S}(\mathbb{R})$ as a function of t: Indeed, since $f(t\gamma+u) \in \mathcal{S}(\mathbb{R}^3)$, for every positive integer N, there exists constants A_N such that

$$(1+|u|)^N(1+|t|^N)|f(t\gamma+u)| \le A_N$$

This shows

$$(1+|t|^N)(\mathcal{R}f)(t,\gamma) \le A_N \int_{\mathbb{R}^2} \frac{1}{(1+|u|)^N}\, du$$

which converges when $N \ge 3$. By a similar estimates on the derivatives of $\mathcal{R}f$, we see that $\mathcal{R}f \in \mathcal{S}(\mathbb{R})$. Moreover, we have the following relation $\hat{\mathcal{R}}f(s,\gamma) = \hat{f}(s\gamma)$: Indeed, by definition,

$$\begin{aligned}
\hat{\mathcal{R}}f(s,\gamma) &= \int_{\mathbb{R}} \left(\int_{P_{t,\gamma}} f \right) e^{-2\pi i s t}\, dt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^2} f(t\gamma+u) e^{-2\pi i s t}\, du\, dt \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}^2} f(t\gamma+u) e^{-2\pi i s \gamma\cdot(t\gamma+u)}\, du\, dt \\
&= \int_{\mathbb{R}^3} f(x) e^{-2\pi i s\gamma\cdot x}\, dx
\end{aligned}$$

where the third equality follows because $\gamma\cdot u = 0$ and $|\gamma| = 1$. Now the injectivity is an immediate consequence of the above relation applying to $f-g$. Moreover, with the above relation, we are ready to prove the reconstruction formula. By Fourier inversion formula,

$$(\mathcal{R}f)(t,\gamma) = \int_{\mathbb{R}} \hat{f}(s\gamma) e^{2\pi i s t}\, ds$$

By the definition of $\mathcal{R}^*$,

$$(\mathcal{R}^*\mathcal{R}f)(x) = \int_{\mathbb{S}^2} \hat{f}(s\gamma) e^{2\pi i s(\gamma\cdot x)}\, ds\, d\sigma(\gamma).$$

Since $\Delta(e^{2\pi is(\gamma\cdot x)}) = -4\pi^2 s^2 e^{2\pi is(\gamma\cdot x)}$, we have

$$\begin{aligned}(\Delta\mathcal{R}^*\mathcal{R}f)(x) &= -4\pi^2 \int_{\mathbb{R}}\int_{\mathbb{S}^2} \hat{f}(s\gamma)e^{2\pi is(\gamma\cdot x)}s^2\, ds\, d\sigma(\gamma) \\ &= -8\pi^2 \int_{\mathbb{R}+}\int_{\mathbb{S}^2} \hat{f}(s\gamma)e^{2\pi is(\gamma\cdot x)}s^2\, ds\, d\sigma(\gamma) \\ &= f(x)\end{aligned}$$

where the last equality follows again by Fourier inversion on $\mathcal{S}(\mathbb{R}^3)$. □

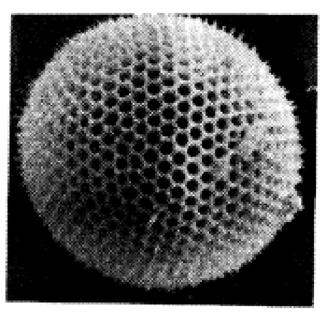
Question 1. Why are the dimensions of the stable and the unstable bundles the same?

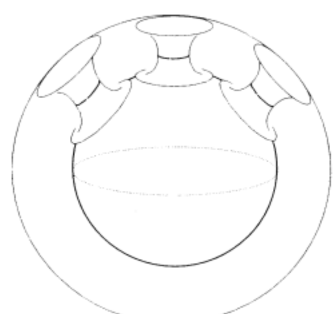
We have the following exercise problem from *Introduction to the Modern Theory of Dynamical Systems* by Katok and Hasselblatt [6.4.3]: Let (M,ω) be a symplectic manifold, $U \subset M$ open, and $f : U \to M$ a symplectic diffeomorphism. Let $\Lambda \subset U$ be a hyperbolic set for f. Prove that $\dim E^+ = \dim E^-$ for all $x \in \Lambda$, $E_x^\pm$ are Lagrangian subspaces of T_xM and that $W^s(x)$ and $W^u(x)$ are Lagrangian submanifolds of M. In the case of $\dim M = 2$ and $\dim SM = 3$, we automatically have $\dim E_s = \dim E_u = \dim \mathbb{R}X = 1$.

