Notes for Math 788 - Stacks
Emory University, Fall 2012

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0　How these notes came to exist

In Spring 2007, Martin Olsson taught "Math 274—Topics in Algebra—Stacks" at UC Berkeley. Anton Geraschenko LATEXed these notes in class,[1] and edited them with other people in the class. They still get modified sometimes. They should be available at

http://math.berkeley.edu/~anton/index.php?m1=writings

The most recent version of the source files are in in the SVN repository

svn://sheafy.net/courses/stacks_sp2007

You can make commits to the repository with the username "guest" and the empty password, but I would rather you email me (geraschenko@gmail.com) so I can set up a user for you.

– When something doesn't make sense to me, I mark it with three big, eye-catching stars [[★★★ like this]]. If you can clear any of these up for me, let me know.

– If you have notes that I'm missing or if you have a correct/clear explanation for something which is incorrect/unclear, let me know (either tell me what you'd like to modify, give me some notes to go on, or update the tex yourself and send me a copy). Real (mathematical) errors should be fixed because it would be immoral to let them propagate (er . . . that is, sit there), and typographical errors hardly take any time to fix, so you shouldn't be shy about telling me about them.

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[1]With the exception of two lectures (originally 30 and 31, but the content has been mixed up with nearby lectures), which were reconstructed from the notes of Tony Varilly, Ed Carter, and Anne Shiu, along with the usual conversation that went into editing.

1 Motivation: non-representable functors

You should do homework (especially if you're enrolled). You'll have to do a lot of work, even if you already know a lot. We'll try to organize a discussion section. The prerequisite is schemes. The references are good; you should look at them. Vistoli's notes on Grothendieck topologies are good; Knutson's book is good, so we'll try to put it on reserve in the library.

In a (Nov. 5 1959) letter from Grothendieck to Serre, Grothendieck talks about moduli spaces and says that he keeps running up against the problem that objects have automorphisms.

A main point for today is that many interesting functors are not representable. If X is a scheme, then we get a functor $h_X : \mathsf{Sch}^{op} \to \mathsf{Set}$ given by $Y \mapsto \mathrm{Hom}(Y, X)$.

Lemma 1.1 (Yoneda). *The functor $h_- : \mathsf{Sch} \to \mathsf{Fun}(\mathsf{Sch}^{op}, \mathsf{Set})$ is fully faithful.*

Definition 1.2. A functor $F : \mathsf{Sch}^{op} \to \mathsf{Set}$ is *representable* if $F \simeq h_X$ for some X. When you represent a functor F, you give the scheme X together with the natural isomorphism $F \simeq h_X$. When you do this, X is unique up to unique isomorphism. ◇

Example 1.3. $\mathbb{A}^n : Y \mapsto \Gamma(Y, \mathcal{O}_Y)^n$. On morphisms, $\mathbb{A}^n$ takes $g : Y' \to Y$ to the pull-back map $g^* : \Gamma(Y, \mathcal{O}_Y)^n \to \Gamma(Y', \mathcal{O}_{Y'})^n$. Let $X = \mathrm{Spec}\,\mathbb{Z}[x_1, \ldots, x_n]$. Then $\mathrm{Hom}(Y, X) \cong \mathrm{Hom}_{alg}\big(\mathbb{Z}[x_1, \ldots, x_n], \Gamma(Y, \mathcal{O}_Y)\big) \cong \Gamma(Y, \mathcal{O}_Y)^n = \mathbb{A}^n(Y)$ is a natural isomorphism, so X represents $\mathbb{A}^n$. ◇

Example 1.4. $\mathbb{A}^n \smallsetminus \{0\}$ should be a sub-functor of $\mathbb{A}^n$. We define it by $Y \mapsto \big\{(y_1, \ldots, y_n) \in \Gamma(Y, \mathcal{O}_Y)^n |$for every $p \in Y$, the images of the y_i are not all zero in $k(p)\big\}$. In the homework, you will show that this functor is representable. ◇

Definition 1.5. An *elliptic curve* over a scheme Y is a diagram $E \underset{f}{\overset{e}{\rightleftarrows}} Y$ where e is a section of f and the fibers of f are genus 1 curves. ◇

Example 1.6. Let $\mathcal{M}_{1,1}$ be the functor defined by $Y \mapsto \{$isoclasses of elliptic curves over $Y\}$. If $g : Y' \to Y$ is a morphism of schemes, then we define $\mathcal{M}_{1,1}(g) : \mathcal{M}_{1,1}(Y) \to \mathcal{M}_{1,1}(Y')$ to be the usual pull-back g^*.

$$
\begin{array}{ccc}
\mathcal{M}_{1,1}(g)(E) = g^*E & \longrightarrow & E \\
{\scriptstyle g^*f}\downarrow\ \ \ \ \ & \raisebox{3pt}{$\ulcorner$} & \ \downarrow{\scriptstyle f} \\
Y' & \underset{g}{\longrightarrow} & Y
\end{array}
\qquad\qquad
\begin{array}{ccc}
g^*E & \longrightarrow & E \\
{\scriptstyle \exists!}\uparrow\ \ \nearrow{\scriptstyle e\circ g} & & \uparrow{\scriptstyle e} \\
Y' & \underset{g}{\longrightarrow} & Y
\end{array}
$$

The section $Y' \to g^*E$ is induced by id_Y and $e \circ g$ by the universal property of pull-backs. ◇

Proposition 1.7. $\mathcal{M}_{1,1}$ *is not representable.*

The intuitive reason that $\mathcal{M}_{1,1}$ is not representable is the following lemma. It says that you cannot have any kind of twisting of bundles.

Lemma 1.8. *If F is a representable functor, with $s_1, s_2 \in F(Y)$ and a covering $Y = \bigcup U_i$ such that $s_1|_{U_i} = s_2|_{U_i}$ for all i, then $s_1 = s_2$.*

Proof. We have that $F \cong h_X$ for some X, and s_1 and s_2 are given by morphisms $Y \to X$ that they agree on a cover of Y. Since morphisms glue, $s_1 = s_2$. $\qquad\square$

Unfortunately, to show that $\mathcal{M}_{1,1}$ is not representable via this Lemma, you need to generalize your notion of covering (to étale covers). We'll see these later. For now we'll give another proof.

Proof of 1.7. Assume there is a scheme M and an isomorphism $\mathcal{M}_{1,1} \simeq h_M$. Let k be an algebraically closed field of characteristic not 2. Consider $R = k[\lambda]_{\lambda(1-\lambda)}$, so $\operatorname{Spec} R$ is $\mathbb{A}^1_k$ with 0 and 1 removed. Let $E \subseteq \mathbb{P}^2_R$ be the closed subscheme defined by $y^2 z = x(x-z)(x-\lambda z)$, so the fibers E_λ of the natural map $E \to \operatorname{Spec} R$ are genus 1 curves. We define $e : \operatorname{Spec} R \to D(z) \cong \operatorname{Spec} R[x,y]$ by the map $R[x,y] \to R$, $x, y \mapsto 0$ and observe that the image of e lies in E, and e is a section of $E \to \operatorname{Spec} R$.

Observe that there is an action of S_3 on R generated by $\sigma_0 : \lambda \mapsto 1/\lambda$ and $\sigma_1 : \lambda \mapsto 1/(1-\lambda)$. Let $j = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2} \in k(\lambda)$.

Lemma 1.9. *The fixed points $k(\lambda)^{S_3}$ are exactly the elements of $k(j)$.*

Proof. First check that $j \in k(\lambda)^{S_3}$. Then we have that $k(j) \subseteq k(\lambda)^{S_3} \subseteq k(\lambda)$. By Galois theory, the second extension is degree 6, and the total extension is degree 6, so the first two fields are equal. $\qquad\square_{\text{Lemma}}$

There are two steps remaining in the proof.
(1) Let E_η be the generic fiber of $E \to \operatorname{Spec} R$. It is given by a map $\phi : \operatorname{Spec} k(\lambda) \to M$. We claim that this map has to factor through the obvious map $g : \operatorname{Spec} k(\lambda) \to \operatorname{Spec} k(j)$.

As we see from the diagram on the left, a factorization of ϕ is the same thing as a morphism $\operatorname{Hom}(g, M) : \phi \mapsto \psi$. By the natural isomorphism $h_M \simeq \mathcal{M}_{1,1}$, this is the same as a morphism $\mathcal{M}_{1,1}(g) : \tilde{E} \mapsto E$, where $\tilde{E}$ is the elliptic curve over $k(j)$ defined by ψ, as shown in the diagram on the right.

Thus, we have $E_\eta = \tilde{E} \times_{\operatorname{Spec} k(j)} \operatorname{Spec} k(\lambda)$. This implies that the action of S_3 can be lifted to E_η by acting on the second factor.
(2) Then we check that the S_3 action cannot lift to E_η.

(1) Say $x = \phi(\eta)$, then we have

$$\prod_{\sigma : k(\lambda) \hookrightarrow \overline{k(j)}} \overline{k(j)} \xleftarrow{\overline{\prod \sigma}} k(\lambda) \longleftarrow k(x)$$

$$\uparrow \text{diagonal} \qquad \qquad \uparrow$$

$$\overline{k(j)} \longleftarrow \overline{k(j)}$$

where σ runs over all embeddings of $k(\lambda)$ into $\overline{k(j)}$ over $k(j)$ (there are six such embeddings). It is enough to show that

$$\coprod_{\sigma : k(\lambda) \hookrightarrow \overline{k(j)}} \operatorname{Spec} \overline{k(j)} \xrightarrow{f} M$$

$$\downarrow \qquad \nearrow$$

$$\operatorname{Spec} \overline{k(j)}$$

[[★★★ why is this factorization enough?]] Fact: If E_1, E_2 have the same j-invariant, then they are isomorphic.

But for any $\sigma : k(\lambda) \hookrightarrow \overline{k(j)}$ the j-invariant of $E_\eta \times_{\operatorname{Spec} k(\lambda), \sigma} \operatorname{Spec} \overline{k(j)}$ is j. [[★★★ j is a regular function on $\operatorname{Spec} R$. The j-invariant of E_η is $j \in k(\eta) = k(\lambda)$. What does j-invariant mean for curves not over points in $\operatorname{Spec} R$?]] So when f is restricted to any particular $\operatorname{Spec} \overline{k(j)}$, it is always the same map, so f factors through $\operatorname{Spec} \overline{k(j)}$.

(2) recall that $E_\eta \to \operatorname{Spec} k(\lambda)$ is the subset of $\mathbb{P}^2_{k(\lambda)}$ defined by $y^2 z = x(x-z)(x-\lambda z)$. Elliptic curves have a group structure and torsion points. In fact, $E_\eta[2]$, the 2-torsion points, are $\infty = [0,1,0], [0,0,1], [1,0,1]$ and $[\lambda, 0, 1]$. If S_3 acts on E_η, then it must preserve the 2-torsion points (and it must fix ∞, which is the identity element). How could we lift $\sigma_0 : \lambda \mapsto 1/\lambda$? It would have to lift

$$E_\eta \xrightarrow{\tilde{\sigma}_0} E_\eta$$

$$\downarrow \qquad \qquad \downarrow$$

$$\operatorname{Spec} k(\lambda) \longrightarrow \operatorname{Spec} k(\lambda)$$

$$1/\lambda \longleftarrow\mapsto \lambda$$

So we need an endomorphism of $k(\lambda)[x,y]/\big(y^2 - x(x-1)(x-\lambda)\big)$ which inverts λ. Working case-by-case, we can show that the only hope you have must be of the form $x \mapsto \lambda x$ and $y \mapsto uy$ for some unit u (this all comes from the fact that we must preserve 2-torsion points). From that we get $u^2 y^2 = \lambda^3 x(x-1)(x-\lambda)$ [[★★★ I don't get that]], so $u^2 = \lambda^3$, so λ^3 has a square root in $k(\lambda)$. Since λ is an indeterminant, this is a contradiction. □

Spec R represents (essentially) $Y \mapsto \big(E/Y$ an elliptic curve, with a basis for its 2-torsion$\big)$ (it actually represents this together with some $\omega \in f_* \Omega'_{E/Y}$).

Consider the functor $G : \mathsf{Sch}^{op} \to \mathsf{Set}$ given by $Y \mapsto$ isomorphisms classes of pairs $\left(E/Y, \mathcal{O}_Y^3 \xrightarrow{\sim} f_* \mathcal{O}_E(3e)\right)$, where $f : E \to Y$.

