Local Conformality and Isothermal Coordinates

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In this self contained write-up, I presented my favorite theorem from Do Carmo's *Differential Geometry of Curves and Surfaces*, and I will give my reason why this is my favorite at the end. No prerequisites more than calculus, linear algebra, and some basic topology, are needed.

Theorem. Any two regular surfaces are locally conformal. (Do Carmo, Section 4-2, pp.230)

Definition 1. A subset $S \subset \mathbb{R}^3$ is a **regular surface** if, for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that

(1) $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$ is differentiable, meaning the component functions have continuous partial derivatives of all orders in U.
(2) $\mathbf{x}$ is a homeomorphism.
(3) For each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.

Remark: Intuitively, a regular surface can be seen as a really nice subset of $\mathbb{R}^3$ on which calculus can be performed. For those who are familiar with the language of manifolds, a regular surface is a 2-dimensional regular submanifold $S \subset \mathbb{R}^3$ with the subspace topology for which the inclusion $\iota : S \hookrightarrow \mathbb{R}^2$ is an embedding.

Definition 2. A diffeomorphism $\varphi : S \to \overline{S}$ is called a **conformal map** if for all $p \in S$ and all $v_1, v_2 \in T_p(S)$ we have

$$\langle d\varphi_p(v_1), d\varphi_p(v_2)\rangle = \lambda^2(p)\langle v_1, v_2\rangle_p$$

where λ^2 is a nowhere-zero differentiable function on S. The surfaces S and $\overline{S}$ are then said to be conformal. A map $\varphi : V \to \overline{S}$ of a neighborhood V of $p \in S$ into $\overline{S}$ is a **local conformal map** at p if there exists a neighborhood $\overline{V}$ of $\varphi(p)$ such that $\varphi : V \to \overline{V}$ is a conformal map. If for each $p \in S$, there exists a local conformal map at p, the surface S is said to be **locally conformal** to $\overline{S}$.

Remark: It is not hard to show that a map is locally conformal if and only if it preserves angles, meaning if unit speed curves $\alpha : I \to S$ and $\beta : I \to S$ intersect at $t = 0$ with angle θ defined as $\cos\theta = \langle \alpha', \beta'\rangle$, then $\varphi \circ \alpha : I \to \overline{S}$ and $\varphi \circ \beta : I \to \overline{S}$ also intersect with angle θ.

Definition 3. The metric in the tangent space of a regular parametrized surface $\mathbf{x} = \mathbf{x}(u, v)$ at $p \in T_p(S)$ locally is given by

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2 \tag{1}$$

$$E(u,v) = \langle \mathbf{x}_u(u,v), \mathbf{x}_u(u,v)\rangle_p, \quad F = \langle \mathbf{x}_u, \mathbf{x}_v\rangle_p, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v\rangle_p$$

The coordinates $u = u(x,y), v = v(x,y)$ are called **isothermal** if the metric becomes

$$ds^2 = \lambda^2(dx^2 + dy^2),$$

with $\lambda = \lambda(x,y) > 0$ smooth. Or equivalently

$$E = G = \lambda^2(x,y), \quad F = 0.$$

The proof of the theorem is based on the existence of an isothermal coordinate in a neighborhood of any point of a regular surface. Once the existence is assumed, S is locally conformal to a plane. Indeed, the isothermal parametrization $\mathbf{x} = \mathbf{x}(x,y)$ maps the straight lines of the plane conformally onto the coordinate curves of the surface. Then by composition, S is locally conformal to any other regular surface.

However, the existence of isothermal coordinate is deep and delicate and can be found in L. Bers, *Riemann Surfaces*, pp.15-35. or Chern, *An elementary proof of the existence of isothermal parameters on a surface*, 1955. Now we give an example of parametrizing $\mathbb{S}^2$ and any surface of revolution S by isothermal coordinates, which establishes the local conformal relation between $\mathbb{S}^2$ and S.

