Harmonic Series

Guangqiu Liang

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Claim.

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Lemma. Suppose $A, B, C, \dots, D, E$ is a finite geometric progression of $n+1$ terms with common ratio r and let S denote its sum, then

$$S = \frac{A^2 - BE}{A - B}$$

Proof. By definition,

$$\frac{A}{B} = \frac{A}{Ar} = \frac{A(1 + r + \dots + r^{n-1})}{Ar(1 + r + \dots + r^{n-1})} = \frac{A + B + C + \dots + D}{B + C + \dots + D + E} = \frac{S - E}{S - A}$$

Solving for S gives the desired result.

Proof. Given any $N \in \mathbb{N}$, we want to show that the partial sum eventually exceeds N. We proceed by showing that we can always remove a finite string of consecutive terms whose sum is 1 or more, independent of where we start.

For the sake of contradiction, we assume there exists an index a such that the sum

$$\frac{1}{a} + \frac{1}{a+1} + \frac{1}{a+2} + \dots < 1$$

But notice the denominators $a, a+1, a+2, \dots$ form an arithmetic progression. Now let's consider a geometric progression beginning with the same first two terms, namely $a, a+1, C, D, \dots, K$ where we can choose $K \geq a^2$ because the common ratio is greater than 1. Then a simple comparison shows that $C > a+2$, $D > a+3$,... Thus, we have

$$\frac{1}{a} + \frac{1}{a+1} + \frac{1}{a+2} + \dots > \frac{1}{a} + \frac{1}{a+1} + \frac{1}{C} + \frac{1}{D} + \dots + \frac{1}{K}$$

where the expression on the left has the same finite number of terms as that on the right. Now summing the geometric progression on the right with the seventeenth-century-fashion lemma we proved above gives,

$$\frac{1}{a} + \frac{1}{a+1} + \frac{1}{a+2} + \dots > \frac{\frac{1}{a^2} - \frac{1}{a+1}\left(\frac{1}{K}\right)}{\frac{1}{a} - \frac{1}{a+1}} \geq \frac{\frac{1}{a^2} - \frac{1}{(a+1)a^2}}{\frac{1}{a} - \frac{1}{a+1}} = 1,$$

a contradiction. Thus, collecting N such strings gives a desired partial sum.

Remark: This little cute but surprisingly elegant argument comes from Jakob Bernoulli (1654-1705) where only elementary summation and comparison are used. Interested readers can discover more details in the book “The Calculus Gallery” by William Dunham.