Proposition 1.10. G *is representable.*

Proof.

$$
\begin{array}{ccc}
 & h_Z & \subseteq \quad h_{\mathbb{P}^2 \times \mathrm{Hilb}} \\
 & \uparrow & \downarrow \\
G & \longrightarrow & h_{\mathrm{Hilb}(\mathbb{P}^2)}
\end{array}
$$

This gives that G is represented by an open sub-scheme of Z. [[★★★ I can't make sense of this]]   □

2   Grothendieck topologies

We ignore all set-theoretic issues in this class; I don't know how to handle them, but they are taken care of elsewhere (e.g. in [SGA]).

Definition 2.1. Let C be a category. A *Grothendieck topology* on C consists of, for each object X in C, a collection $Cov(X)$ of sets $\{X_i \to X\}$ of arrows, called *coverings of X*, such that

1. If $V \to X$ is an isomorphism, then $\{V \to X\} \in Cov(X)$.

2. If $\{X_i \to X\} \in Cov(X)$ and any arrow $Y \to X$, then the fiber products $X_i \times_X Y$ exist, and $\{X_i \times_X Y \to Y\} \in Cov(Y)$.

3. If $\{X_i \to X\} \in Cov(X)$ and $\{V_{ij} \to X_i\} \in Cov(X_i)$ for each i, then $\{V_{ij} \to X_i \to X\} \in Cov(X)$.

A *site* is a category C together with a Grothendieck topology.   ◇

Remark 2.2. This is called a "pre-topology" in SGA4, but I don't see any reason not to call it a topology.   ◇

There is a general framework into which many interesting sites fit. We describe it here in Sch, but it can be done in other categories, like $\mathbf{Top}$. Let $\mathcal{P}$ and $\mathcal{Q}$ be properties of morphisms of schemes. The $\mathcal{P}$-$\mathcal{Q}$ *site* on a scheme Y is the full subcategory of Sch/Y whose objects are $\mathcal{P}$ morphisms to Y, with $\left\{ X_i \xrightarrow[\to Y \leftarrow]{} X \right\} \in Cov(X \to Y)$ if each $X_i \to X$ is $\mathcal{Q}$ and $\coprod X_i \to X$ is surjective. If Y is not specified, it is taken to be the final object $\operatorname{Spec} \mathbb{Z}$. Obviously, this does not form a site in general, but for certain $\mathcal{P}$ and $\mathcal{Q}$, it does.

Example 2.3 (Big and small sites)**.** If $\mathcal{P}$ is vacuous (i.e. all morphisms are $\mathcal{P}$, so we are working in the category Sch/Y), then we get the *big $\mathcal{Q}$ site* on Y. If $\mathcal{P} = \mathcal{Q}$, then we get the *small $\mathcal{Q}$ site* on Y.   ◇

We will often think about the following sites. You should check that they are indeed sites.

– Big/small site of a topological space ($\mathcal{Q}$ = homeomorphism to an open subset).

– Big/small Zariski site of a scheme ($\mathcal{Q}$ = open immersion).[1]

– Big/small étale site on a scheme ($\mathcal{Q}$ = étale). Note that if two schemes are étale over another scheme S, then any S-morphism between them is automatically étale.

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[1] It is better to define the Zariski site by taking $\mathcal{Q}$ to be the property of being "locally an open immersion", where $f : X \to Y$ is locally an open immersion if for every point $x \in X$, there is a neighborhood U of x such that $f : U \to Y$ is an open immersion. This is a better definition because this way we get a chain of topologies, getting progressively finer: Zariski, étale, lisse, fppf, fpqc.

- Big fppf site on a scheme ($\mathcal{Q}$ = fppf = flat, locally of finite presentation).[2]

- Big fpqc site on a scheme ($\mathcal{Q}$ = fpqc[3])

- Lisse-étale[4] site on a scheme ($\mathcal{P}$ = smooth, $\mathcal{Q}$ = étale). Note that Y-morphisms between smooth schemes over Y are not necessarily smooth.

Example 2.4 (Induced site structure on an over category). Let (C, Cov) be a site, and let $X \in Ob(\mathsf{C})$. Then we can define C/X, whose objects are morphisms to X and morphisms are commutative triangles as usual. We define $\left\{ X_i' \underset{\searrow\, X \,\swarrow}{\longrightarrow} X' \right\}$ to be a covering if $\{X_i' \to X'\}$ is a covering in (C, Cov). It is immediate to verify the axioms (the fiber products are all the same). $\diamond$

Example 2.5 (Induced site structure on $(\mathsf{C} \downarrow F)$). Let (C, Cov) be a site. Let Δ be the simplicial category (or any category for that matter). Let $F : \Delta^{op} \to \mathsf{C}$ be a functor. Define C_F as the category whose objects are pairs $\big(\delta, X \to F(\delta)\big)$, where $\delta \in \Delta$ and $X \to F(\delta)$ an arrow in C, and a morphism $\big(\delta', X' \to F(\delta')\big) \xrightarrow{(f, f^\flat)} \big(\delta, X \to F(\delta)\big)$ is a map $f : \delta \to \delta'$ in Δ and a morphism $f^\flat$ making the following diagram commute.

$$
\begin{array}{ccc}
X' & \xrightarrow{\ f^\flat\ } & X \\
\downarrow & & \downarrow \\
F(\delta') & \xrightarrow{\ F(f)\ } & F(\delta)
\end{array}
$$

We define coverings of $\big(\delta, X \to F(\delta)\big)$ to be a set of the form

$$
\left\{ \big(\delta, X_i \to F(\delta)\big) \xrightarrow{(\mathrm{id}, f^\flat)} \big(\delta, X' \to F(\delta)\big) \ \middle|\ \{X_i \xrightarrow{f^\flat} X\} \in Cov_\mathsf{C}(X) \right\}. \qquad \diamond
$$

Remark 2.6. Example 2.4 is a special case of Example 2.5 by taking Δ to be the one point category with only the identity morphism, with $F(*) = X$. $\diamond$

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[2] If we are dealing with locally noetherian schemes, then locally of finite presentation is the same as locally of finite type.

[3] A morphism $f \colon X \to Y$ is said to be fpqc if it is locally faithfully flat and quasi-compact. That is, for every point $x \in X$, there is a quasi-compact open neighborhood of x whose image under f is an open affine of Y.

[4] "Lisse" is French for "smooth".

3 Sheaves and Topoi

Definition 3.1. Let C be a category. A *presheaf of sets* on C is a functor $F : \mathsf{C}^\circ \to \mathsf{Set}$. If C is a site, then F is a *sheaf* on (C, Cov) if it is a presheaf such that for every object $X \in \mathsf{C}$ and every covering $\{X_i \to X\} \in Cov(X)$, the sequence

$$F(X) \longrightarrow \prod F(X_i) \underset{pr_2^*}{\overset{pr_1^*}{\rightrightarrows}} \prod F(X_i \times_X X_j)$$

is exact (an equalizer).[1] ◇

Remark 3.2. Notions of (pre)sheaves of groups, rings, modules, etc. are given by replacing the target category Set by Gp, Ring, Mod, respectivly. That is, for every X, $F(X)$ has the appropriate structure (of a group, for example), and the morphisms go to the appropriate kinds of morphisms. [[★★★ can you sheafify presheaves with an arbitrary category as a target?]] ◇

Example 3.3. Let X be a topological space, and let C be the small site on X. Then this is the usual sheaf condition. ◇

The category of sheaves injects fully faithfully into the category of presheaves via the forgetful funtor.

Theorem 3.4 (Sheafification). *The forgetful functor from sheaves to presheaves has a left adjoint. In particular, if F is a presheaf, there is a morphism to a sheaf $F \to F^a$ so that any morphism from F to a sheaf factors uniquely through F^a.*

Proof. Step 1: define the projection $F \to F^s$,[2] where $F^s(X) := F(X)/\sim$, where $a, b \in F(X)$ are equivalent if there is a covering $\{X_i \to X\}$ such that a and b have the same image in each $F(X_i)$. Here we are forcing the injectivity part of the sheaf condition. If $Y \to X$ is a morphism, then any cover of X pulls back to a cover of Y, and if two sections $a, b \in F(X)$ agree on an open cover of X, their images in $F(Y)$ will agree on the pulled-back cover. Thus, the dashed arrow in the following diagram is well defined, so F^s is a functor.

$$
\begin{array}{ccc}
F(X) & \longrightarrow & F(Y) \\
\downarrow & & \downarrow \\
F^s(X) & \dashrightarrow & F^s(Y)
\end{array}
$$

---

[1] $S \xrightarrow{j} S' \underset{g}{\overset{f}{\rightrightarrows}} S''$ is an equalizer if $f \circ j = g \circ j$ and any $h : X \to S'$ with $f \circ h = g \circ h$ uniquely factors through j. In Set, this just means that j is an injection, with $\{s' \in S' | f(s') = g(s')\} = \operatorname{im} j$. In Ab, this means that j is the kernel of $f - g$.

[2] The "s" stands for "separated", which means that for every X and every covering $\{X_i \to X\}$, the map $F(X) \to \prod F(X_i)$ is injective (the "first half" of the sheaf condition). You could make an intermediate category of separated presheaves. The "a" stands for "associated" sheaf.

Step 2: define $F^s \to F^a$, where

$$F^a(X) = \left\{ \big(\{X_i \to X\}, \{a_i\}\big) \Big| \right.$$
$$\left. \{X_i \to X\} \in Cov(X), \{a_i\} \in Eq\big(\textstyle\prod F^s(X_i) \rightrightarrows \prod F^s(X_i \times_X X_j) \big) \right\} \Big/ \sim$$

where $\big(\{X_i \to X\}, \{a_i\}\big) \sim \big(\{X'_j \to X\}, \{a'_j\}\big)$ if for all i and j, the images of a_i and a'_i in $F(X_i \times_X X'_j)$ are equal. I leave it to you to check that it works (it's actually a lot of work).[3] $\square$

We will see that sheafification arises naturally. For example, affine $(n+1)$-space is the functor $\mathbb{A}^{n+1} : \mathsf{Sch}^\circ \to \mathsf{Set}$ given by $Y \mapsto \Gamma(Y, \mathcal{O}_Y)^{n+1}$. In the homework (Exercise 1.3), you prove that $\mathbb{A}^{n+1} \smallsetminus \{0\} : \mathsf{Sch}^\circ \to \mathsf{Set}$ is given by $Y \mapsto \{(y_1, \ldots, y_{n+1}) | \text{for each} \ y \in Y, \text{not all } y_i \text{ are zero in } k(y)\}$. One might try to define projective space $\mathbb{P}^n$ as the functor $\mathbb{A}^{n+1} \smallsetminus \{0\}/\mathbb{G}_m$, given by $Y \mapsto (\mathbb{A}^{n+1} \smallsetminus \{0\})(Y)/\Gamma(Y, \mathcal{O}_Y)^\times$, but it turns out this is not a sheaf; the correct definition of $\mathbb{P}^n$ is the sheafification.

Definition 3.5. A *topos* is a category equivalent to the category of sheaves on a site. $\diamond$

Grothendieck's insight is that the basic object of study is the topos, not the site. It is often useful to replace a site by another site with the same topos (in some appropriate sense). For example, if we use the Zariski topology, the category of sheaves on Sch (the site of all schemes) is the same as the category of sheaves on Aff (the site of affine schemes), which is perhaps easier to deal with. [[★★★ maybe this is a good place to talk about replacing a topology on a site by the finest topology that gives the same topos, and about the canonical site producing a given topos. maybe better after talking about continuous functors between sites]]

<u>Notation</u>: we use the following notation for topoi. Note that these are *topoi*, not sites.

$\mathbf{Top}_{cl}$	(X_{cl})	classical topos (of a topolocical space X)
Sch_{zar}	(X_{zar})	(small) Zariski topos (of X)
	X_{ZAR}	big Zariski topos of X ($= (\mathsf{Sch}/X)_{zar}$)
Sch_{et}	(X_{et})	(small) étale topos (of X)
	X_{ET}	big étale topos of X ($= (\mathsf{Sch}/X)_{et}$)
$\mathsf{Sch}_{lis\text{-}et}$	$(X_{lis\text{-}et})$	lisse-étale topos (of X)
Sch_{fppf}	(X_{fppf})	(big) fppf topos (of X)

---

[3]If you try to just do step 2, you don't get a sheaf. For example, let $X = \{p, q\}$ have the discrete topology, and let S be a set with $|S| > 1$. If F is the constant presheaf on X associated to S, then $F^a(X) = S$ (a section on p and a section on q "agree on the intersection" only if they are equal because $F(\varnothing) = S$), but the sheafification should give you S^2 [[★★★ can you get an example that doesn't use the empty set in this way?]]. But you do get a separated presheaf, so you could just do step 2 twice to get the sheafification.