Question 2. Are there any nontrivial elementary examples of Anosov manifolds that are not negatively curved?

Yes, there are compact surfaces in $\mathbb{R}^3$ with Anosov geodesic flow. We refer to the paper "Anosov geodesic flows for embedded surfaces" by Donnay, Victor J. & Pugh, Charles C. for an interesting construction.

In particular, they proved for a compact, connected oriented surface M of genus g, let S denote the set of Riemannian metrics g_0 on M for which there exists an isometric imbedding of (M, g_0) into $\mathbb{R}^3$, then if the genus of M is sufficiently large, then one (and hence an open subset) of metrics $g_0 \in S$ must have Anosov geodesic flow. Roughly speaking, such surfaces look like a spherical shell with many holes drilled through them. See picture.





Last but not the least, we mention a beautiful proof to the fact that there are no negatively curved compact embedded surfaces in $\mathbb{R}^3$: Let M be such a surface, then M being compact implies M is bounded. Let $\mathbb{S}_r \subset \mathbb{R}^3$ be a sphere with radius r centered at zero osculating M at at least one point p. Since the curvature of $\mathbb{S}_r$ at p is $\frac{1}{r^2} > 0$, by continuity M has nonpositive curvature at p as well.

2022.6.14 Guillarmou Invariant Distribution Continued (Guangqiu)

Recall from last time we are on the way to show the s-injectivity of the geodesic ray transform over divergence free symmetric cotensor of order m on Anosov surfaces. The key insight in the paper is the use of an intermediate operator $\Pi : C^\infty(SM) \to C^{-\infty}(SM)$, which we will define in details for today's talk. The goal is to show the desired properties of Π_m (modified Π defined on $C^\infty(SM)$). We start by recalling the definition:

Definition 1. (X-ray transform on symmetric cotensors) Let $C^\infty(M, \otimes_S^m T^*M)$ denote the space of symmetric cotensor on M of order m, then there is a natural pullback map $\pi_m^* : C^\infty(M, \otimes_S^m T^*M) \to C^\infty(SM)$ such that

$$(\pi_m^* f)(x, v) = \langle f(x), \otimes^m v \rangle.$$

Define the *X-ray transform* on symmetric cotensors I_m as

$$I_m(f)(\gamma) := \int_0^T \langle f(\gamma(t)), \otimes^m \dot{\gamma}(t) \rangle \, dt$$

where γ is a closed geodesic on M and $\dot{\gamma}(t)$ is its time derivative. We saw from last time $\ker I_m$ contains all $f \in C^\infty(M, \otimes_S^m T^*M)$ of the form $f = Dh$ for some $h \in C^\infty(M, \otimes_S^{m-1} T^*M)$. Thus, let $D = \sigma\nabla$ denotes the symmetrized covariant derivative, then $D^* = -\tau D$ is in fact the divergence operator. Then the s-injectivity of I_m is equivalent to $\ker I_m \cap \ker D^* = 0$.

The operator Π

Now we define the operator Π through the resolvent operators $R_+(\lambda) = (-X + \lambda)^{-1}$ and $R_-(\lambda) = (-X - \lambda)^{-1}$ where $\lambda \in \mathbb{R}$ and X is the geodesic vector field generated by the geodesic flow $\varphi_t : SM \to SM$. Let $d\mu$ be a smooth invariant measure on SM with respect to φ_t (there is natural choice $d\mu$ for Anosov flow φ_t). Then integration by part shows $\langle Xu, v \rangle_{L^2(SM,d\mu)} = -\langle u, Xv \rangle_{L^2(SM,d\mu)}$ for all $u, v \in C^\infty(SM)$.

