Arithmetic-geometric Mean, π, Perimeter of Ellipse, and Beyond

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Abstract

From the shape that people are worst at drawing by hand to the orbits that most planets cannot escape, the appearance of circles and ellipses is ubiquitous in the universe. The study of these two geometric objects hence never stops. Among countless properties they have, π and elliptic perimeter are two of the brightest ones that mathematicians have been pursuing for thousands of years. In this paper, we will explore a surprising but elegant relation between arithmetic-geometric mean and the two of them, which somehow makes the world more approachable to us.

We will begin by closely examining arithmetic-geometric mean as the cornerstone for our discussion. Then, with modest knowledge of elliptic integrals, we will reach the first highlight of the paper, namely a formula for π that is highly suitable for computation. After the exposure, we restore ourselves with the tool of Landen's Transformations, which will serve as a medium tool of relating different elliptic integrals. We will climax with the intangible nature of ellipse, displayed as some exact but not simple or simple but not exact formulas for the perimeter. Last but not the least, we explore a special type of elliptic functions which arises naturally from elliptic integrals and peek at some of its interesting properties.

This paper is an explicit review of the references listed in the end and should be approachable to students armed with elementary calculus and patience with heavy algebraic manipulations.

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The proofs and ideas in this paper are mainly credited to Almkvist and Berndt [1], while the contributions of other mathematicians will be explicitly stated as the discussion proceeds.

1 Arithmetic-geometric Mean

We begin our journey with a brief discussion of the arithmetic-geometric mean. The definition, rate of convergence, and implication of the mean will be presented below.

Definition 1.1. *Given two positive real numbers a and b where $a < b$, define the following recursion:*

$$a_0 = a, \quad b_0 = b,$$

$$a_{n+1} = \sqrt{a_n b_n}, \quad b_{n+1} = \frac{a_n + b_n}{2}$$

The number the two sequences converge to is called the ***arithmetic-geometric mean*** *of a and b, which is often denoted as $M(a, b)$.*

To show the validity of the above definition, we need to prove limits of the two sequences exit and are the same: Notice by AM-GM,

$$a_{n+1} = \sqrt{a_n b_n} < \frac{a_n + b_n}{2} = b_{n+1}$$

which implies the n-th term of $\{a_n\}$ is strictly less than that of $\{b_n\}$. Thus,

$$a_{n+1} = \sqrt{a_n b_n} > \sqrt{a_n \cdot a_n} = a_n,$$

$$b_{n+1} = \frac{a_n + b_n}{2} < \frac{b_n + b_n}{2} = b_n$$

Thus, $\{a_n\}$ is increasing and bounded above by b, and $\{b_n\}$ is decreasing and bounded below by a. By monotone convergence theorem, each sequence therefore converges. Moreover, by the elementary identity intrinsically embedded in AGM, namely $(a+b)^2 - 4ab = (a-b)^2$, we have

$$\frac{b_1 - a_1}{b - a} = \frac{b - a}{4(b_1 + a_1)} = \frac{b - a}{2(a + b) + 4a_1} < \frac{1}{2}$$

Performing induction on n by the exact procedure as above gives,

$$b_n - a_n = \left(\frac{1}{2}\right)^n (b - a)$$

Clearly, as $n \to \infty$, the sequence $\{b_n - a_n\}$ tends to 0. Therefore, $\{a_n\}$ and $\{b_n\}$ have the same limit. Take a closer look at the rapidity of the convergence by defining,

$$c_n = \sqrt{b_n^2 - a_n^2}, \quad n \geq 0$$

Then,

$$c_{n+1} = \sqrt{b_{n+1}^2 - a_{n+1}^2} = \frac{1}{2}(b_n - a_n) = \frac{c_n^2}{4b_{n+1}} \le \frac{c_n^2}{4M(a,b)}$$

Thus, $\{c_n\}$ tends to 0 quadratically. This is a fast rate of convergence and well adapted to numerical computations. Later in **Chapter 3**, we will see the rapid convergence of AGM gives rise to Gauss-Legendre algorithm which can produce 45 million correct digits of π with only simply 25 iterations.

2 Introduction to Elliptic Integral

As for now, we step into the discussion of elliptic integral where the definitions of several special types of elliptic integral will be given. Then, we will delicately explore a both surprising and essential representation of arithmetic-geometric mean in terms of elliptic integral, which is due to Gauss, of course.