Definition 3.6. A *morphism of topoi* $f : T \to T'$ is an isomorphism class of triples[4] (f_*, f^*, ϕ), where $f_* : T \to T'$, $f^* : T' \to T$ are functors, and $\phi(F) : \mathrm{Hom}_T(f^*F', F) \xrightarrow{\sim} \mathrm{Hom}_{T'}(F', f_*F)$ is an adjunction between them. Also, f^* must commute with finite projective limits. $\diamond$

Example 3.7. In the classical case, if $X \xrightarrow{f} X'$ is a continuous map of topological spaces, we have $f^{-1} : \mathrm{Opens}(X') \to \mathrm{Opens}(X)$. If F is a sheaf on X, we define $(f_*F)(U') := F\big(f^{-1}(U')\big)$. This has a left adjoint which commutes with finite projective limits, as we will show next two lectures (Proposition 4.3 and Theorem 5.1) $\diamond$

Limits

Let C be a category, and $F : I \to \mathsf{C}$ be a functor. For $X \in \mathsf{C}$ define $k_X : I \to \mathsf{C}$ to be the functor sending each object to X and each morphism to id_X. We define $\varprojlim F : C \to \mathsf{Set}$ as the functor given by $X \mapsto \mathrm{Nat}(k_X, F)$. We are usually interested in whether this functor is representable.

Example 3.8. Letting I be different things, we can construct some familiar friends.

1. (equalizers) $I = (\cdot \rightrightarrows \cdot)$, with $F(I) = X_1 \underset{f_2}{\overset{f_1}{\rightrightarrows}} X_2$, we have $\varprojlim F(X) = \{X \xrightarrow{g} X_1 | f_1 \circ g = f_2 \circ g\}$. This functor is represented by the equalizer of f_1 and f_2.

2. (products) I is a set (i.e. a category with only identity morphisms), and $F(I) = \{X_i\}_{i \in I}$, then $\varprojlim F(X) = \big\{\{g_i : X \to X_i\}_{i \in I}\big\}$. This functor is represented by the product $\prod_{i \in I} X_i$.

3. (fibred products) $I = (\cdot \to \cdot \leftarrow \cdot)$, with $F(I) = X_1 \xrightarrow{f_1} X_2 \xleftarrow{f_3} X_3$. Then

$$\varprojlim F(X) = \big\{X_1 \xleftarrow{g_1} X \xrightarrow{g_3} X_3 \,\big|\, f_1 \circ g_1 = f_3 \circ g_3\big\}.$$

   This functor is represented by the fibred product $X_1 \times_{X_2} X_3$. $\diamond$

Lemma 3.9. *Let C be a category, then the following are equivalent.*

1. *Projective limits (resp. finite projective limits) in C are representable.*

2. *Products (resp. finite products) and equalizers are representable.*

3. *Products and fiber products (resp. finite products and fiber products) are representable.*

---

[4]Originally, we defined a morphism to be *a triple*, but if we do that, we run into the problem that a continuous functor of (appropriate) sites doesn't induce a well-defined morphism of topoi. This will be the topic of the next two lectures.

Proof. $1 \Rightarrow 2$ and $1 \Rightarrow 3$ are immediate, as (finite) products, equalizers, and fiber products are (finite) projective limits.

$2 \Rightarrow 1$ Given $F : I \to \mathsf{C}$, define

$$\varprojlim F = Eq\Big(\prod_{i \in Ob(I)} F(i) \xrightarrow[p_2]{p_1} \prod_{u \in Mor(I)} F\big(\text{target}(u)\big) \Big)$$

where we need to define p_1 and p_2. Defining a morphism to a product is equivalent to defining a morphism to each factor, so fix a morphism $u \in Mor(I)$. Let $p_1 : \prod_{i \in Ob(I)} F(i) \to F\big(\text{target}(u)\big)$ be projection onto the target of u and let $p_2 : \prod_{i \in I} F(i) \to F\big(\text{source}(u)\big) \xrightarrow{F(u)} F(\text{target}(u))$ be projection to the source followed by $F(u)$.

$3 \Rightarrow 2$ It is enough to observe that an equalizer is a fiber product over a product: $Eq(X \underset{v}{\overset{u}{\rightrightarrows}} Y)$ is the limit of the diagram $X \xrightarrow{(1,u)} X \times Y \xleftarrow{(1,v)} X$. □

Proposition 3.10. *Let T be a topos and $F : I \to T$, then $\varprojlim F$ is representable.*

Proof. Recall that if $F : I \to T$, $\varprojlim F : X \mapsto \text{Nat}(k_X, F)$. By the lemma, it is enough to check that equalizers and products are representable. To check these cases, we can choose a site C whose category of sheaves is equivalent to T (by definition of a topos, there is such a site). We define a product $\prod_i F_i$ as $U \mapsto \prod F_i(U)$, then we just have to check that this is a sheaf (this is left as an exercise). Similarly, we can define

$$Eq\Big(F_1 \underset{g}{\overset{f}{\rightrightarrows}} F_2 \Big)(U) = Eq\Big(F_1(U) \xrightarrow[g(U)]{f(U)} F_2(U) \Big)$$

and check that this is a sheaf. □

Remark 3.11. The same proof shows that in the category $\hat{\mathsf{C}}$ of *presheaves* on C, finite projective limits are representable. ◇

4   Continuous functors between sites

Recall that if we have a map of topoi $f : T \to T'$, this means that we have functors $f_* : T \to T'$ and $f^* : T' \to T$, together with an adjunction $\phi : \operatorname{Hom}_T(f^*F, G) \xrightarrow{\sim} \operatorname{Hom}_{T'}(F, f_*G)$, and f^* commutes with finite projective limits. That is, the natural map $f^*(\varprojlim F) \to \varprojlim(f^*F)$ is an isomorphism. Heuristically, a "continuous map of sites" should induce a map of topoi.

Definition 4.1. Let C and C' be sites. A functor $f : \mathsf{C}' \to \mathsf{C}$ is *continuous* if

1. for every $X' \in \mathsf{C}'$ and every $\{X_i' \to X'\} \in Cov_{\mathsf{C}'}(X')$, we have $\{f(X_i') \to f(X')\} \in Cov_{\mathsf{C}}\big(f(X')\big)$, and

2. f commutes with fiber products when they exist in C'.

$\diamond$

Let T and T' be the categories of sheaves of C and C', respectively. Then given a continuous functor f, we get a functor $f_* : T \to T'$ defined by $F \mapsto \big(X' \mapsto F(f(X'))\big)$.[1] We need to check that this satisfies the sheaf axiom: let $\{X_i' \to X'\} \in Cov_{\mathsf{C}'}(X')$, then we have

$$
\begin{array}{ccccc}
f_*F(X') & \longrightarrow & \prod_i f_*F(X_i') & \rightrightarrows & \prod_{i,j} f_*F(X_i' \times_{X'} X_j') \\
\| & & \| & & \| \\
F(f(X')) & \longrightarrow & \prod_i F(f(X_i')) & \rightrightarrows & \prod_{i,j} F\big(f(X_i') \times_{f(X')} f(X_j')\big)
\end{array}
$$

where the last vertical equality follows from the fact that the continuous functor f commutes with projective limits when they exist. The bottom sequence is exact since F is a sheaf, so the top sequence is also exact. Thus, f_*F is a sheaf.

Example 4.2. Recall the fppf, étale, and Zariski topologies on Sch, then the following identity functors are continuous: Zariski site $\xrightarrow{\mathrm{id}}$ étale site $\xrightarrow{\mathrm{id}}$ fppf site. These induce functors on topoi $\mathsf{Sch}_{Zar} \xleftarrow{\mathrm{id}_*} \mathsf{Sch}_{et} \xleftarrow{\mathrm{id}_*} \mathsf{Sch}_{fppf}$. $\diamond$

Proposition 4.3. *Let $f : \mathsf{C}' \to \mathsf{C}$ be continuous, then the functor $f_* : T \to T'$ has a left adjoint (f^*, ϕ).*

Proof. Note that f_* is obtained by restricting the functor $\hat{f}_* : \hat{\mathsf{C}}' \to \hat{\mathsf{C}}$ defined by $(\mathsf{C}^\circ \xrightarrow{F} \mathsf{Set}) \mapsto (\mathsf{C}'^\circ \xrightarrow{f^\circ} \mathsf{C}^\circ \xrightarrow{F} \mathsf{Set})$. It is enough to show that $\hat{f}_*$ has a left adjoint $\hat{f}^*$ because then $F \mapsto (\hat{f}^*F)^a$ is a left adjoint to f_*:

$$
\begin{aligned}
\operatorname{Hom}_T\big((\hat{f}^*F)^a, G\big) &= \operatorname{Hom}_{\hat{\mathsf{C}}}(\hat{f}^*F, G) && \text{(universal property of } -^a) \\
&= \operatorname{Hom}_{\hat{\mathsf{C}}'}(F, \hat{f}_*G) = \operatorname{Hom}_{T'}(F, f_*G).
\end{aligned}
$$

---

[1]If you like to think about topological spaces, you should think of f as the map on open sets (which pulls open sets back) corresponding to a map of topological spaces. See Example 3.7.

Let F be a presheaf on C', and let $U \in \mathsf{C}$. Then we define

$$\hat{f}^* F(U) = \varinjlim_{U \to f(U')} F(U').^2$$

It is a messy exercise to check that this gives you an adjoint. [[★★★ it isn't too bad]]   □

So a continuous functor $f : \mathsf{C}' \to \mathsf{C}$ induces an adjoint pair (f^*, f_*, ϕ). Unfortunately, f^* need not commute with finite projective limits, as is illustrated by the following example.

Example 4.4. Let k be a field. Take $X = \mathbb{A}^1_k$ and $Y = \operatorname{Spec} k$. Consider $f : Y \hookrightarrow X$, the inclusion of the origin. Recall that Lis-Et(X) has objects smooth X-schemes and coverings are étale coverings. Then we get a functor Lis-Et$(X) \xrightarrow{f}$ Lis-Et(Y) given by $(U \to X) \mapsto (U \times_X Y \to Y)$. This is a continuous functor: if $\{U_i \to U\} \in Cov(U)$, then $\{U_i \times_X Y \to U \times_X Y\} \in Cov(U \times_X Y)$ because the pull-back of an étale morphism is étale. However, f^* does not commute with finite projective limits.

Let $\mathcal{O}_X(U \to X) = \Gamma(U, \mathcal{O}_U)$. This is a presheaf on Lis-Et(X), and in fact it is a representable sheaf, represented by $\mathbb{A}^1_X \to X$. If you go through the adjunction, you'll see that $f^*\mathcal{O}_X$ is represented by $\mathbb{A}^1_Y$:

$$\begin{aligned}
\operatorname{Hom}(h_{\mathbb{A}^1_X}, f_*G) &= f_*G(\mathbb{A}^1_X) && \text{(Yoneda's Lemma)} \\
&= G(\mathbb{A}^1_Y). && (\mathbb{A}^1_X \times_X Y = \mathbb{A}^1_Y)
\end{aligned}$$

We have that $X = \operatorname{Spec} k[t]$. We have a map $\times t : \mathcal{O}_X \to \mathcal{O}_X$ which is injective (if $k[t] \to R$ is flat, then multiplication by t is injective on R, in particular if it is smooth). When you pull this map back to $f^*\mathcal{O}_X = \mathcal{O}_Y$, we get $\times t : \mathcal{O}_Y \to \mathcal{O}_Y$ which is the zero map because $V \to \operatorname{Spec} k \xrightarrow{t=0} \operatorname{Spec} k[t]$ [[★★★]].