By Stone's theorem on one-parameter unitary group, the generator $-iX$ of the unitary operator $e^{tX} : L^2(SM) \to L^2(SM)$ defined by $(e^{tX} f)(y) = f(\varphi_t(y))$ is self-adjoint on $L^2(SM, d\mu)$. It follows from the spectral theorem that $\mathrm{Spec}_{L^2}(-iX) \subset \mathbb{R}$. The resolvents $R_+(\lambda) = (-X + \lambda)^{-1}$ and $R_-(\lambda) = (-X - \lambda)^{-1}$ for $\mathfrak{Re}(\lambda) > 0$ are well-defined and can be recovered by taking the Laplace transform of the unitary group e^{tX}, namely

$$R_+(\lambda) f(y) = \int_0^\infty e^{-\lambda t} e^{tX}(f(y)) \, dt = \int_0^\infty e^{-\lambda t} f(\varphi_t(y)) \, dt.$$

and

$$R_-(\lambda) f(y) = \int_0^\infty e^{-\lambda u} e^{-uX}(f(y)) \, dt = -\int_{-\infty}^0 e^{\lambda t} f(\varphi_t(y)) \, dt$$

Moreover, the spectral projector $1_{[a,b]}(-iX)$ and $1_{(a,b)}(-iX)$ can be expressed in terms of the resolvent

$$\frac{1}{2}\left(1_{[a,b]}(-iX) + 1_{(a,b)}(-iX)\right) = \frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \int_a^b \left(R_+(i\lambda + \varepsilon) - R_-(-i\lambda + \varepsilon)\right) d\lambda \qquad (4)$$

Since X is skew-adjoint, we have $(-X-\lambda)^* = (X-\overline{\lambda}) = -(-X+\overline{\lambda})$ on $C^\infty(SM)$, which implies they satisfy

$$R_-(\overline{\lambda})^* = -R_+(\lambda) \tag{5}$$

on $L^2(SM)$ for $\mathfrak{Re}(\lambda) > 0$. By a theorem of Faure-Sjöstrand, the Schwartz kernel of the resolvent operator $R_-(\lambda) : C^\infty(SM) \to C^{-\infty}(SM)$ admits a meromorphic extension to $\mathbb{C}$ as an element in $C^{-\infty}(SM \times SM)$ with poles of finite multiplicity as a bounded operator. The next theorem which relates mixing of the geodesic flow to the poles of $R_\pm(\lambda)$ is the key in defining and analyzing the operator Π.

Theorem 2. Let X be a smooth Anosov vector field on a compact manifold M and let $d\mu$ be an invariant measure with respect to the flow of X. Then the flow is mixing if and only if the only pole of $R_\pm(\lambda)$ on the line $i\mathbb{R}$ is $\lambda = 0$ and it is a simple pole with residue $\pm(1 \otimes 1)$.

The proof is not very involved and is based on estimates using the mixing properties and the spectral projector identity (4). Using the theorem together with (5), we can write the Laurent expansion of $R_\pm(\lambda)$ at $\lambda = 0$ as

$$R_+(\lambda) = \frac{1 \otimes 1}{\lambda} + R_0 + O(\lambda), \quad R_-(\lambda) = -\frac{1 \otimes 1}{\lambda} - R_0^* + O(\lambda). \tag{6}$$

where $R_0, R_0^* : H^s(SM) \to H^{-s}(SM)$ are bounded (The existence of R_0 and R_0^* is guaranteed by Faure-Sjöstrand). Then the bounded self-adjoint operator $\Pi : H^s(SM) \to H^{-s}(SM)$ for all $s > 0$ is defined as

$$\Pi := R_0 + R_0^*$$

and satisfies the following properties

1. $\mathrm{Ran}\,(\Pi)$ is of infinite dimension and is dense in the space of invariant distributions $\mathcal{I} := \{w \in C^{-\infty}(SM) : Xw = 0\}$.
2. For all $f \in H^s(SM), X\Pi f = 0$.
3. For all $f \in H^{s+1}(SM)$ such that $Xf \in H^s(SM)$, $\Pi X f = 0$.
4. If $f \in H^s(SM)$ with $\langle f, 1\rangle = 0$, then $f \in \ker \Pi$ if and only if there exists a solution $H^s(SM)$ to the cohomological equation $Xu = f$, and u is unique modulo constants.

The boundedness of Π comes from the boundedness of R_0 and R_0^*. Moreover, properties 2 & 3 are obtained from manipulating equation (6), namely

$$\mathrm{Id} = (-X+\lambda)R_+(\lambda) = 1 \otimes 1 - XR_0 + O(\lambda),$$
$$\mathrm{Id} = (-X-\lambda)R_-(\lambda) = 1 \otimes 1 + XR_0^* + O(\lambda)$$

Taking $\lambda \to 0$ gives

$$-XR_0 = \mathrm{Id} - 1 \otimes 1 = -R_0X, \quad XR_0^* = \mathrm{Id} - 1 \otimes 1 = R_0^*X.$$

Therefore,

$$X\Pi f = X(R_0 + R_0^*)f = (1 \otimes 1 - \mathrm{Id})(f) + (\mathrm{Id} - 1 \otimes 1)(f) = 0$$

Similarly, we have $\Pi X f = 0$ for all $f \in H^{s+1}(SM)$ such that $Xf \in H^s(SM)$.