Example 1. (Mercator's projection) Let $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, where

$$U = \{(\theta, \varphi) \in \mathbb{R}^2 : 0 < \theta < \pi, 0 < \varphi < 2\pi\}$$

$$\mathbf{x}(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

be a parametrization of the unit sphere $\mathbb{S}^2$. Let

$$\log\tan\frac{1}{2}\theta = u, \quad \varphi = v,$$

then a new parametrization of the coordinate neighborhood $\mathbf{x}(U) = V$ can be given by

$$\mathbf{y}(u,v) = (\mathrm{sech} u\cos v, \mathrm{sech} u\sin v, \tanh u)$$

By direct computation, the coefficients of the metric induced by $\mathbf{y}$ are given by

$$E = \langle \mathbf{x}_u, \mathbf{x}_u\rangle = \mathrm{sech}^2 u, \quad F = 0, \quad G = \langle \mathbf{x}_v, \mathbf{x}_v\rangle = \mathrm{sech}^2 u$$

Thus, $\mathbf{y}^{-1} : V \subset \mathbb{S}^2 \to \mathbb{R}^2$ is a conformal map which takes the meridians and parallels of $\mathbb{S}^2$ onto straight lines of the plane.

(Lagrange 1779) Let $\mathbf{z} : U \subset \mathbb{R}^2 \to S$ be the parametrization of a surface of revolution S:

$$\mathbf{x}(\theta, \varphi) = (f(\varphi)\cos\theta, f(\varphi)\sin\theta, g(\varphi)), \quad f(\varphi) > 0,$$

$$U = \{(\theta, \varphi) \in \mathbb{R}^2 : 0 < \theta < 2\pi, a < \varphi < b\}$$

Since the map $\phi : U \to \mathbb{R}^2$ given by

$$\phi(u,v) = \left(u, \int \frac{\sqrt{(f'(v))^2 + (g'(v))^2}}{f(v)} dv \right)$$

is a local diffeomorphism (here we forgo the cumbersome but standard calculation). Now let

$$u = \theta, \quad v = h(\varphi) = \int \frac{\sqrt{(f'(\varphi))^2 + (g'(\varphi))^2}}{f(\varphi)} dv$$

Then a new parametrization of the coordinate neighborhood $\mathbf{x}(U) = V$ can be given by

$$\mathbf{y}(u,v) = (f(h^{-1}(v)) \cos u, f(h^{-1}(v)) \sin u, g(h^{-1}(v)))$$

By direct computation (avoid calculating h^{-1} by using inverse function theorem), the coefficients of the metric induced by $\mathbf{y}$ are given by

$$E = G = (f'(v))^2 > 0, \quad F = 0$$

Thus, $\mathbf{y}^{-1} : V \subset S \to \mathbb{R}^2$ is a conformal map which again takes the meridians and parallels of the surface of revolution S onto the straight lines of the plane. Thus, we have established the conformal relation between the unit sphere $\mathbb{S}^2$ and any surface of revolution S.

Summary

I hope the readers have not yet been baffled by the tedious computations of finding isothermal coordinates. Indeed, it is in general hard to find an isothermal coordinate for a given regular surface. However, the existence of one provides a huge amount of insights in the theory of surface. Some of these insights and the reason why I like this theorem are presented below:

1. Isothermal coordinate takes a very elegant form, assuming which will enable us to simplify a lot of proofs tremendously. For instance, the proof of Local Gauss-Bonnet Theorem (Do Carmo, *Differential Geometry of curves & surfaces* , pp. 272-273) assumes the existence of an isothermal parametrization of an oriented regular surface so that lots of simplifications can be made, without which I presume the amount of work might be doubled.

2. As we saw, isothermal coordinates arise naturally from conformal mapping, and the later is a set of important techniques in complex analysis. The classical approach of imposing a Riemann surface structure on any smooth orientable compact surface in $\mathbb{R}^3$ involves the use of isothermal coordinates. That is, upon finding an isothermal coordinate, any such surface will admit a conformal structure on which complex analysis can be performed. Interested readers may consult Schlag, *A Course in Complex Analysis and Riemann Surfaces*, pp.135-138.

3. Finally, the reason why this is my favorite theorem from Chapter 2, 3, and 4 of Do Carmo is its simplicity. Behind this simple line of statement, there is a beautiful story I found: I am personally very interested in complex analysis and keen to play with conformal maps. They are, according to Evan Chen in *An Infinitely Large Napkin*, the nicest functions on earth, which means they are also super rare outside the realm of complex analysis. However, this theorem tells me if I look close enough, golds are in fact everywhere.