So we have that $Eq(\mathcal{O}_X \underset{0}{\overset{\times t}{\rightrightarrows}} \mathcal{O}_X) = \{0\}$ by $Eq(\mathcal{O}_Y \underset{0}{\overset{\times t}{\rightrightarrows}} \mathcal{O}_Y) = \mathcal{O}_Y$, so this f^* doesn't commute with projective limits.   ◇

Remark 4.5. This is kind of bad. Why not think about Sch/X with étale topology? [[★★★ then something]] If you look at Sch/X with the Zariski topology, then something behaves badly.

Note that if we look at the Lis-Lis site, then you still have the same problem ... we didn't use anything about the coverings being étale.   ◇

---

[2]To do this precisely, define a category I_U whose objects are pairs $(U', U \to f(U'))$ and whose morphisms $(U'_1, U \to f(U'_1)) \xrightarrow{g} (U'_2, U \to f(U'_2))$ are morphisms $g : U'_2 \to U'_1$ in C' such that the diagram $f(U'_2) \xleftarrow[f(g)]{\overset{U}{\longrightarrow}} f(U'_1)$ commutes.

Now define $F_U : I \to \mathsf{Set}$ by $(U', U \to f(U')) \mapsto F(U')$. Then we have $\varinjlim F_U := \operatorname{Nat}(F_U, k_-)$. Since this is a direct limit of sets, it is represented by a set, and that is the set we want to define $\hat{f}^* F(U)$ to be.

Theorem 4.6. *If $f : \mathsf{C}' \to \mathsf{C}$ is continuous and finite projective limits are representable in C', then f^* commutes with finite projective limits.*

$$Z \hookrightarrow U_1 \rightrightarrows U_2$$
$$\searrow \downarrow \swarrow$$
$$X$$

The equalizer need not be smooth even if the two maps are smooth.

5   When f^* commutes with finite limits

Theorem 5.1. *If $f : \mathsf{C}' \to \mathsf{C}$ is continuous and finite projective limits are representable in C', then f^* commutes with finite projective limits. In particular, $f : T \to T'$ is a morphism of topoi in this case.*

Proof. [[★★★ this proof needs to be completed and cleaned up]] By the same argument used to prove Lemma 3.9, it is enough to show that f^* commutes with products of two sheaves (and thus finite products) and equalizers.

Recall that $f^*F = (\hat{f}^*F)^a$. One can check that $-^a$ commutes with finite projective limits,[1] so it is enough to show that $\hat{f}^* : \hat{\mathsf{C}} \to \hat{\mathsf{C}}'$ commutes with finite products and equalizers.

(Products) If $F_1, F_2 \in \hat{\mathsf{C}}$, and $U' \in \mathsf{C}'$. Recall the definition of $\hat{f}^*$ from the proof of Proposition 4.3. We have

$$\hat{f}^*(F_1 \times F_2)(U') \qquad\qquad (\hat{f}^*F_1 \times \hat{f}^*F_2)(U')$$

$$\|\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\xrightarrow{\ \Gamma\ }$$

$$\varinjlim_{U' \to f(U)} F_1(U) \times F_2(U) \qquad \left(\varinjlim_{U' \to f(U_1)} F_1(U_1) \right) \times \left(\varinjlim_{U' \to f(U_2)} F_2(U_2) \right)$$

where Γ is defined in the obvious way.[2] First we check injectivity. If $\sigma = \big(U' \to f(U), s_1 \in F_1(U), s_2 \in F_2(U) \big)$ and $\tau = \big(U' \to f(V), t_1 \in F_1(V), t_2 \in F_2(V) \big)$. If $\Gamma(\sigma) = \Gamma(\tau)$, then there is a diagram

$$
\begin{array}{ccc}
U' \longrightarrow f(U_1) & \qquad & U' \longrightarrow f(U_2) \\
\downarrow \quad\swarrow\ \ \downarrow & & \downarrow \quad\swarrow\ \ \downarrow \\
f(U) \qquad f(V) & & f(U) \qquad f(V)
\end{array}
$$

such that s_1, t_1 have the same image in $F(U_1)$ and the images of s_2, t_2 are the same in $F(U_2)$.[[★★★ if you take a limit over a *filtering* category, then it commutes with limits? something is fishy; what is it]] We'd like to say that we can take U_1 and U_2 to be the same.

We have the diagram

$$
\begin{array}{c}
U_3 \\
\downarrow\ \ \searrow \\
U_1 \qquad U_2 \\
\downarrow\ \ \swarrow \\
U \times V
\end{array}
$$

---

[1]It doesn't commute with infinite products, by the way.

[2]By the universal properties of $\varinjlim$, it is enough to define a map from each $F_1(U) \times F_2(U)$ to some $F_1(U_1)$ and some $F_2(U_2)$. Take $U_1 = U_2 = U$, and then take the obvious projections.

That is, we can find some U_3 so that blah[[★★★]]. For surjectivity, you use the existence of products.

(Equalizers) Say we have $F_1 \xrightarrow[g]{h} F_2$, then we are looking at

$$\hat{f}^* Eq\left(F_1 \overset{h}{\underset{g}{\Rightarrow}} F_2\right)(U') \qquad\qquad Eq\left(\hat{f}^* F_1 \overset{\hat{f}^* h}{\underset{\hat{f}^* g}{\Rightarrow}} \hat{f}^* F_2\right)(U')$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\xrightarrow{\qquad}$$

$$\varinjlim_{U' \to f(U)} Eq\left(F_1(U) \overset{h(U)}{\underset{g(U)}{\Rightarrow}} F_2(U)\right) \qquad Eq\left(\varinjlim_{U' \to f(U)} F_1(U) \overset{\hat{f}^* h(U')}{\underset{\hat{f}^* g(U')}{\Rightarrow}} \varinjlim_{U' \to f(U)} F_2(U)\right)$$

Same sort of arguments to show that this is a bijection.   □

There are lots of sites that have the property that finite projective limits exist.

Example 5.2. Sch has finite projective limits (because we have finite products and fiber products). Thus, all big sites have finite projective limits.   ◇

Example 5.3. The small étale site has finite projective limits. We know how to produce products and fiber products

$$Z \hookrightarrow U_1 \rightrightarrows U_2$$

with maps down to X.

The equalizer need not be smooth even if the two maps are smooth, but if the two maps are étale, then so is the equalizer.   ◇

Example 5.4. If X is a scheme, then we get $X_{\text{lis-et}} \xrightarrow{\varepsilon} X_{et}$ is a morphism of topoi, and ε_* is exact, so you can compute cohomology in either topos.   ◇

Faithfully flat descent

If $Y \to X$ is a morphism of schemes, then we get a functor of points $h_Y : (\mathsf{Sch}/X)^{op} \to$ Set which is a presheaf. The main point is the following.

Theorem 5.5. h_Y is a sheaf in the fppf topology (and therefore also in the étale topology).

By the Yoneda embedding, we have $(\mathsf{Sch}/X) \hookrightarrow (\mathsf{Sch}/X)_{et} = X_{ET}$. We will define *algebraic space* to be an object in X_{ET} which is more general than a scheme, but in which you can still do geometry (i.e. where you can redo EGA).

Definition 5.6. A morphism of schemes $f : X \to Y$ is *flat* if for all $x \in X$ the map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. It is *faithfully flat* if it is flat and surjective. A map of rings $A \to B$ is *faithfully flat* if the map of spectra $\operatorname{Spec} B \to \operatorname{Spec} A$ is. $\qquad \diamond$

Key case: $Y = \mathbb{A}^1_{\mathbb{Z}}$ is flat over $X = \operatorname{Spec} \mathbb{Z}$. Then

$$h_Y(\operatorname{Spec} A) \longrightarrow h_Y(\operatorname{Spec} B) \rightrightarrows h_Y(\operatorname{Spec} B \otimes_A B)$$

$$\begin{array}{ccccc}
\| & & \| & & \| \\
A & \longrightarrow & B & \xrightrightarrows[b \mapsto 1 \otimes b]{b \mapsto b \otimes 1} & B \otimes_A B
\end{array}$$

If [[★★★ iff?]] $A \to B$ is faithfully flat, then the bottom sequence is exact.

6   Representable functors are fppf sheaves

The main result for today is Theorem 6.10, that representable functors are sheaves in the fppf topology.

Recall the following three propositions. The first one is standard, and the other two are in EGA.

Proposition 6.1. *Let $A \to B$ be a ring morphism. Then the following are equivalent.*

1. *$A \to B$ is faithfully flat.*

2. *A sequence of A-modules $M' \to M \to M''$ is exact if and only if $M' \otimes_A B \to M \otimes_A B \to M'' \otimes_A B$ is exact.*

3. *A homomorphism $M' \to M$ of A-modules is injective if and only if $M' \otimes_A B \to M \otimes_A B$ is injective.*

4. *B is flat over A and $\left(M \otimes_A B = 0 \implies M = 0 \right)$.*

Proposition 6.2 ([EGA, IV.1.10.4]). *A flat morphism that is locally of finite presentation is open.*

Proposition 6.3 ([EGA, [[★★★ somewhere in IV]]]). *If $f : X \to Y$ is faithfully flat and quasi-compact, then a subset $U \subseteq Y$ is open if and only if $f^{-1}(U) \subseteq X$ is open (i.e. Y has the induced topology).*

The following corollary allows us to deal fppf morphisms.

Corollary 6.4. *Let $f : X \to Y$ be faithfully flat and locally of finite presentation, and let $Y = \bigcup_i U_i$ be a Zariski open covering, with each U_i affine. Then for each i, there is a Zariski covering $f^{-1}(U_i) = \bigcup_j V_{ij}$ with V_{ij} quasi-compact and $f(V_{ij}) = U_i$.*

Proof. Let $p \in f^{-1}(U_i)$, and let $W_{ip} \subseteq f^{-1}(U_i)$ be an affine open neighborhood of p. Given any open affine set $W_{iq} \subseteq f^{-1}(U_i)$, $f(W_{iq})$ is open by proposition 6.2. Since U_i is affine, it is quasi-compact, so we can choose a finite set $\{W_{iq_k}\}_{k=1}^n$ so that the $f(W_{ip}) \cup \bigcup_k f(W_{iq_k}) = U_i$. Now we can define $V_{ip} := W_{ip} \cup \bigcup_k W_{iq_k}$. Since V_{ip} is a finite union of affines, it is quasi-compact. Furthermore, $p \in V_{ip}$, so $\{V_{ip}\}_{p \in f^{-1}(U_i)}$ covers $f^{-1}(U_i)$. $\qquad \square$

[[★★★ a flat morphism satisfying the conclusion of this corollary is said to be *fpqc*]]

Proposition 6.5. *If $f : A \to B$ is faithfully flat, then the following sequence is exact.*

$$A \xrightarrow{\;\;f\;\;} B \mathrel{\substack{\xrightarrow{b \mapsto b \otimes 1} \\[-0.3em] \xrightarrow[b \mapsto 1 \otimes b]{}}} B \otimes_A B$$

Proof. We use the following trick. Since B is faithfully flat over A, exactness of the sequence in question is equivalent to exactness of the sequence obtained by tensoring with B.

$$B \xrightarrow{\;b' \mapsto 1 \otimes b'\;} B \otimes_A B \xrightarrow[\;b \otimes b' \mapsto 1 \otimes b \otimes b'\;]{\;b \otimes b' \mapsto b \otimes 1 \otimes b'\;} B \otimes_A B \otimes_A B$$

The first map is injective because the multiplication map $m : B \otimes_A B \to B$ is a section. Now we check exactness in the middle: if the two maps from $B \otimes_A B$ agree on some element, $\sum b_i \otimes b_i'$, then we have $\sum b_i \otimes 1 \otimes b_i' = \sum 1 \otimes b_i \otimes b_i'$. Applying $1 \otimes m$, we get $\sum b_i \otimes b_i' = \sum 1 \otimes b_i b_i' = 1 \otimes \sum_i b_i b_i'$, so the original element $\sum b_i \otimes b_i'$ is in the image of the first map, proving exactness in the middle. $\qquad\square$

Remark 6.6 (Faithfully flat extensions)**.** We will use the above trick again. The fact that we could tensor with B and get multiplication, as section of $B \to B \otimes_A B$, effectively makes it so that we can *assume* we have a section of $A \to B$. Sometimes, we'll base extend by an fppf cover and get a section for some map, but to save writing we'll say that you can assume the original map has a section. Here is how the above argument would look:

Base extending by B, we may assume we have a section g of f; in particular, f is injective. Let $b \in B$ with $1 \otimes b = b \otimes 1$. Applying $fg \otimes \mathrm{id}$, we have that $b = fg(b)$, so b is in the image of f, proving exactness in the middle. $\qquad\diamond$

Corollary 6.7. *If $V \to U$ is a faithfully flat map of affine schemes, and X is an affine scheme, then the sequence $h_X(U) \to h_X(V) \rightrightarrows h_X(V \times_U V)$ is exact.*

Proof. Let $U = \operatorname{Spec} A$, $V = \operatorname{Spec} B$, and $X = \operatorname{Spec} R$. By Proposition 6.5, we have the exact sequence $A \to B \rightrightarrows B \otimes_A B$, and we wish to show exactness of the sequence $\operatorname{Hom}(R, A) \to \operatorname{Hom}(R, B) \rightrightarrows \operatorname{Hom}(R, B \otimes_A B)$.