We mention without proving that property 4 follows from the theory of anisotropic Sobolev space and propagation of singularities. As for property 1, the infinite dimensionality of $\mathrm{Ran}\,(\Pi)$ follows from, for each closed geodesic γ, taking a smooth function f supported in an arbitrarily small tubular neighborhood (open band) which $f = 1$ on γ and $\langle f, 1\rangle = 0$. Such an f is not in $\ker \Pi$, for otherwise $\Pi f = 0$ would imply by property 4 that $Xu = f$ for some $u \in C^\infty(SM)$ and hence $\int_\gamma f = 0$, a contradiction.

Since Anosov manifolds have countably many disjoint closed geodesics γ_k, let $\{f_k\}$ with disjoint supports so that $\int_{\gamma_j} f_k = \delta_{jk}$. By property 4 of the operator Π again, we have $\dim \mathrm{span}\,\{\Pi f_k : k \le N\} = N$ for any $N > 0$. This shows $\dim \mathrm{Ran}\,(\Pi) = \infty$.

Moreover, the density of $\mathrm{Ran}\,(\Pi)$ in the space of invariant distribution $\mathcal{I}$ can be obtained by looking at the restriction into $H^{-s}(SM)$ for each $s > 0$. Let

$$I_{-s} := \{w \in H^{-s}(SM) : Xw = 0\},$$

then the closure in $H^{-s}(SM)$ of $\mathrm{Ran}\,(\Pi|_{H^s}) := \{\Pi f \in H^{-s}(SM) : f \in H^s(SM)\}$ is I_{-s}.

Relation between Π and I_m

Now we see the relation between the X-ray transform I_m and the operator Π: Given $f \in C^\infty(SM)$ such that $I_m f = 0$, then by Livsic theorem, there exists $u \in C^\infty(SM)$ such that $Xu = f$, which implies $\Pi f = \Pi X u = 0$ by property 2. Thus, $\ker I_m \subset \ker \Pi$.

To study $\ker \Pi$, we look at it acting on symmetric m-tensors. In particular, we define the operator $\Pi_m : C^\infty(M, \otimes_S^m T^*M) \to C^{-\infty}(M, \otimes_S^m T^*M)$ as

$$\Pi_m := \pi_{m*}\, \Pi\, \pi_m^*$$

where $\pi_{m*} : C^{-\infty}(SM) \to C^{-\infty}(M, \otimes_S^m T^*M)$ is the push-forward induced by π_m^* on distributions, namely

$$\langle \pi_{m*} u, \psi\rangle = \langle u, \pi_m^* \psi\rangle_g$$

Since the case for all $m > 0$ is rather complicated and similar to the case when $m = 0$, we only present the results for $m = 0$, namely $\Pi_0 : C^\infty(M) \to C^{-\infty}(M)$. We have the following theorem characterizing Π_0 and $\ker \Pi_0$:

Theorem 3. If (M, g) has Anosov geodesic flow, the operator Π_0 is an elliptic self-adjoint pseudo-differential operator of order -1, with principal symbol

$$\sigma(\Pi_0)(x, \xi) = C_n |\xi|_{g_x}^{-1}$$

where C_n is a non-zero constant depending only on n. As a consequence, the kernel $\ker \Pi_0 := \{f \in C^{-\infty}(M) : \Pi_0 f = 0\}$ is finite dimensional and its elements are smooth.

The proof involves a detailed examination using microlocal analysis and pseudo-differential calculus, which we will not go into. However, it worth mentioning that the finite dimensionality of $\ker \Pi_0$ comes from the ellipticity.

In the case when $m = 0$, we can show further that $\ker \Pi\pi_0^* : C^{-\infty}(M) \to C^{-\infty}(SM)$ is trivial, which directly implies the injectivity of I_0. The difficulty comes in when $m \neq 0$, in which case we can only show $\ker \Pi\pi_m^* : C^{-\infty}(M, \otimes_S^m T * M) \cap \ker D^* \to C^{-\infty}(SM)$ is finite dimensional.