Definition 2.1. *Define the complete elliptic integral of the first kind as:* $|x| < 1$,

$$K(x) = \int_0^{\pi/2} (1 - x^2 \sin^2 \varphi)^{-\frac{1}{2}} d\varphi = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}} \tag{1}$$

Such type of integral naturally arises in the computation of the arc length of a lemniscate or the period of a pendulum. Later in this chapter we shall see that elliptic integral of the first kind can be directly expressed as a function of the arithmetic-geometric mean.

Definition 2.2. *Define the complete elliptic integral of the second kind as:* $|x| < 1$,

$$E(x) = \int_0^{\pi/2} (1 - x^2 \sin^2 \varphi)^{\frac{1}{2}} d\varphi = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}} dt \tag{2}$$

Notice, it's easy to tell that the elliptic integral of the second kind describes the perimeter of an ellipse. Liouville proved in 1834 that both the first and the second kind are nonelementary, meaning that there are no such antiderivatives of these integrals in terms of elementary functions.

Gauss proved the theorem below with his ingenious idea of equating the series expansion of both sides of the equation, which can be found in [1] pp.588-589. However, here we present another short and elegant proof that is given by Newman [8]:

Theorem 2.1 *Let $|x| < 1$, then*

$$M(1+x, 1-x) = \frac{\pi}{2K(x)}$$

Proof. Before proving theorem 2.1, we give a reformulation of it. Define

$$I(a,b) = \int_0^{\pi/2} (a^2 \cos^2 \varphi + b^2 \sin^2 \varphi)^{-1/2} d\varphi$$

Clearly,

$$I(a,b) = \frac{1}{a} K(x), \quad x = \frac{1}{a}\sqrt{a^2 - b^2}$$

Since

$$M(a,b) = M(a_1, b_1) \quad \text{and} \quad M(ca, cb) = cM(a,b)$$

for any constant c, we have

$$M(1-x, 1+x) = \frac{1}{a} M(a + \sqrt{a^2 - b^2}, a - \sqrt{a^2 - b^2}) = \frac{1}{a} M(a,b)$$

Then, the reformulation of Theorem 2.1 follows immediately as

Theorem 2.1' *Let $a > b > 0$. Then*

$$M(a,b) = \frac{\pi}{2I(a,b)}$$

Proof. With a change of variable, here we omit the cumbersome calculations, we can rewrite $I(a,b)$ as

$$I(a,b) = \int_0^{\infty} \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2 + a^2)(x^2 + b^2)}}$$

Observe that the elliptic integral on the right hand side above is invariant under the transformation

$$(a,b) \to \left(\frac{a+b}{2}, \sqrt{ab} \right)$$

since by a change of variable $t = \frac{1}{2}(x - ab/x), x \in (0, \infty), t \in (-\infty, \infty)$, moreover

$$dt = \frac{x^2 + ab}{2x^2}, \quad t^2 + \left(\frac{a+b}{2}\right)^2 = \frac{(x^2+a^2)(x^2+b^2)}{4x^2}, \quad t^2 + ab = \frac{(x^2+ab)^2}{4x^2}$$

under the transformation,

$$\int_{-\infty}^{\infty} \frac{dt}{\sqrt{(t^2 + \left(\frac{a+b}{2}\right)^2)(t^2+ab)}} = 2\int_0^{\infty} \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}}$$

the elliptic integral is preserved.

Therefore, if we continue the process of taking arithmetic-geometric mean of a and b, which we denote as M, we find that all the integrals have the same value

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2+a_n^2)(x^2+b_n^2)}}$$
$$\to \int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2+M^2)(x^2+M^2)}} = \int_{-\infty}^{\infty} \frac{dx}{x^2+M^2} = \frac{\pi}{M}$$

and we obtain the formula

$$M(a,b) = \pi / \int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}} = \frac{\pi}{2I(a,b)}$$

Theoretically this formula is already fascinating in the sense that it bridges elliptic integral and arithmetic-geometric mean. In practice, we shall see in **Chapter 3** and **Chapter 5** that theorem 2.1 is the foundation of fast computations of π and elliptic perimeter.

3 Calculation of π

In this section, we present an efficient method to compute π with the use of the fast convergence of the arithmetic-geometric mean. Before giving the explicit formula, we need two important theorems that relate the integrals $E(x)$ and $K(x)$.