Since A injects into B, two maps from R to A which agree in B are the same. If $f : R \to B$ satisfies $1 \otimes f(r) = f(r) \otimes 1$, then $f(r)$ lies in A, so f is obtained from a map $R \to A$. $\qquad\square$

Lemma 6.8. *Let $F : \mathsf{Sch}^{op} \to \mathsf{Set}$ be a presheaf satisfying the following conditions.*

1. *F is a sheaf in the big Zariski topology.*

2. *If $V \to U$ is a faithfully flat morphism of affine schemes, then the sequence $F(U) \to F(V) \rightrightarrows F(V \times_U V)$ is exact.*

Then F is a sheaf for the fppf topology.

[[★★★ if you remove the word "faithfully" from the second condition, then you can conclude that F is a sheaf in the fpqc topology]]

Remark 6.9. The following proof also works if you work over some scheme X. $\qquad\diamond$

Proof of Lemma 6.8. Let $\{U_i \to U\} \in Cov_{fppf}(U)$, and let $V = \coprod U_i$, then we have the diagram

$$
\begin{array}{ccc}
F(U) \longrightarrow & F(V) \rightrightarrows & F(V \times_U V) \\
\parallel & \downarrow \wr & \downarrow \wr \\
F(U) \longrightarrow & \prod_i F(U_i) \rightrightarrows & \prod_{i,j} F(U_i \times_U U_j)
\end{array}
$$

where the vertical isomorphisms follow from the fact that F is a Zariski sheaf. Thus, it is enough to consider coverings consisting of a single morphism $\{V \to U\}$.

<u>Note</u>: If $\{U_i \to U\} \in Cov(U)$ is a finite set of maps with U_i and U affine, then $V = \coprod U_i$ is also affine. The top sequence is exact by assumption, so the sheaf condition (exactness of the bottom sequence) is verified.

For a general (single element) fppf covering $f : V \to U$, choose a Zariski cover $V = \bigcup V_i$ with V_i quasi-compact and $f(V_i) = U_i$ affine (we can do this by Corollary 6.4). Write each $V_i = \bigcup_a V_{ia}$ as a finite union of affines. Then consider the following diagram.

The U_i cover U and the V_{ia} cover V in the usual Zariski sense. Since F is a Zariski sheaf, the two vertical columns are exact. For a fixed i, the V_{ia} are a finite number of affines which cover the affine U_i, so by the note, the middle horizontal sequence is exact. We wish to show that the top sequence is exact.

Since $\beta \circ \delta$ is injective, we must have that γ is injective. Observe that this shows that F is separated in the fppf topology. In particular, since $\{V_{ia} \cap V_{jb} \to U_i \cap U_j\}_{a,b} \in Cov_{fppf}(U_i \cap U_j)$, α must be injective!

Now we check exactness at $F(V)$ by a diagram chase, illustrated above.[1] Let $x \in F(V)$ be taken to $y \in F(V \times_U V)$ by both maps. Then d must be taken by both maps to c, so by exactness of the middle row, it comes from some e. Since d is the image of x, it is taken to some g by both maps. The two images f and f' must both be taken to g; since α is injective, we must have $f = f'$. Therefore, e must be the image of some $h \in F(U)$. Since ε is injective and $\varepsilon(x) = d = \varepsilon \circ \gamma(h)$, we get $\gamma(h) = x$. $\square$

Theorem 6.10. *Let X be a scheme. Then $h_X : \mathsf{Sch}^{op} \to \mathsf{Set}$ is a sheaf for the fppf topology.*

---

[1]The starting object is circled. A solid arrow indicates that an object at one end is defined by the object at the other end. A dotted arrow indicates that commutativity of the diagram forces the two objects to be related as described. For example, we know (by commutativity of the diagram) that both horizontal maps take d to c; therefore, by exactness of the middle row, d defines the element e.

Proof. (Affine case) First assume that X is affine. Combining Lemma 6.8 with Corollary 6.7, we have that h_X is a sheaf in the fppf topology.

(General case) Now let X be any scheme. Write $X = \bigcup X_i$ as a union of open affine subschemes. By Lemma 6.8, it is enough to consider an fppf covering of the form $t : V \to U$, where U and V are affine.

We have to check the exactness of

$$h_X(U) \xrightarrow{\ \alpha\ } h_X(V) \rightrightarrows h_X(V \times_U V).$$

First we do injectivity. Suppose $f, g \in h_X(U)$ are identified by α. Then we have $V \xrightarrow{t} U \underset{g}{\overset{f}{\rightrightarrows}} X$ with $ft = gt$. In particular, the maps of sets must agree; since t is surjective, f and g must be set-theoretically equal. Now consider $U_i = f^{-1}(X_i) = g^{-1}(X_i)$. By the affine case, $f|_{U_i} = g|_{U_i}$ scheme-theoretically. Therefore, we get $f = g$.

Now we check exactness in the middle. We will denote the forgetful functor to $\mathsf{Sch} \to \mathbf{Top}$ by $|\cdot|$. Let $f \in h_X(V)$ with $fp_1 = fp_2 \colon V \times_U V \underset{p_2}{\overset{p_1}{\rightrightarrows}} V \xrightarrow{f} X$. Applying the forgetful functor, we get the diagram

The dashed arrow exists by the universal property of $|V| \times_{|U|} |V|$. For some reason, we have that $|f|\pi_1 = |f|\pi_2$ [[★★★ why?]]. If v_1 and v_2 are two points in V which lie over the same point in U, then $(v_1, v_2) \in |V| \times_{|U|} |V|$, and we get $f(v_1) = |f|\pi_1(v_1, v_2) = |f|\pi_2(v_1, v_2) = f(v_2)$. Thus, we get a well-defined map $h : |U| \to |X|$ given by $u \mapsto f\big(t^{-1}(u)\big)$. By Proposition 6.3, the topology on U is induced by t, so h is continuous.

Let $V_i = f^{-1}(X_i)$ and $U_i = h^{-1}(X_i)$. Then $V_i \to U_i$ are fppf coverings. By the affine case, we have (unique) morphisms of schemes $h_i : U_i \to X_i$ so that $f|_{V_i} = h_i \circ t|_{V_i}$. Covering the intersections $X_i \cap X_j$ by affines and using the uniqueness, we have that the h_i agree on intersections $U_i \cap U_j$. Therefore, we get a morphism of schemes $h : U \to X$ so that $f = h \circ t$. $\qquad\square$

7 Descent

Descent in general

For an object Y (in some category), let $\mathsf{C}(Y)$ be some category associated to Y. For a morphism $f : X \to Y$, assume we have a pullback functor $f^* : \mathsf{C}(Y) \to \mathsf{C}(X)$. The question is, *"which objects in $\mathsf{C}(X)$ come from objects in $\mathsf{C}(Y)$ via f^*?"*

First we do a sanity check: if $E \in \mathsf{C}(X)$ is to be of the form f^*D for some $D \in \mathsf{C}(Y)$, then pulling E back along morphisms coequalized by f had better produce isomorphic results. In particular, f coequalizes the two projections $p_1, p_2 : X \times_Y X \to X$, so there had better be some isomorphism $\sigma : p_2^*E \xrightarrow{\sim} p_1^*E$. Moreover, this σ would have to satisfy the cocycle condition $p_{13}^*\sigma = p_{12}^*\sigma \circ p_{23}^*\sigma$. The general form of descent theorems is this: if f is a nice morphism, and E passes the above sanity check, then it is f^*D for some $D \in \mathsf{C}(Y)$. Here is a more precise formulation.

Definition 7.1. Let D be a category in which fiber products are representable (like Sch), and let $\mathsf{C} : \mathsf{D}^{op} \to \mathsf{Cat}$ be a lax 2-functor.[1] For a morphism g in D, denote $\mathsf{C}g$ by g^*. Let $f : X \to Y$ be a morphism in D. We define the category $\mathsf{C}(X \xrightarrow{f} Y)$ as follows. The objects are pairs (E, σ), where E is in $\mathsf{C}(X)$ and $\sigma : p_2^*E \xrightarrow{\sim} p_1^*E$ is an isomorphism, where p_1 and p_2 are the projections $X \times_Y X \to X$. Furthermore, if p_{12}, p_{13}, and p_{23} are the projections $X \times_Y X \times_Y X \to X \times_Y X$, we require the diagram on the left to commute (the "equalities" are really canonical isomorphisms).

$$
\begin{array}{ccc}
& p_{13}^*p_2^*E \xrightarrow{p_{13}^*\sigma} p_{13}^*p_1^*E & \\
p_{23}^*p_2^*E & & p_{12}^*p_1^*E \\
\xrightarrow{p_{23}^*\sigma} p_{23}^*p_1^*E = p_{12}^*p_2^*E \xrightarrow{p_{12}^*\sigma} & &
\end{array}
\qquad
\begin{array}{ccc}
p_2^*E' & \xrightarrow{p_2^*\varepsilon} & p_2^*E \\
\sigma' \downarrow & & \downarrow \sigma \\
p_1^*E' & \xrightarrow{p_1^*\varepsilon} & p_1^*E
\end{array}
$$

A morphism $(E, \sigma) \xrightarrow{\varepsilon} (E', \sigma')$ is a morphism $\varepsilon : E \to E'$ in $\mathsf{C}(X)$ such that the diagram on the right commutes. We call σ *descent data* for the object E. $\diamond$

Remark 7.2. If $\{X_i \to Y\}$ is a set of morphisms, we can define $\mathsf{C}(\{X_i \to Y\})$ similarly, but in most of the sites we care about and for most applications, we can always replace $\{X_i \to Y\}$ by the single morphism $X = \coprod X_i \to Y$. $\diamond$

Note that if $F \in \mathsf{C}(Y)$, then $(f^*F, \mathrm{can}) \in \mathsf{C}(X \to Y)$, where can is the canonical isomorphism $p_2^*f^*F \cong (f \circ p_2)^*F = (f \circ p_1)^*F \cong p_1^*f^*F$. That is, the functor $f^* : \mathsf{C}(Y) \to \mathsf{C}(X)$ factors through $\mathsf{C}(X \to Y)$. In general, descent theorems say that if f is sufficiently nice, then $f^* : \mathsf{C}(Y) \to \mathsf{C}(X \to Y)$ is an equivalence of categories. Here are some examples of descent theorems.

---

[1]That is, if f and g are composable morphisms in D, then we do not require the isomorphism $(fg)^* \cong g^*f^*$ to be an equality, but we *do* require that the isomorphism is natural in f and g. Moreover, we require that the two isomorphisms $f^*g^*h^* \cong (gf)^*h^* \cong (hgf)^*$ and $f^*g^*h^* \cong f^*(hg)^* \cong (hgf)^*$ agree.

- (Theorem 7.5) If $f : X \to Y$ is a covering in some site C (which has projective limits), then a sheaf on C/X with descent data is equivalent to a sheaf on C/Y.

- (Proposition 7.3) "Sheaf Hom of fppf sheaves is an fppf sheaf." [[★★★ this works in any site]] In particular morphisms between schemes over S can be defined fppf locally on S.

- (Theorem 7.13) If $X \to Y$ is a quasi-compact fppf covering of schemes, then a quasi-coherent sheaf on X together with descent data is equivalent to a quasi-coherent sheaf on Y.

- (Theorem 8.2) If $g \geq 2$ and $X \to Y$ is a quasi-compact fppf covering of schemes, then a genus g curve over X with descent data is equivalent to a genus g curve over Y.