However, the finite dimensionality of $\Pi\pi_m^*$ is still sufficient to get a key ingredient of showing s-injectivity of I_m: The existence of invariant distributions with prescribed push-forward π_{m*}, namely for each $f \in H^s(M, \otimes_S^m T^*M) \cap \ker D^*$ with $\langle f, k\rangle_{L^2} = 0$ for all $k \in \ker \Pi\pi_m^* \cap \ker D^*$, there exists $w \in C^{-\infty}(SM)$ such that $Xw = 0$ and $\pi_{m*}w = f$.

Once we have the existence of such invariant distribution, we obtain the s-injectivity: For any $f \in C^\infty(M, \otimes_S^m T^*M)$ with $I_m(f) = 0$, then by Livsic theorem there exists $u \in C^\infty(SM)$ such that $Xu = \pi_m^* f$. By the GK Fourier decomposition method on the sphere bundle, we get further $u = \pi_{m-1}^* q$ for some $q \in C^\infty(M, \otimes_S^{m-1} T^*M)$, or equivalently $f = Dq$, which implies $f \in \ker D^*$. (think more about the existence of prescribed pushforward)

Appendix.

Stone's Theorem on One-parameter Unitary Groups Let $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous one-parameter unitary group. Then there exists a unique operator $A : \mathcal{D}_A \to \mathcal{H}$, that is self-adjoint on $\mathcal{D}_A$ and such that $U_t = e^{itA}$ for all $t \in \mathbb{R}$. The domain of A is defined by

$$\mathcal{D}_A = \left\{ \psi \in \mathcal{H} : \lim_{\varepsilon \to 0} \frac{-i}{\varepsilon}(U_\varepsilon(\psi) - \psi) \text{ exists } \right\}$$

Conversely, let $A : \mathcal{D}_A \to \mathcal{H}$ be a self-adjoint operator on $\mathcal{D}_A \subset \mathcal{H}$. Then the one-parameter family U_t of unitary operators defined by $U_t := e^{itA}$ for all $t \in \mathbb{R}$ is a strongly continuous one-parameter group.

Resolvent Functional Calculus The Hille-Yosida theorem relates the resolvent through a Laplace transform to an integral over the one-parameter group generated by A. In particular, if A is self-adjoint and $U(t)$ is the associated one-parameter group of unitary operators, then the resolvent of iA can be recovered as the Laplace transform

$$R(z; iA) = (iA - z)^{-1} = \int_0^\infty e^{-zt} U(t)\, dt.$$

Schwartz Kernel Theorem Let X and Y be open sets in $\mathbb{R}^n$. Every distribution $k \in \mathcal{D}'(X \times Y)$ defines a continuous linear map $K : \mathcal{D}(Y) \to \mathcal{D}'(X)$ such that

$$\langle k, u \times v\rangle = \langle Kv, u\rangle$$

for every $u \in \mathcal{D}(X), v \in \mathcal{D}(Y)$. Conversely, for every such continuous linear map K, there exists a unique distribution $k \in \mathcal{D}'(X \times Y)$ such that the above equality holds. In particular, we call the distribution k the kernel of the map K.

Fredholm Operator Given Banach spaces X and Y, a bounded linear operator $T : X \to Y$ is Fredholm if range(T) is closed, $\dim\ker T < \infty$, and $\dim\operatorname{coker} T < \infty$. Moreover, the index of a Fredholm operator is the integer

$$\text{ind}T := \dim\ker T - \dim\operatorname{coker} T$$

where the cokernel coker$(T) = Y/\text{range}(T)$.

A useful characterization of Fredholm operators goes as follow: A bounded operator $T : X \to Y$ is Fredholm if and only if it is invertible modulo compact operators, i.e. there exists a compact operator $S : Y \to X$ such that $Id_X - ST$ and $Id_Y - TS$ are compact on X and Y respectively.

Moreover, the set of Fredholm operators from X to Y is open in the Banach space $L(X, Y)$ of bounded linear operator, equipped with the operator norm, and the index is locally constant.

Exercise 1. Define the symmetric derivative operator $D := \sigma \circ \nabla : C^\infty(M; S^mT^*M) \to C^\infty(M; S^{m+1}T^*M)$, then the divergence operator is its formal adjoint given by $D^*f := -\text{Tr}(\nabla f)$.