The first one below is also known as the Legendre's Relation:

Theorem 3.1. *Let $x' = \sqrt{1-x^2}$, where $0 < x < 1$. Then*

$$K(x)E(x') + K(x')E(x) - K(x)K(x') = \frac{\pi}{2} \tag{3}$$

where K and E are the complete elliptic integrals of the first and second kinds defined as (1) and (2) respectively.

The proof given below follows the idea to show that the derivative of the left hand side of the equation above is 0, thereby showing that it is a constant. Then, we find the constant by taking limits.

Proof. To make notations clear, we let $c = x^2$ and $c' = 1 - c$. Also, we denote $E(c)$ and $K(c)$ as E and K respectively. We begin by differentiating $(E - K)$ with respect to c,

$$\frac{d}{dc}(E-K) = \frac{d}{dc}\left(\int_0^{\pi/2} \frac{-c\sin^2\varphi}{(1-c\sin^2\varphi)^{1/2}}d\varphi\right) \tag{4}$$

$$= \frac{E}{2c} - \frac{1}{2c}\int_0^{\pi/2} \frac{1}{(1-c\sin^2\varphi)^{1/2}}d\varphi \tag{5}$$

Notice such relation,

$$\frac{d}{d\varphi}\left(\frac{\sin\varphi\cos\varphi}{(1-c\sin^2\varphi)^{1/2}}\right) = \frac{1}{c}(1-c\sin^2\varphi)^{1/2} - \frac{c'}{c}(1-c\sin^2\varphi)^{-3/2}$$

we substitute it into the (5) and yield

$$\frac{d}{dc}(E-K) = \frac{E}{2c} - \frac{E}{2cc'} + \frac{1}{2c'}\int_0^{\pi/2} \frac{d}{d\varphi}\left(\frac{\sin\varphi\cos\varphi}{(1-c\sin^2\varphi)^{1/2}}\right)d\varphi$$
$$= \frac{E}{2c}\left(1 - \frac{1}{c'}\right) = -\frac{E}{2c'}$$

To keep the notation consistent, we let denote $K(c')$ and $E(c')$ as K' and E' respectively. Since $c' = 1 - c$, by exact the same procedure as above with a reflection of c, we have

$$\frac{d}{dc}(E' - K') = \frac{E'}{2c}$$

Moreover,

$$\frac{dE}{dc} = -\frac{1}{2}\int_0^{\pi/2} \frac{\sin^2\varphi}{\sqrt{1-c\sin^2\varphi}}d\varphi = \frac{E-K}{2c}$$

Similarly,
$$\frac{dE'}{dc} = -\frac{E'-K'}{2c'}$$
Now, we are in good shape to launch the first conclusion. If L denotes the left hand side of (3), then we can write L as

$$L = EE' - (E-K)(E'-K')$$

Combining all the derivatives above, we find that

$$\frac{dL}{dc} = \frac{(E-K)E'}{2c} - \frac{E(E'-K')}{2c'} + \frac{E(E'-K')}{2c'} - \frac{(E-K)E'}{2c} = 0$$

Thus, L is a constant. Then, we find its value by letting c tend to 0,

$$E - K = -c\int_0^{\pi/2} \frac{\sin^2\varphi}{(1-c\sin^2\varphi)^{1/2}}d\varphi = O(c)$$

and

$$K' = \int_0^{\pi/2}(1-c'\sin^2\varphi)^{-1/2}d\varphi \leq \int_0^{\pi/2}(1-c')^{-1/2}d\varphi = O(c^{-1/2})$$

Thus,

$$\lim_{c\to 0} L = \lim_{c\to 0}\{(E-K)K' + E'K\} = \lim_{c\to 0}\left(O(c^{1/2}) + 1\cdot\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Then, we turn to look at the second key theorem:
Theorem 3.2. *For $a > b > 0$, define*

$$J(a,b) = \int_0^{\pi/2}(a^2\cos^2\varphi + b^2\sin^2\varphi)^{1/2}d\varphi \tag{6}$$

and recall c_n is defined in Section 1. Then

$$J(a,b) = \left(a^2 - \frac{1}{2}\sum_{n=0}^{\infty}2^n c_n^2\right)I(a,b)$$

where $I(a,b)$ is defined in Section 2.

