Arzelà-Ascoli Theorem

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Definition 1. A family $\mathcal{F}$ of continuous functions on a region $\Omega \subset \mathbb{C}$ is said to be **normal** on Ω if every sequence $\{f_n\} \subset \mathcal{F}$ contains a subsequence which converge uniformly on compact subsets of Ω.

Definition 2. A family of functions $\mathcal{F}$ defined on a set $E \subset \mathbb{C}$ is **equicontinuous** on E if for each $w \in E$ and each $\epsilon > 0$, there exists a $\delta > 0$ so that if $z \in E$ and $|z - w| < \delta$, then $|f(z) - f(w)| < \epsilon$ for all $f \in \mathcal{F}$.

Theorem. A family of continuous functions is normal on a region $\Omega \subset \mathbb{C}$ if and only if (i) $\mathcal{F}$ is equicontinuous on Ω and (ii) there is a $z_0 \in \Omega$ so that the collection $\{f(z_0) : f \in \mathcal{F}\}$ is a bounded subset of $\mathbb{C}$.

The following proof is given in Complex Analysis, Marshall. The spirit of most of the proofs available is the use of diagonalization argument and the completeness of the underlying field, which is $\mathbb{C}$ in this case but can certainly be extended to arbitrary complete metric space.

Proof. Suppose $\mathcal{F}$ is normal and we proceed by contradiction. If $\mathcal{F}$ is not equicontinuous at $z_0 \in \Omega$ then there is an $\epsilon > 0$, $z_n \in \Omega$ and $f_n \in \mathcal{F}$ such that $|z_n - z_0| < 1/n$ and $|f_n(z_n) - f_n(z_0)| \geq \epsilon$ for $n \geq n_0$.

By the normality of $\mathcal{F}$, $\{f_n\}$ has a subsequence that converges uniformly on $\{|z - z_0| \leq 1/n_0\}$ to a continuous function f. Relabeling the original sequence, we may assume the above results still hold. Now

$$\begin{aligned}\epsilon &\leq |f_n(z_n) - f_n(z_0)| \\ &\leq |f_n(z_n) - f(z_n)| + |f(z_n) - f(z_0)| + |f(z_0) - f_n(z_0)|\end{aligned}$$

By uniform convergence, we can choose n large enough so that the first and the third terms on the right hand side above are less than $\epsilon/3$. Moreover, since f is the uniform limit of a sequence of continuous function, we know f is continuous on $\{|z - z_0| \leq 1/n_0\}$, forcing the second term to be less than $\epsilon/3$ as well. This contradiction shows $\mathcal{F}$ is equicontinuous on Ω. For any $z_0 \in \Omega$, then $S = \{f(z_0) : f \in \mathcal{F}\}$ is a bounded set, because if $f_n \in \mathcal{F}$ but $|f_n(z_0)| \to \infty$, then there is no convergent subsequence of $\{f_n\}$, a contradiction.

For the other direction, suppose (i) and (ii) in the statement hold. We first show that (ii) holds for all points in Ω by a typical connectedness argument: For any $w \in \Omega$, by equicontinuity there exists a closed disk $\Delta \subset \Omega$ whose interior contains w and such that $|f(z) - f(w)| < 1$ for all $z \in \Delta$ and $f \in \mathcal{F}$.

Now let $U = \{z_0 \in \Omega : \text{(ii) holds}\}$. For any $w \in U$, the existence of Δ ball implies that U is open. Moreover, for any $w \in \Omega \setminus U$, the Δ ball also implies that $\Omega \setminus U$ is open. Since $U \neq \emptyset$ and Ω is connected, we see that $U = \Omega$.

Now we extract a diagonal sequence: For any sequence of functions $\{f_n\} \subset \mathcal{F}$. Let $D = \{z_k\}$ be a countable dense subset of Ω, we can do this because $\mathbb{C} \sim \mathbb{R}^2$ is separable. Since $\{f(z_1) : f \in \mathcal{F}\}$ is bounded, by Bolzano-Weierstrass theorem, we can find a convergent subsequence $\{f_n^{(1)}\}$ such that $\{f_n^{(1)}(z_1)\}$ converges. Likewise we can find a sequence $\{f_n^{(2)}\} \subset \{f_n^{(1)}\}$ such that $\{f_n^{(2)}(z_2)\}$ converges, and indeed there is a sequence $\{f_n^{(k)}\} \subset \{f_n^{(k-1)}\}$ such that $\{f_n^{(k)}(z_k)\}$ converges. Then, the sequence $\{f_n^{(n)}\}$ converges at z_k for each k. We may relabel this sequence as $\{f_n\}$.

Fix a closed disk $\Delta \subset \Omega$ and fix $\epsilon > 0$. By assumption (i) the family $\mathcal{F}$ is uniformly equicontinuous on Δ, so we can find $\delta > 0$ such that if $f \in \mathcal{F}$ and if $z, w \in \Delta$ with $|z - w| < \delta$ then

$$|f(z) - f(w)| < \epsilon/3$$

Choose a finite set $z_k(1), \ldots, z_k(N) \in D \bigcap \Delta$ so that for each $z \in \Delta$ we have $\min_j\{|z - z_{k(j)}|\} < \delta$. We can do this because Δ is compact. Now by the convergence of $\{f_n\}$, we can find $M < \infty$ so that for $m, p \geq M$ and $1 \leq j \leq N$,

$$|f_m(z_{k(j)}) - f_p(z_{k(j)})| < \epsilon/3$$

Now given $z \in \Delta$ and $m, p \geq M$, choose $z_{k(j)}$ so that $|z - z_{k(j)}| < \delta$. Then

$$|f_m(z) - f_p(z)| \leq |f_m(z) - f_m(z_{k(j)})| + |f_m(z_{k(j)}) - f_p(z_{k(j)})| + |f_p(z_{k(j)}) - f_p(z)|$$

Since the first and the third terms are less than $\epsilon/3$ by the uniform equicontinuity and the middle term is also less than $\epsilon/3$ by the convergence of $\{f_n\}$, we have that $|f_m(z) - f_p(z)| < \epsilon$ on Δ when $m, p > M$. Since ϵ is arbitrary, this proves $\{f_n\}$ is Cauchy in the uniform topology on Δ. Since $\mathbb{C}$ is complete, our relabeled subsequence $\{f_n\}$ converges on Δ. Moreover, due to the fact that normality is a local property, we have shown that $\mathcal{F}$ is normal on Ω because $\mathcal{F}$ is normal on every closed disk contained in Ω.

As we mentioned above, Arzelà-Ascoli theorem can be largely generalized. Here we provide two generalizations from Munkres, and the proofs are using exactly the same ideas as above.

Theorem 45.4 (Ascoli's theorem) Let X be a compact space; let $(\mathbb{R}^n, d)$ denote euclidean space in either the square metric or the euclidean metric; give $\mathcal{C}(X, \mathbb{R}^n)$ the corresponding uniform topology. A subspace $\mathcal{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ is precompact if and only if is equicontinuous and pointwise bounded under d.

Theorem 47 Let X be a space and let (Y, d) be a metric space. Given $\mathcal{C}(X, Y)$ the topology of compact convergence; let $\mathcal{F}$ be a subset of $\mathcal{C}(X, Y)$.

(a) If $\mathcal{F}$ is equicontinuous under d and the set

$$\mathcal{F}_a = \{f(a) | f \in \mathcal{F}\}$$

is precompact for each $a \in X$, then $\mathcal{F}$ is contained in a compact subspace of $\mathcal{C}(X, Y)$.

(b) The converse holds if X is locally compact Hausdorff.