Proof. Given a linear differential operator $L : \Gamma(E) \to \Gamma(F)$ where E and F are vector bundles over the Riemannian manifold M, then L admits a **formal adjoint** $L^* : \Gamma(F) \to \Gamma(E)$ defined by

$$\int_M \langle Lf, g\rangle \, dV_g = \int_M \langle f, L^*g\rangle \, dV_g$$

where $f \in \Gamma(E)$ and $g \in \Gamma(F)$ are compactly supported. In our case, that the formal adjoint of $D = \sigma \circ \nabla$ is indeed the divergence operator follows from an integration by part formula, i.e. If ω is any k-tensor field and η any $k+1$-tensor field,

$$\int_M \langle \nabla\omega, \eta\rangle \, dV = -\int_M \langle \omega, \text{Tr}(\nabla g)\rangle \, dV + \int_{\partial M} \langle \omega \otimes N, \eta\rangle \, d\tilde{V}$$

where the pairing $\langle \cdot, \cdot \rangle$ is with respect to the musical isomorphisms and N is the outward normal vector on ∂M.

In the case of closed manifold M, the last term on the right hand side above vanishes automatically; in the general case of manifold with boundary, the term still vanishes because in the definition of formal adjoint, testing functions are taken to be compactly supported. □

Exercise 2. With the operator D defined as above, show that for each $f \in C^k(M; S^mT^*M)$, we have $\pi_{m+1}^* Df = X\pi_m^* f$ where X is the geodesic vector field.

Proof. We first note that for any m-tensor g on the Riemannian manifold M, we have $\pi_m^* g = \pi_m^* \sigma g$: Indeed, let $g = g_{i_1 \cdots i_m} dx^{i_1} \otimes \cdots \otimes dx^{i_m}$ and $\sigma g = g_{i_1 \cdots i_m} dx^{i_1} \cdots dx^{i_m}$ where

$$dx^{i_1} \cdots dx^{i_m} = \frac{1}{m!} \sum_{\sigma \in S_m} dx^{i_{\sigma(1)}} \otimes \cdots \otimes dx^{i_{\sigma(m)}}$$

Then for any $(x, v) \in SM$,

$$\pi_m^*(\sigma g)(x, v) = g_{i_1 \cdots i_m}(x) \left(\frac{1}{m!} \sum_{\sigma \in S_m} v^{i_{\sigma(1)}} \cdots v^{i_{\sigma(m)}} \right) = g_{i_1 \cdots i_m}(x) v^{i_1} \cdots v^{i_m} = \pi_m^* g(x, v)$$

where the second to last equality follows because $v^{i_{\sigma(1)}}, \ldots, v^{i_{\sigma(m)}}$ are simply real numbers, so rearranging order does not change their products. Hence, for our claim, it suffices to show $\pi_{m+1}^*(\nabla f) = X \pi_m^* f$.

For any $(x_0, v) \in SM$, we introduce normal coordinates $(x_1, x_2, \ldots, x_n)$ of x_0, the geodesic vector field then takes the form:

$$X_{(x_0, v)} = \sum_{k=1}^{n} v^k \frac{\partial}{\partial x^k}$$

Let $f = f_{i_1 \cdots i_m} dx^{i_1} \cdots dx^{i_m}$ be a symmetric m-tensor on M, then

$$\pi_m^* f(x_0, v) = f_{i_1 \cdots i_m}(x_0) v^{i_1} \cdots v^{i_m}$$

and

$$\nabla f(x_0) = \sum_{i_{m+1}=1}^{n} \frac{\partial f_{i_1 \cdots i_m}(x_0)}{\partial x^{i_{m+1}}} dx^{i_1} \cdots dx^{i_m} \otimes dx^{m+1}$$

Together we have

$$X \pi_m^* f(x_0, v) = \sum_{i_{m+1}=1}^{n} \frac{\partial f_{i_1 \cdots i_m}(x_0)}{\partial x^{i_{m+1}}} v^{i_1} \cdots v^{i_m} \cdot v^{i_{m+1}} = \pi_{m+1}^*(\nabla f)(x_0, v)$$

□

2022.6.21 Guillarmou & Lefeuvre, Marked Length Spectrum, 2019 (Haozhe)

Today we examine the last paper in the series, which gives the first local marked length spectrum rigidity result on Anosov manifolds of all dimensions. The main theorem states that the marked length spectrum of a Riemannian manifold of Anosov geodesic flow and non-positive curvature locally determines the metric, in the sense that two close enough metrics with the same marked length spectrum are isometric.