Here we forego a cumbersome proof that involves heavy algebra, which can be found in Borwein's book [3], pp 13-15. Notice that

$$J(a,b) = aE(x), \quad x = \frac{1}{a}\sqrt{a^2 - b^2}$$

With the power of theorem 3.1 and 3.2, we are now in a decent position to pull out the formula for π which is highly suitable for computation.

Theorem 3.3. *With c_n defined as in Section 1, namely $c_n = \sqrt{a_n^2 - b_n^2}$*
,

$$\pi = \frac{4M^2(1, 1/\sqrt{2})}{1 - \sum_{n=1}^{\infty} 2^{n+1} c_n^2}$$

Proof. We begin by recalling the relations of the elliptic integrals in Sections 2 and 3, namely

$$I(a,b) = \frac{1}{a}K(x), \quad E(x) = \frac{1}{a}J(a,b)$$

Letting $x = x' = 1/\sqrt{2}$ in theorem 1, we have

$$2K\left(\frac{1}{\sqrt{2}}\right)E\left(\frac{1}{\sqrt{2}}\right) - K^2\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{2}$$

Now if we let $a = 1$ and $b = 1/\sqrt{2}$ in theorem 2, we have

$$E\left(\frac{1}{\sqrt{2}}\right) = \left(1 - \sum_{n=0}^{\infty} 2^n c_n^2\right) K\left(\frac{1}{\sqrt{2}}\right)$$

Moreover, with theorem 2.2 using the same setting for a and b as above,

$$M(1, 1/\sqrt{2}) = \frac{\pi}{2K(1/\sqrt{2})}$$

Solving for π completes the proof.

Since, in Section 1, we've shown that the arithmetic-geometric mean and thus c_n converge quadratically, this formula computes π in a quadratical manner.

In fact, theorem 3.3 is the foundation of most of the modern algorithms that efficiently compute π, and the current world record is 31 trillion digits by Emma Haruka Iwao, in particular it's 31,415,926,535,897 digits.

Example 3.1 Notice the constant $M(1, 1/\sqrt{2})$ has been constantly appearing in the context, which implies its close connections with π. Gauss calculated it with accuracy up to 19 decimal places:

$$M(\sqrt{2}, 1) = 1.198140234735922074\dots$$

with the following table in [7, vol. III, pp. 361-371]:

n	a_n	b_n
0	1.4142135623730950488	1.0000000000000000000
1	1.2071067811865475244	1.1892071150027210667
2	1.1981569480946342955	1.1981235214931201226
3	1.1981402347938772090	1.1981402346773072057
4	1.1981402347355922074	1.1981402347355922074

Moreover, Gauss's constant in mathematics, which I believe is necessary to specify, is defined as

$$G = \frac{1}{M(1, \sqrt{2})} = 0.83462684167\dots$$

which he discovered in 1799 that

$$G = \frac{2}{\pi} \int_0^1 \frac{dt}{\sqrt{1 - t^4}}$$

4 Landen's Transformation

In this section, we will explicitly show the Landen's Transformation for the complete elliptic integral of the first kind (Theorem 4.1). Then, we will state other forms of Landen's transformation.

We first present a series representation of $K(x)$, the complete elliptic integral of the first kind defined as (1):

Lemma 4.1.

$$K(x) = \frac{\pi}{2} \sum_0^\infty \frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2} x^{2k}$$

where $(\alpha)_k$ is defined by

$$(\alpha)_k = (\alpha + 1)(\alpha + 2)\dots(\alpha + k - 1)$$

Proof. Expanding the integrand of $k(x)$ is a binomial series gives

$$(1 - x^2 \sin^2 \varphi)^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n e^{in} \sin^{2n} \varphi$$

By Weierstrass M-Test, the series converges uniformly and absolutely. Thus, we can integrate term by term, which gives

$$\begin{aligned} K(x) &= \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n e^{in} \int_0^{\pi/2} \sin^{2n} \varphi d\varphi \\ &= \sum_{j=0}^{\infty} \frac{(2j-1)!!}{(2j)!!} e^{2j} \frac{(2j-1)!!}{(2j)!!} \frac{\pi}{2} \\ &= \frac{\pi}{2} \sum_{0}^{\infty} \left(\frac{(2j-1)!!}{(2j)!!} e^j \right)^2 = \frac{\pi}{2} \sum_{0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2} x^{2k} \end{aligned}$$

Now, with the lemma in hand, we turn to show the Landen's transformation for the complete elliptic integral of the first kind,