Descent for morphisms of sheaves/schemes

The following proposition shows that morphism of schemes over S can be defined locally *over S* in the fppf topology. More generally, morphisms of sheaves on S can be defined locally in the fppf topology.

Proposition 7.3 ("Sheaf Hom is already a sheaf"). *Let F and G be fppf sheaves on a scheme S, let $S' \to S$ be an fppf cover, and let $S'' = S' \times_S S'$. Note that we have two S-morphism $p_1, p_2 : S'' \to S'$. If $f' : F|_{S'} \to G|_{S'}$ is a morphism such that $p_1^* f' = p_2^* f' : F|_{S''} \to G|_{S''}$, then f' is induced by a unique morphism of fppf sheaves $f : F \to G$.*

[[★★★ this works in any site]]

Proof. For any S-scheme U, let $U' = U \times_S S'$ and $U'' = U \times_S S''$. Then we get a diagram

$$
\begin{array}{ccc}
F(U) \longrightarrow F(U') \rightrightarrows F(U'') & \qquad & \textcircled{x} \longrightarrow y \rightrightarrows z \\
\Big\downarrow{\scriptstyle \exists! f} \quad \Big\downarrow{\scriptstyle f'} \quad \Big\downarrow{\scriptstyle f''=p_1^* f=p_2^* f} & \qquad & \Big\downarrow \quad \Big\downarrow \quad \Big\downarrow \\
G(U) \longrightarrow G(U') \rightrightarrows G(U'') & \qquad & a \longmapsto b \rightrightarrows c
\end{array}
$$

Since $S' \to S$ is an fppf cover, so is $U' \to U$ (one of the axioms of a site). Note also that

$$U' \times_U U' \cong U' \times_U (U \times_S S') \cong U' \times_U S' \cong U' \times_{S'} (S' \times_S S') = U' \times_S S'' = U''.$$

Since F and G are fppf sheaves, the two horizontal sequences are exact. Now a diagram chase produces f. (See the footnote in lecture 6 for how to read the chase.) $\qquad \square$

Corollary 7.4. *Let X and Y be schemes over S, let $S' \to S$ be an fppf cover, let $S'' = S' \times_S S'$, let $X' = X \times_S S'$, $X'' = X \times_S S''$, and define Y' and Y'' similarly. Note that we have two S-morphism $p_1, p_2 : S'' \to S'$. If $f' : X' \to Y'$ is a morphism such that $p_1^* f' = p_2^* f' : X'' \to Y''$, then f' is induced by a unique S-morphism $f : X \to Y$.*

Proof. By the Yoneda lemma, it is enough to find a morphism $f : h_X \to h_Y$ inducing f'. Note that the universal properties of X', X'', Y', and Y'', we have that $h_X|_{S'} = h_{X'}|_{S'}$, $h_X|_{S''} = h_{X''}|_{S''}$, etc. Now the proposition produces f. $\square$

Descent for sheaves in a site

The main point of this section is roughly that "a sheaf on a site can be defined locally in the topology of that site".

Let C be a site in which finite projective limits are representable. For any object X in C, we can form the site C/X (see Example 2.4). Let $\mathsf{Sh}(X)$ be the category of sheaves on C/X. If $f : X \to Y$ is morphism in C, then we get an induced morphism of sites $\mathsf{C}/Y \to \mathsf{C}/X$, given by $(Z \to Y) \mapsto (Z \times_Y X \to X)$. It is immediate to check that this induced functor is continuous. By Theorem 5.1, it induces a morphism of topoi $f : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$. For a sheaf E on C/X, $f_* E(Z \to Y) = E(Z \times_Y X \to X)$. For a sheaf F on C/Y, $f^* F(Z \to X) = \lim_W G(W \to Y)$, where the limit is taken over objects $W \to Y$ such that the following diagram commutes.

$$
\begin{array}{ccc}
Z & \longrightarrow & W \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

But this system has an initial object,[2] namely $Z \to Y$. Thus, $f^* F(Z \to X) = F(Z \to Y)$, so the functor f^* is just given by restriction of the sheaf F.

Theorem 7.5. *Let* $f : X \to Y$ *be a covering of* Y. *Then the functor* $f^* : \mathsf{Sh}(Y) \to \mathsf{Sh}(X \to Y)$, *is an equivalence of categories.*

Proof. To show that f^* is an equivalence, it is enough to show that it is fully faithful (induces isomorphisms on Hom sets) and is essentially surjective (every isomorphism class is in the image).

(Full faithfulness) It is enough to construct a left inverse. Given $(E, \sigma) \in \mathsf{Sh}(X \to Y)$, define

$$
h(E, \sigma) = Eq\left(f_* E \underset{\longrightarrow}{\overset{\longrightarrow}{}} \begin{array}{c} f_* p_{1*} p_1^* E \\ f_* p_{2*} p_2^* E \end{array} \underset{g_*(\sigma)}{\overset{\longrightarrow}{\underset{\longrightarrow}{}}} g_* p_1^* E \right).
$$

Next we show that h is right adjoint to f^*. Since f^* is left adjoint to f_*, a morphism $f^* F \to E$ corresponds to a morphism $F \to f_* E$. The condition that $F \to f_* E$ equalizes the two maps above turns out to be equivalent the the condition that the morphism $f^* F \to E$ is compatible with the descent data. [[★★★ check this some time]]

We get a unit of adjunction, the natural transformation $\mathrm{id} \to h f^*$. If we can show that this natural transformation is an isomorphism on each object, then it follows

---

[2]Yeah, I know, it looks like you want a *terminal* object, but you apply G and then take the limit, and G is contravariant, so what you really want to find is an initial object *before* you apply G (or a terminal object after you apply G.

that hf^* is isomorphic to the identity functor. First note that for $F \in \mathsf{Sh}(Y)$ and $Z \to Y$, we get that $hf^*F(Z \to Y) = f^*F(Z \times_Y X \to X) = F(Z \times_Y X \to Y)$, and the morphism from $F(Z \to Y)$ is induced by the Y-morphism $Z \times_Y X \to Z$ [[★★★ good way to see this?]]. Thus, we wish to check that the obvious morphism $F(Z \to Y) \to F(Z \times_Y X \to Y)$ is the equalizer in question, i.e. that the following sequence is exact.

$$
\begin{array}{ccccc}
F(Z) & \longrightarrow & f_*f^*F(Z) & \rightrightarrows & g_*g^*F(Z) \\
\| & & \| & & \| \\
F(Z) & \longrightarrow & F(Z \times_Y X) & \rightrightarrows & F(Z \times_Y X \times_Y X)
\end{array}
\tag{7.6}
$$

But $Z \times_Y X \to Z$ is a covering (since $X \to Y$ is a covering), and $(Z \times_Y X) \times_Z (Z \times_Y X) = Z \times_Y X \times_Y X$, so the sequence is exact by the sheaf axiom on F.

(Essential surjectivity) Given (E, σ), let $F = h(E, \sigma)$. We wish to show that $(E, \sigma) \cong (f^*F, \mathrm{can})$. Since both of these are sheaves on X, it is enough to check that these sheaves are isomorphic when restricted to some cover $W \to X$.[3] Using this, we will now reduce to the case where $f : X \to Y$ has a section.

Base extending by $f : X \to Y$, we change the names of things in the following way.

$$
\begin{array}{ccccc}
{}`X{}' \times_{`Y'} {}`X{}' & = (X \times_Y X) \times_Y X \xrightarrow{p_{12}} X \times_Y X & \\
{}^{`p_1{}'}\big\downarrow\big\downarrow{}^{`p_2{}'} & \quad p_{13}\big\downarrow\big\downarrow p_{23} \qquad\qquad p_1\big\downarrow\big\downarrow p_2 \\
{}`E{}',\quad {}`f{}'^*{}`F{}' \qquad {}`X{}' & = \quad X \times_Y X \xrightarrow{\;\;p_1\;\;} X \qquad\quad E,\quad f^*F \\
{}^{`s{}'}\Big\uparrow\Big\downarrow{}^{`f{}'} & \quad \Delta\Big\uparrow\Big\downarrow p_2 \qquad\qquad\quad \Big\downarrow f \\
{}`F{}' = {}`h{}'({}`E{}', {}`\sigma{}') \qquad {}`Y{}' & = \quad X \xrightarrow{\quad f \quad} Y \qquad F = h(E, \sigma)
\end{array}
$$

E_X and σ pull back along p_1 to give a sheaf on $X \times_Y X$ with descent data with respect to p_2, and F pulls back along f to give ${}`F{}'$. We have that $p_1^*(f^*F) \cong p_2^* f^*F = {}`f{}'^*{}`F{}'$. Moreover, commutativity of the following diagram (in which the pairs of vertical arrows are adjoint functors) tells us that ${}`h{}'({}`E{}', {}`\sigma{}') = {}`F{}'$. This concludes the reduction to the case where f has a section.

$$
\begin{array}{ccccc}
\mathsf{Sh}(X \times_Y X \xrightarrow{p_2} X) & = \mathsf{Sh}({}`X{}' \xrightarrow{`f'} {}`Y{}') & \xleftarrow{\;p_1^*\;} & \mathsf{Sh}(X \to Y) \\
{}_{p_2^* = `f'^*}\Big\uparrow\Big\downarrow{}^{`h'} & & {}_{f^*}\Big\uparrow\Big\downarrow{}^{h} \\
\mathsf{Sh}(X) = \mathsf{Sh}({}`Y{}') & \xleftarrow{\qquad f^* \qquad} & \mathsf{Sh}(Y)
\end{array}
$$

---

[3]For any $Z \to X$ and any sheaf $G \in \mathsf{Sh}(X)$, the sheaf axiom forces $G(Z \to X)$ to be the equalizer of $G(Z \times_X W \to X) \rightrightarrows G(Z \times_X W \times_X W \to X)$, but these two morphisms factor through $W \to X$, so their values are known once we know G restricted to W.

If you feel like we're assuming the result of the theorem we're trying to prove, think about it this way. This statement is saying "it is enough to verify that a morphism of sheaves is an isomorphism by looking on a basis for the topology". The theorem is saying "given an open cover, with sheaves defined on each open set so that the restrictions to the intersections agree compatibly, there is a unique sheaf on the whole space which restricts to the given sheaves on each of the open sets".

In what follows, we suppress the descent data in the notation. We have an isomorphism

$$E = (sf \times 1)^* p_2^* E \xrightarrow[(sf \times 1)^* \sigma]{\sim} (sf \times 1)^* p_1^* E = f^* s^* E.$$

$$X \xrightarrow{sf \times 1} X \times_Y X \mathrel{\substack{p_2 \\ \longrightarrow \\ \longrightarrow \\ p_1}} X$$

By the first part of this proof, h is a left inverse to f^*, so we have that $F = hE \cong hf^* s^* E \cong s^* E$. Applying f^*, we get the isomorphism $f^* F \cong f^* s^* E \cong E$, as desired. $\square$

Descent for sheaves of modules

[[★★★ Notation. In class, we use f^* for what I would usually call f^{-1}, and I don't want to use notation different from what we use in class. Unfortunately, for sheaves of modules, there is a different f^*, and sometimes the distinction is important. Since I can't come up with a good solution, I'm going to use f^* to mean f^{-1} (for all sheaves) and $f^\star$ to mean pullback for sheaves of modules. Let me know if you have a better solution.]]

Let X and Y be objects in C, and let $\mathcal{O}_X \in \mathsf{Sh}(X)$ and $\mathcal{O}_Y \in \mathsf{Sh}(Y)$ be sheaves of rings. For a morphism $f : X \to Y$, assume we also have a morphism of sheaves of rings $f^* \mathcal{O}_Y \to \mathcal{O}_X$. We get that f induces a continuous morphism of sites $\mathsf{C}/Y \to \mathsf{C}/X$, given by $(Y' \to Y) \mapsto (X \times_Y Y' \to X)$. Since finite projective limits are representable in C/Y, Theorem 5.1 tells us that this induces a morphism of topoi $(f_*, f^*) : \mathsf{Sh}(X) \to \mathsf{Sh}(Y)$. Note that f_* is also a morphism of the categories $\mathcal{O}_X$-mod $\to \mathcal{O}_Y$-mod, where $\mathcal{O}_Y$ acts on $f_* \mathcal{F}$ via the map $\mathcal{O}_Y \to f_* \mathcal{O}_X$. However, f^* is not left adjoint to f_* (it doesn't even give a functor the other way). But there is a left adjoint, which we call $f^\star$. It is given by $f^\star \mathcal{G} = f^* \mathcal{G} \otimes_{f^* \mathcal{O}_Y} \mathcal{O}_X$ (you have to sheafify after your take the tensor product). [[★★★ it would be nice to prove that $f^\star$ is left adjoint to f_*]]

$$X_{et} \mathrel{\substack{f_* \\ \longleftarrow \\ \longrightarrow \\ f^*}} Y_{et}$$

$$\mathcal{O}_X\text{-mod} \mathrel{\substack{f_* \\ \longrightarrow \\ \longleftarrow \\ f^\star}} \mathcal{O}_Y\text{-mod}$$

Remark 7.7. For regular Zariski sheaves on schemes (or topological spaces for that matter), if $X \to Y$ is an open map, then it is really easy to understand f^* and $f^\star$. There is an analogous statement in this situation.