We will see that the proof mainly uses the analysis of the operator Π from the last paper of the first author together with some derivative estimates. We start with the definitions of the length spectrum and marked length spectrum:

Definition 1. Let (M, g) be a Riemannian manifold of Anosov geodesic flow, then the set of lengths (counting multiplicities) of closed geodesics is discrete and is called the **length spectrum** of g.

Moreover, let $\mathcal{C}$ denote the set of free-homotopy classes of M, or equivalently the set of conjugacy classes of the fundamental group $\pi_1(M)$. For a compact negatively curved Riemannian manifold (M, g), it is well known that for each $c \in \mathcal{C}$, there is a unique closed geodesic γ_c of g in the class c. Hence, we can add a marking to each element of the length spectrum and obtain the marked length spectrum:

Definition 2. The **marked length spectrum** of a compact negatively curved Riemannian manifold (M, g) is a map $L_g : \mathcal{C} \to \mathbb{R}^+$ defined as $L_g(c) = \ell_g(\gamma_c)$ where γ_c is the unique closed geodesic corresponding to c and $\ell_g(\gamma_c)$ the length of γ_c with respect to g.

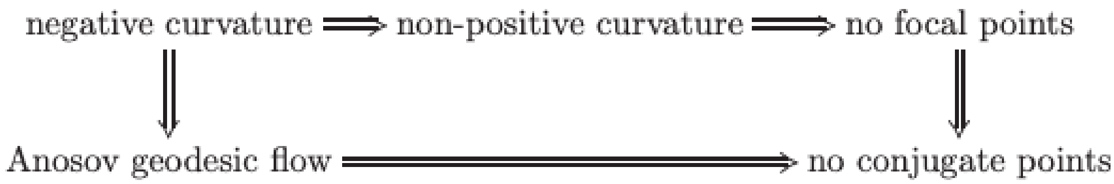
It worth mentioning that on compact negatively curved Riemannian manifolds, the Laplacian spectrum determines the length spectrum, but not necessarily vice versa. However, it is known in the case of hyperbolic surfaces that they do determine each other.

The marked length spectrum as a map certainly contains more information than its image, i.e. the length spectrum. For instance, Sunada's construction of non-isometric isospectral manifolds breaks the marked length spectrum but not the length spectrum.

Since Vigeras has constructed examples of non-isometric hyperbolic surfaces with the same length (or equivalently the same Laplacian) spectrum, rigidity questions are better raised in the setting of the marked length spectrum. For instance, the following long-standing conjecture by Burns-Katok is widely believed to be true:

Conjecture 3. If g and g_0 are two negatively curved metrics on a closed manifold M, and if they have the same marked length spectrum, then they are isometric.

The difficulty of approaching this problem lies in the non-linearity (with respect to g) of the marked length spectrum L_g, and results have been few and centered around the case when $\dim M = 2$. The best result by Croke-Fathi-Feldman has shown the conjecture for g having non-positive curvature and g_0 having no conjugate points. Here we provide a diagram which shows the relations of the different assumptions we have run into:



Naturally, attention has been given to the linearized problem with the linearized operator dL_g, which by a straightforward computation is precisely the geodesic X-ray transform I_2, whose injectivity and microlocal properties for Anosov surfaces has been examined by Guillarmou in the last paper. What remains is a few smoothness estimates which gurantees the local rigidity.

Theorem 4. Let $\mathcal{L} : \mathcal{U}_{g_0} \to \ell^\infty(\mathcal{C})$ be the g_0-normalized marked length spectrum and defined as

$$\mathcal{L}(g)(c) := \frac{L_g(c)}{L_{g_0}(c)}$$

where $\mathcal{U}_{g_0}$ is a $C^N(M; S^2_+ T^*M)$ neighborhood of the metric g_0. Then $\mathcal{L}$ is C^2 near g_0 with the $C^3(M; S^2_+ T^*M)$ topology. In particular, there is a neighborhood $\mathcal{U}_{g_0} \subset C^3(M; S^2_+ T^*M)$ of g_0 and $C = C(g_0) > 0$ such that for all $g \in \mathcal{U}_{g_0}$,

$$\|\mathcal{L}(g) - \mathbf{1} - d\mathcal{L}_{g_0}(g - g_0)\|_{\ell^\infty(C)} \leq C \|g - g_0\|^2_{C^3(M)}$$

(unfinished...) [3]

[3] Last updated: 7/18/2022