Theorem 4.1 *If* $0 \le x < 1$*, then*

$$K\left(\frac{2\sqrt{x}}{1+x}\right) = (1+x)K(x)$$

where $K(x)$ *is the complete elliptic integral of the first kind*

Proof. Similar as proving the lemma, we expand the left hand side by binomial series,

$$\begin{aligned} K\left(\frac{2\sqrt{x}}{1+x}\right) &= \frac{1}{2} \int_0^{\pi} \left(1 - \frac{4x}{(1+x)^2} \sin^2 \varphi\right)^{-1/2} d\varphi \\ &= \frac{1}{2} \int_0^{\pi} \left(1 - \frac{2x}{(1+x)^2} (1 - \cos 2\varphi)\right)^{-1/2} d\varphi \\ &= \frac{1}{2}(1+x) \int_0^{\pi} (1 + x^2 + 2x \cos 2\varphi)^{-1/2} d\varphi \\ &= \frac{1}{2}(1+x) \int_0^{\pi} (1 + xe^{2i\varphi})^{-1/2} (1 + xe^{-2i\varphi})^{-1/2} d\varphi \end{aligned}$$

Now we are in good shape to employ the binomial expansion for the integrand. Since the manipulations are cumbersome and similar to the proof of the lemma, we omit them here and conclude

$$K\left(\frac{2\sqrt{x}}{1+x}\right) = \frac{\pi}{2}(1+x)\sum_{0}^{\infty}\frac{\left(\frac{1}{2}\right)_k^2}{(k!)^2}x^{2k} = (1+x)K(x)$$

It may seem that Landen's transformation is simply a change of variable formula. However, its importance lies in its power to compute elliptic integrals in an iterative manner. Moreover, the transformation made possible the theory of more general transformations, leading up to the theories of modular equations, complex multiplication, and singular moduli [9].

To illustrate another form of Landen's transformation, we introduce Gauss ordinary hypergeometric series,

Definition 4.1.

$$F(a,b;c;x) = \sum_{0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}x^k, \quad |x| < 1$$

where a, b, *and* c *denote arbitrary complex number*

then the Landen's transformation for hypergeometric series is given as

$$F\left(a,b;2b;\frac{4x}{(1+x)^2}\right) = (1+x)^{2a}F\left(a,a-b+\frac{1}{2};b+\frac{1}{2};x^2\right) \qquad (7)$$

where when $a = b = \frac{1}{2}$, we obtain the special case by Theorem 2.1 and 4.1,

$$F\left(\frac{1}{2},\frac{1}{2};1;\frac{4x}{(1+x)^2}\right) = (1+x)F\left(\frac{1}{2},\frac{1}{2};1;x^2\right)$$

We will see the importance of these transformation in the next section when we derive the exact by not simple formulas for the perimeter of ellipse.

5 Approximation for the perimeter of ellipse

In this section, we will explore the relation between arithmetic-geometric mean and elliptical perimeter. As mentioned in the abstract, we will first examine the exact but not simple formulas, and then the simple but not exact ones, among which Indian mathematician Ramanujan's result has been one of the most accurate.

Definition 5.1. *If an ellipse is parameterized by* $x = a\cos\varphi$ *and* $y = b\sin\varphi, 0 \le \varphi \le 2\pi$*, then from elementary calculus,*

$$L(a,b) = \int_0^{2\pi} (a^2\cos^2\varphi + b^2\sin^2\varphi)^{1/2} d\varphi = 4J(a,b)$$

where $J(a,b)$ *is as defined (6).*

Now we can immediately see from theorem 2.1 and 3.2 that arithmetic-geometric mean and elliptical perimeter are intrinsically related. But before presenting the relation, let's first take a look at two exact formulas.

Theorem 5.1. *Let* $x = a\cos\varphi$ *and* $y = b\sin\varphi, 0 \le \varphi \le 2\pi$*,* $e = (1/a)\sqrt{a^2-b^2}$*, the eccentricity of the ellipse. Then the perimeter of the ellipse can be expressed in terms of Gauss's hypergeometric series*

$$L(a,b) = 2\pi a F\left(\frac{1}{2}, -\frac{1}{2}; 1; e^2\right) \tag{8}$$

$$= \pi(a+b) F\left(-\frac{1}{2}, -\frac{1}{2}; 1; \lambda^2\right) \tag{9}$$

where

$$\lambda = \frac{a-b}{a+b}$$

Proof. Similarly, for (8), we expand of the integrand of $L(a,b)$ in a binomial series, and integrate term by term,

$$\begin{aligned} L(a,b) &= 4a\int_0^{\pi/2} (1 - e^2\cos^2\varphi)^{1/2} d\varphi \\ &= 4a\sum_{n=0}^{\infty} \frac{(-1/2)_n}{n!} e^{2n} \int_0^{\pi/2} \cos^{2n}\varphi d\varphi \\ &= 2\pi a F\left(\frac{1}{2}, -\frac{1}{2}; 1; e^2\right) \end{aligned}$$