If $\mathcal{O}_X$ and $\mathcal{O}_Y$ are obtained by restricting some sheaf of rings $\mathcal{O}$ on C, $f : X \to Y$ is a morphism in C, $\mathcal{G}$ is a sheaf on Y, and $U \to X$ is an element of C/X, then $f^* \mathcal{G}(U \to X) = \varinjlim_{U \to V} \mathcal{G}(V \to Y) = \mathcal{G}(U \to Y)$. That is, $f^* \mathcal{G}$ is obtained simply by restricting $\mathcal{G}$ to C/X. In particular, $f^* \mathcal{O}_Y = \mathcal{O}_X$. If $\mathcal{G}$ is an $\mathcal{O}_Y$-module, then $f^\star \mathcal{G} = f^* \mathcal{G} \otimes_{f^* \mathcal{O}_Y} \mathcal{O}_X = f^* \mathcal{G}$ is given by restricting $\mathcal{G}$ to C/X.

In some sense, this says that the bigger your site is, the easier it is to understand f^* and $f^\star$. In the case of Zariski sheaves on schemes, f^* and $f^\star$ are hard because most morphisms between schemes are not open immersions (i.e. f is usually not in C). ◇

Corollary 7.8. *Let C be a site with projective limits, let $f : X \to Y$ be a covering, and let $\mathcal{O}_X \in \mathsf{Sh}(X)$ and $\mathcal{O}_Y \in \mathsf{Sh}(Y)$ be sheaves of rings, with $\mathcal{O}_X = f^*\mathcal{O}_Y$. Then $f^* : \mathcal{O}_Y\text{-}\mathsf{mod} \to \mathcal{O}_{X \to Y}\text{-}\mathsf{mod}$ (interpret this in the obvious way) is an equivalence of categories.*

Proof. The module structure comes along for the ride along f_*, f^*, and h.   □

Descent for quasi-coherent sheaves

For any scheme X, define the presheaf $\mathcal{O}_{X_{fppf}}$ on Sch/X by $(T \to X) \mapsto \Gamma(T, \mathcal{O}_T)$. Note that $\mathcal{O}_{X_{fppf}}$ is represented by $\mathbb{A}^1_X$, so by Theorem 6.10, it is an fppf sheaf. Note that this is the restriction of the sheaf $\mathcal{O}_{\mathrm{Spec}\,\mathbb{Z}}$ on Sch.

Given a quasi-coherent sheaf F on X (i.e. in X_{zar}), we define an $\mathcal{O}_{X_{fppf}}$-module $F_{fppf} : \mathsf{Sch}/X \to \mathsf{Set}$ by $(T \xrightarrow{h} X) \mapsto \Gamma(T, h^\star F)$.

Lemma 7.9. *F_{fppf} is a sheaf in the fppf topology on X.*

Proof. We will apply Lemma 6.8. For $T \xrightarrow{h} X$, $h^\star F$ is a (Zariski) sheaf on T, so F_{fppf} is a sheaf in the big Zariski topology on X. Next we need to check the sheaf condition for an fppf cover of the form $\mathrm{Spec}\,B \to \mathrm{Spec}\,A$ over X. Since F is quasi-coherent, $h^\star_A F$ is quasi-coherent on $\mathrm{Spec}\,A$, so it is $\tilde{M}$ for some A-module M; then $h^\star_B F \cong (B \otimes_A M)^\sim$ and $h^\star_{B \otimes_A B} F \cong (B \otimes_A B \otimes_A M)^\sim$. Thus, the sheaf condition is equivalent to the sequence $M \to B \otimes_A M \rightrightarrows B \otimes_A B \otimes_A M$ being exact. The proof of Lemma 6.5 works almost verbatim.   □

There is a sort of inverse procedure. If $\mathcal{F}$ is *any* sheaf of $\mathcal{O}_{X_{fppf}}$-modules on $(\mathsf{Sch}/X)_{fppf}$, then for any X-scheme $T \to X$, we get a sheaf $\mathcal{F}_T$ (in T_{zar}) by restricting $\mathcal{F}$ to the small Zariski site of T (an open subset of T is a scheme over X). Moreover, if we have an X-morphism $T' \xrightarrow{g} T$, we get a morphism $\mathcal{F}_T \to g_*\mathcal{F}_{T'}$, given by

$$\mathcal{F}_T(U) = \mathcal{F}(U \to T) \xrightarrow{\mathcal{F}p_1} \mathcal{F}(U \times_T T' \to T') = g_*\mathcal{F}_{T'}(U).$$

By the adjunction, this induces a morphism $g^\star\mathcal{F}_T \to \mathcal{F}_{T'}$. Furthermore, if we have morphisms $T'' \xrightarrow{f} T' \xrightarrow{g} T$ over X, then $\mathrm{Hom}(\mathcal{F}_T, g_*f_*\mathcal{F}_{T''}) \cong \mathrm{Hom}(g^\star\mathcal{F}_T, f_*\mathcal{F}_{T''}) \cong \mathrm{Hom}(f^\star g^\star\mathcal{F}_T, \mathcal{F}_{T''}) \cong \mathrm{Hom}((gf)^\star\mathcal{F}_T, \mathcal{F}_{T''})$, so for some reason [[★★★]] $(gf)^\star\mathcal{F}_T \to \mathcal{F}_{T''}$ is the same as the composition $f^\star g^\star\mathcal{F}_T \xrightarrow{f^\star(-)} f^\star\mathcal{F}_{T'} \to \mathcal{F}_{T''}$.

Remark 7.10. If F is a quasi-coherent sheaf on X_{zar}, then given morphisms $T' \xrightarrow{g} T \xrightarrow{h} X$ we get $(F_{fppf})_T = h^\star F$. Then the map $g^\star(F_{fppf})_T \to (F_{fppf})_{T'}$ is just the isomorphism $g^\star h^\star F \cong (hg)^\star F$.   ◇

Definition 7.11. $\mathsf{Qcoh}(X_{fppf})$ is the full subcategory of $\mathcal{O}_{X_{fppf}}$-mod whose objects are $\mathcal{F}$ such that

1. for all $T \to X$, $\mathcal{F}_T$ is quasi-coherent, and

2. for all $g : T' \to T$, the map $g^\star \mathcal{F}_T \to \mathcal{F}_{T'}$ is an isomorphism.   ◇

Proposition 7.12. $\pi^\star : \mathsf{Qcoh}(X_{zar}) \to \mathsf{Qcoh}(X_{fppf})$, given by $F \mapsto F_{fppf}$ is an equivalence of categories.

Proof. Define $\pi_* : \mathsf{Qcoh}(X_{fppf}) \to \mathsf{Qcoh}(X_{zar})$, sending $\mathcal{F}$ to $\mathcal{F}_X$. It is clear that $\pi_* \circ \pi^\star = \mathrm{id}$, and we compute

$$(\pi^\star \pi_* \mathcal{F})(T \xrightarrow{h} X) = (h^\star \mathcal{F}_X)(T) \xrightarrow{\sim} \mathcal{F}_T(T) = \mathcal{F}(T \xrightarrow{h} X)$$

where the isomorphism in the middle is because $\mathcal{F}$ is in $\mathsf{Qcoh}(X_{fppf})$.   □

There is a stronger statement, which is what people properly call descent of quasi-coherent sheaves. For a morphism $f : X \to Y$ of schemes, define $\mathsf{Qcoh}((X \to Y)_{zar})$ as in Definition 7.1, but with all instances of $-^*$ replaced with $-^\star$. In other words, it's the category of quasi-coherent (Zariski) sheaves E on X together with an isomorphism $p_2^\star E \cong p_1^\star E$ satisfying a cocycle condition (which involves $p_{ij}^\star$ instead of p_{ij}^*).

Theorem 7.13. *Let $X \xrightarrow{f} Y$ be an fppf cover with f quasi-compact and quasi-separated. Then $f^\star : \mathsf{Qcoh}(Y_{zar}) \to \mathsf{Qcoh}((X \xrightarrow{f} Y)_{zar})$ is an equivalence of categories.*

Proof. The trick is to reinterpret the statement in the fppf site. We identify $\mathsf{Qcoh}(Y_{zar})$ with $\mathsf{Qcoh}(Y_{fppf})$ as above, and we identify $\mathsf{Qcoh}((X \to Y)_{zar})$ with $\mathsf{Qcoh}(X \to Y)$ (where the objects are quasi-coherent fppf sheaves with regular descent data). Then $f^\star : \mathsf{Qcoh}(Y_{zar}) \to \mathsf{Qcoh}((X \xrightarrow{f} Y)_{zar})$ is identified with $f^* : \mathsf{Qcoh}(Y_{fppf}) \to \mathsf{Qcoh}(X \to Y)$.

By the hypotheses, f_*, p_{1*}, and p_{2*} preserve quasi-coherence [Har77, II.5.8], and all the pullbacks preserve quasi-coherence as usual. Since kernels of maps of quasi-coherent sheaves are quasi-coherent, we have that the functor h from the proof of Theorem 7.5 preserves quasi-coherence. Now Theorem 7.5 applies to prove the result.   □

Remark 7.14. The hypothesis that f is quasi-compact and quasi-separated is actually unnecessary. Let $Y = \bigcup_i Y_i$, with each Y_i affine, and let $f^{-1}(Y_i) = \bigcup_j X_{ij}$ with X_{ij} quasi-compact and $f(X_{ij}) = Y_i$ for each j (we can do this by Corollary 6.4). Then $f|_{X_{ij}} : X_{ij} \to Y_i$ is an fppf cover for each i and j, so the above argument proves descent for quasi-coherent sheaves [[★★★ how do you see that $X_{ij} \to Y_i$ are quasi-separated?]]. Similarly, we can cover $Y_i \cap Y_j$ and $Y_i \cap Y_j \cap Y_k$ by affine schemes, and cover their pre-images by quasi-compact schemes so that we get descent for quasi-coherent sheaves there.

Now given a quasi-coherent sheaf $\mathcal{G}$ on X with descent data, we descend $\mathcal{G}|_{X_{ij}}$ to a quasi-coherent sheaf on Y_i. The descent data tells us [[★★★ somehow]] that

the resulting sheaf is independent of j, so we'll call it $\mathcal{F}_i$. On the affine cover of the intersections $Y_i \cap Y_j$, $\mathcal{F}_i$ and $\mathcal{F}_j$ must both restrict to the sheaf same sheaf (the descended restriction of $\mathcal{G}$), so they are isomorphic, and we get a cocycle condition by the same sort of argument on the triple intersections. Thus, the $\mathcal{F}_i$ glue together to give a quasi-coherent sheaf $\mathcal{F}$, which pulls back to $\mathcal{G}$. ◇

Remark 7.15. We can replace "quasi-coherent sheaves" by "quasi-coherent sheaves of ideals", "quasi-coherent sheaves of algebras", or "locally free sheaves", and the proof still works. ◇

Example 7.16 (Descent for closed subschemes)**.** If X is a scheme and $\mathcal{I} \subseteq \mathcal{O}_X$ is a quasi-coherent sheaf of ideals, then the closed subscheme defined by $\mathcal{I}$ is the sheaf given by $T \mapsto \{g : T \to X \mid \text{the composition } \mathcal{I} \hookrightarrow \mathcal{O}_X \to g_*\mathcal{O}_T \text{ is zero}\}$. Let $f : X \to Y$ be a quasi-compact fppf cover, let F be an fppf sheaf on schemes with a map $F \to Y$, and assume that $Z := F \times_Y X$ is the closed subscheme of X defined by some quasi-coherent sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$. We wish to show that F is a closed subscheme of Y.

$$
\begin{array}{ccc}
Z & \xrightarrow{\mathcal{I}} & X \\
\downarrow & \ulcorner & \downarrow f \\
F & \longrightarrow & Y
\end{array}
$$

Since p_2^*Z and p_1^*Z are both isomorphic to $(fp_1)^*F$, they are the same closed subscheme of $X \times_Y X$, so we have that $p_2^\star\mathcal{I} = p_1^\star\mathcal{I}$ (and we get the cocycle condition similarly). By descent for quasi-coherent sheaves of ideals, we have that $\mathcal{I} \cong f^\star\mathcal{J}$ for some quasi-coherent sheaf of ideals $\mathcal{J} \subseteq \mathcal{O}_Y$. Let $W \subseteq Y$ be the closed subscheme defined by $\mathcal{J}$.