Moreover, we show (9) by first setting the Landen's transformation defined as (7) $a = -1/2, b = 1/2, x = \lambda$, then

$$F\left(-\frac{1}{2}, \frac{1}{2}; 1; e^2\right) = F\left(\frac{1}{2}, -\frac{1}{2}; 1; e^2\right) = \frac{a+b}{2a} F\left(-\frac{1}{2}, -\frac{1}{2}; 1; \lambda^2\right)$$

Then, plugging (8) into the above formula, we prove (9).

Clearly, the above two formulae for the perimeter of ellipse involve the hypergeometric series and are thereby hard to compute. But with theorems 2.1' and theorem 3.2, we are able to generate approximations to the perimeter. Combining the two theorems gives,

$$L(a,b) = 4J(a,b) = \frac{2\pi}{M(a,b)}\left(a^2 - \frac{1}{2}\sum_{n=0}^{\infty} 2^n c_n^2\right)$$

Now one of the simple approximations can be found by replacing $M(a,b)$ by a_2 and neglecting the terms with $n \geq 2$,

$$L(a,b) \approx \frac{2\pi}{a_2}\left(a^2 - \frac{c_0^2}{2} - c_1^2\right) = \frac{2\pi a_1^2}{a_2} = 2\pi\left(\frac{a+b}{\sqrt{a}+\sqrt{b}}\right)^2$$

As you may notice, when we only consider the first two terms with $n < 2$, the formula is simple enough but exhibiting the complicate sign of taking the square root. It is reasonable to expect that the higher order approximations of this method is not suitable in a computational sense anymore.

Now we want to examine the second highlight of this paper, namely the highly accurate approximation formula given by India's mathematician S. Ramanujan.

Theorem 5.2 *Ramanujan, 1914: Suppose $L(a,b)$ denotes the circumference of the ellipse with major axis a and minor axis b, and $\lambda = (a-b)/(a+b)$, then*

$$\frac{L(a,b)}{\pi(a+b)} \approx 1 + \frac{3\lambda^2}{10+\sqrt{4-3\lambda^2}} \tag{10}$$

which Ramanujan himself claimed that the formula was found empirically, and exactly how he discovered it still remains a mystery. However, we are able to appreciate how marvellous this approximation is by looking closely at the error.

Denote the right hand side of (10) as $A(\lambda)$, then we can examine the accuracy by comparing the approximation $A(\lambda)$ with the exact series expansion, namely

$$A(\lambda) - \frac{L(a,b)}{\pi(a+b)} = A(\lambda) - F\left(-\frac{1}{2}, -\frac{1}{2}; 1; \lambda^2\right) \tag{11}$$

where for convenience we can simplify the hypergeometric series as

$$F\left(-\frac{1}{2},-\frac{1}{2};1;\lambda^2\right)=1+\frac{1}{4}\lambda^2+\frac{1}{4^3}\lambda^4+\frac{1}{4^4}\lambda^6+\frac{25}{4^7}\lambda^8+\frac{49}{4^8}\lambda^{10}+\dots$$

Then, it can be found that the first nonzero term of the series on the right hand side of (11) is $-\frac{3}{2^{17}}\lambda^{10}$. Moreover, since

$$\lambda=\frac{a-b}{a+b}=\frac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}}\approx\frac{e^2}{4}$$

the error for $A(\lambda)$ is

$$\pi(a+b)\frac{3\lambda^{10}}{2^{17}}\approx 3\pi a(1+\sqrt{1-e^2})\frac{(e^2/4)^{10}}{2^{17}}<2\pi a\frac{e^{20}}{2^{37}}=3\pi a\frac{e^{20}}{2^{36}}$$

Clearly, the approximation is really good when the eccentricity is not too large, in other words more circular.

For example, for the orbit of Mercury ($e=0.206$), the error of this approximation is about 1.5×10^{-13} meters. Taking Pluto, whose orbit used to be the most elliptic ($e=0.250$) in our solar system, the error is still less than 10^{-6} meters. The approximation blows up when we consider a very extreme case of the orbit of the Halley's comet, which has eccentricity $e=0.967$, and still yields a decent error of 2585 meters, Cook [6].

6 Jacobi Elliptic Functions

This concluding section serves as an exploration of some of the topics discussed above. We will give a definition of Jacobi elliptic function, examine some of its interesting properties, and explore its further implications. Most of the contents in this section is due to [4].

The motivation of elliptic functions comes from the well-defined trigonometric functions that arise from inverting with reference to the circle. It is then reasonable to expect that elliptic functions are a generalized version of trigonometric functions with respect to conic sections, in particular ellipse.

Indeed, consider the function

$$f(t)=\frac{1}{\sqrt{1-t^2}}$$

From elementary calculus class, we define the inverse sine function by

$$F(x) = \int_0^x \frac{1}{\sqrt{1-t^2}} dt = \sin^{-1} x, \quad -1 \le x \le 1$$

i.e. $F(\sin\varphi) = \varphi$. In a similar sense, we first define a corresponding elliptic version of F, namely the incomplete elliptic integral of the first kind.

Definition 6.1 *Let the elliptic modulus k satisfy $0 \le k^2 \le 1$, and $0 \le \varphi \le \pi/2$. The incomplete elliptic integral of the first kind is defined as:*

$$F(\phi, k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^{\sin\phi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{12}$$

where the second equality follows from a change of variable $x = \sin\phi$ from the first one.

Notice, the only difference between the complete version defined in Section 2 and this incomplete version is that we now have another freedom of variable ϕ. In fact, this upper bound of integration is referred to as the **Jacobian amplitude (amp)**.

Since there are three distinctive trigonometric functions, with the extra freedom in hand, it is reasonable to expect that there are more in the case of elliptic functions. In fact, there are twelve Jacobi elliptic functions, which we will be only presenting three of them as representatives.

Definition 6.2 *Let $u = F(\phi, k)$ where F is defined as in (12), then the inversion the elliptic integral gives*

$$\phi = F^{-1}(u, k) = \text{amp}(u, k)$$

and from this we can define

$$\sin\phi = \sin(\text{amp}(u, k)) = \mathbf{sn(u,k)}$$

$$\cos\phi = \cos(\text{amp}(u, k)) = \mathbf{cn(u,k)}$$

$$\sqrt{1-k^2\sin^2\phi} = \sqrt{1-k^2\sin^2(\text{amp}(u,k))} = \mathbf{dn(u,k)}$$

*where **sn(u,k)**, **cn(u,k)**, and **dn(u,k)** are known as the elliptic sine, cosine, and delta functions respectively.*

By definition, these functions satisfy the properties as the normal sine and cosine functions. Namely,

$$\mathrm{sn}^2(u,k) + \mathrm{cn}^2(u,k) = 1, \quad \mathrm{dn}^2(u,k) + k^2\mathrm{sn}^2(u,k) = 1$$

and when $k = 0$, $\mathrm{sn}(u,0) = \sin u, \mathrm{cn}(u,0) = \cos u, \mathrm{dn}(u,0) = 1$. Thus, they serve as nice generalizations of the trigonometric functions.

In general, this is where this paper ends, but we want to make a few marks here: Following our discussion, we will be entering a whole new world of elliptic function, in which a lot of substantial connections and practical results can be found. For example, another way to define elliptic functions is to consider them in the complex plane, where they are characterized by doubly periodic meromorphic functions [10].

Moreover, the ordinary nonlinear differential equation that describes the motion of a simple pendulum is also closely related to elliptic functions. In fact, we will not only be able to produce an exact solution [2] but also give a fast convergence approximation based on the relation between AGM and elliptic integral [5].

There is another interesting question that occurs to the author "is there a generalized version of arithmetic-geometric mean?" Namely, given n constants, can we construct n series that converge to the same limit?

Last but not the least, the transformation in Section 4 that is named after the English mathematician John Landen (1719-1790). He certainly is not as well-known as some of his contemporaries such as Gauss and Legendre. However, for those readers who are interested in his history and works, they definitely would find Watson's paper enjoyable to read [11].

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