Given $g : T \to X$, the composition $\mathcal{J} \hookrightarrow \mathcal{O}_Y \to f_*g_*\mathcal{O}_T$ is equal to zero if and only if the composition $f^\star\mathcal{J} = \mathcal{I} \hookrightarrow f^\star\mathcal{O}_Y = \mathcal{O}_X \to g_*\mathcal{O}_T$ is zero (since the adjunction $f^\star \vdash f_*$ is a group isomorphism). Thus, we get that $Z \cong f^*W$. By Exercise 2.4, $\mathsf{Sch}_{fppf}/h_X \cong X_{fppf}$, so we may think of Z as an fppf sheaf on X, and we may think of F and W as fppf sheaves on Y. Since $f^*F \cong Z \cong f^*W$, descent for sheaves in a site tells us that $F \cong W$. Thus, F is the closed subscheme of Y defined by $\mathcal{J}$. ◇

8 Descent for $\mathcal{M}_g$, $g \geq 2$

Definition 8.1. For an integer g and a scheme S, $\mathcal{M}_g(S)$ is the category whose objects are smooth proper maps $\pi : C \to S$ all of whose geometric fibers are connected genus g curves, and whose morphisms are isomorphisms over S.

For a morphism of schemes $X \to Y$, the category $\mathcal{M}_g(X \to Y)$ has objects pairs (C_X, σ), where $C_X \in \mathcal{M}_g(X)$ and σ is an isomorphism $\sigma : p_2^* C_X \xrightarrow{\sim} p_1^* C_X$, with the compatibility hexagon like the one in the previous lecture. A morphism in this category, $(C_X', \sigma') \xrightarrow{\varepsilon} (C_X, \sigma)$ is a morphism $\varepsilon : C_X' \to C_X$ such that the following diagram commutes.

$$
\begin{array}{ccc}
p_2^* C_X' & \xrightarrow{\ p_2^* \varepsilon\ } & p_2^* C_X \\
{\sigma'} \downarrow & & \downarrow {\sigma} \\
p_1^* C_X' & \xrightarrow{\ p_1^* \varepsilon\ } & p_1^* C_X
\end{array}
$$

$\diamond$

Proposition 8.2. *If $g \geq 2$ and $f : X \to Y$ is a quasi-compact fppf cover, then the pull-back functor $f^* : \mathcal{M}_g(Y) \to \mathcal{M}_g(X \xrightarrow{f} Y)$, given by $C \mapsto (f^*C, \mathrm{can})$, is an equivalence of categories.*

The following lemma is the key. It is the only part of the proof that uses $g \geq 2$.

Lemma 8.3. *If $g \geq 2$, then for any $(\pi : C \to S) \in \mathcal{M}_g(S)$, $\Omega_{C/S}^{\otimes 3}$ is a relatively very ample sheaf.*[1]

Proof. On fibers, the canonical sheaf is very ample by Riemann-Roch.[2] We also have that $H^1(C, \Omega_{C/S}^{\otimes 3}) = 0$ [[★★★ somehow]]. By [[★★★ something]] from [Har77, III §9], we get that $\Omega_{C/S}^{\otimes 3}$ is relatively very ample. $\qquad\qquad\square$

Proof of Proposition 8.2. We need to show that the pull-back functor is fully faithful and essentially surjective. By Proposition 7.4, morphisms of pull-back curves in $\mathcal{M}_g(X \to Y)$ "glue" to give morphisms of the originals in $\mathcal{M}_g(Y)$, so f^* is injective on Hom sets. Since isomorphisms also glue in the fppf topology, we have that f^* is injective on objects. Thus, f^* is fully faithful.

Now we show essential surjectivity. Let $(C_X, \sigma) \in \mathcal{M}_g(X \to Y)$. By the lemma, $\Omega_{C_X/X}^{\otimes 3}$ is relatively very ample. [[★★★ $\pi_* \Omega_{C_X/X}$ is a locally free sheaf, and this construction commutes with base change for some reason ... where do we use this?]].

---

[1] A sheaf $\mathcal{E}$ on C is *relatively very ample* if there is an open cover of S so that $\mathcal{E}$ is very ample over each open set. This means that there is a closed immersion $C \hookrightarrow \mathbb{P}(\pi_* \mathcal{E})$ (this is a twisted projective space over S).

[2] Let K be the canonical divisor and let $l(D) = \dim \Gamma(\mathcal{L}(D))$. Riemann-Roch for curves states that $l(D) - l(K - D) = \deg D + 1 - g$. $\mathcal{L}(D)$ is very ample if and only if $l(D - P - Q) = l(D) - 2$ for all points P and Q (see [Har77, IV §3]). If $\deg D < 0$, then $l(D) = 0$. Thus, if $\deg D > 2g$, $\mathcal{L}(D)$ will be very ample. For $g \geq 2$, $\deg 3K = 6g - 6 > 2g$, so the third tensor power of the canonical sheaf is very ample.

Let $E_X = \pi_* \Omega^{\otimes 3}_{C_X/X}$. The isomorphism σ induces an isomorphism $\sigma_E : p_2^* E_X \xrightarrow{\sim} p_1^* E_X$. The cocycle condition on σ induces the cocycle condition for σ_E.

$$p_1^* E_X \xleftarrow[\sigma_E]{\sim} p_2^* E_X$$

$$p_1^* C_X \xleftarrow[\sigma]{\sim} p_2^* C_X \longrightarrow X \times_Y X$$

$$E_X \quad C_X \longrightarrow X$$

By descent for locally free sheaves (see Remark 7.15), there is a locally free sheaf E on Y which pulls back to E_X. It follows that $\mathbb{P}E$ pulls back to $\mathbb{P}E_X$. Consider the following diagram.

$$p_1^* C_X = p_2^* C_X \lhook\joinrel\longrightarrow \mathbb{P}E_X \times_{\mathbb{P}E} \mathbb{P}E_X \longrightarrow X \times_Y X$$

$$C_X \lhook\joinrel\longrightarrow \mathbb{P}E_X \longrightarrow X$$

$$C \lhook\joinrel\dashrightarrow \mathbb{P}E \longrightarrow Y$$

with vertical maps labeled fppf qcompact.

The closed immersion of C_X into $\mathbb{P}E_X$ is the one induced by $\Omega^{\otimes 3}_{C_X/X}$. [[★★★ why do we get the equality of subschemes $p_1^* C_X = p_2^* C_X$?]] Let $\mathcal{I}_{C_X}$ be the quasi-coherent sheaf of ideals of C_X within $\mathbb{P}E_X$. The equality of subschemes $p_1^* C_X = p_2^* C_X$ induces an isomorphism $\sigma_{\mathcal{I}} : p_2^* \mathcal{I}_{C_X} \xrightarrow{\sim} p_1^* \mathcal{I}_{C_X}$ which satisfies the cocycle condition. By descent for quasi-coherent sheaves of ideals along the quasi-compact fppf cover $\mathbb{P}E_X \to \mathbb{P}E$, we have that $\mathcal{I}_{C_X}$ is the pull-back of some quasi-coherent sheaf of ideals $\mathcal{I}_C$ on $\mathbb{P}E$. Let C be the subscheme defined by $\mathcal{I}_C$. Then we have that C pulls back to C_X. Since smoothness and properness descend along fppf covers, $C \to Y$ is smooth and proper. For any geometric point y in Y, there is a corresponding geometric point x in X, and the fiber C_y is equal to the fiber $(C_X)_x$, so it is a connected genus g curve. $\square$

Remark 8.4. The moral is that when you want to talk about descent of anything, you do it via descent of quasi-coherent sheaves. The key to this proof was the construction of a canonical embedding of the curve into some projective space, so that we could use the quasi-coherent sheaf of ideals.

For $g = 0$, the anti-canonical sheaf is very ample, so essentially the same proof works to prove descent for $\mathcal{M}_0$. However, for $g = 1$, there is no canonical projective embedding, and in fact the descent result doesn't hold! See Lecture Notes in Mathematics 179, Raynaud, Faisceaux ample.... The counterexample has a normal base of dimension at least 2. $\diamond$

9   Separated schemes: a warmup for algebraic spaces

The question for today is, "How can we characterize (separated) schemes among presheaves on Aff_{zar} (of Aff_{fppf})?"

Definition 9.1. Let C be a category, and we have two functors $F, G : \mathsf{C}^{op} \to \mathsf{Set}$. A morphism of functors $f : F \to G$ is called *relatively representable* if for every $X \in \mathsf{C}$, and for every $g : h_X \to G$, the fiber product $h_X \times_G F : \mathsf{C}^{op} \to \mathsf{Set}$ is representable.   ◇

Remark 9.2. $f : h_X \to G$ is the same as an element of $G(X)$.   ◇

In our case, take $C = \mathsf{Aff}$.

Definition 9.3. A morphism $f : F \to G$ is an *affine open (resp. closed) immersion* if (1) f is relatively representable, and (2) for all $X \in \mathsf{Aff}$ and $g : h_X \to G$, the map $F \times_G h_X \to h_X$ is an open (resp. closed) immersion.   ◇

Proposition 9.4. *A functor $F : \mathsf{Aff}^{op} \to \mathsf{Set}$ is representable by a separated scheme if and only if the following hold*

   1. *F is a sheaf with respect to the big Zariski topology,*

   2. *the diagonal morphism $\Delta : F \to F \times F$ is an affine closed immersion, and*

   3. *there exists a family of objects $\{X_i\}$ in Aff and morphisms $\pi_i : h_{X_i} \to F$ which are affine open immersions and such that the map of Zariski sheaves $\coprod_i h_{X_i} \to F$ is surjective. (note that this is not the same as a surjection of functors ... things only have to be locally in the image)*

Proof. ($\Longrightarrow$) Say X is separated, and $F = h_X$. (1) is clear. For (2), the important point is that the Yoneda embedding commutes with products, so $h_X \times h_X \cong h_{X \times X}$

$$
\begin{array}{ccc}
h_P & \longrightarrow & h_Y \\
\downarrow & & \downarrow \\
h_X & \longrightarrow & h_{X \times X}
\end{array}
\qquad\qquad
\begin{array}{ccc}
P & \overset{r}{\longrightarrow} & Y \\
\downarrow & & \downarrow \\
X & \overset{\Delta}{\longrightarrow} & X \times X
\end{array}
$$

(2) is equivalent to saying that r is a closed immersion for all affine Y. Since X is separated, Δ is a closed immersion, so r is a closed immersion.

   (3) Let $X = \bigcup X_i$ be an open covering by affines, so $h_{X_i} \to h_X$ are affine open immersions.[1] We wish to show local surjectivity of $\coprod h_{X_i} \to h_X$. That is, for every affine Y, and for every element of $g \in h_X(Y)$ (which we may think of as a morphism

---

[1]We use that X is separated here. Otherwise, $h_{X_i} \to h_X$ might not be relatively representable (e.g. the intersection of two affine open sets may not be affine).

$h_Y \to h_X$), there is a covering of Y so that g factors through $\coprod h_{X_i}$. It is clear that we may take the covering $Y = \bigcup g^{-1}(X_i)$.

$$
\begin{array}{ccc}
h_{g^{-1}(X_i)} & \xrightarrow{\ g_i\ } & h_{X_i} \\
\downarrow & & \downarrow \\
h_Y & \xrightarrow{\ g\ } & h_X
\end{array}
\qquad\qquad
\begin{array}{ccc}
\coprod h_{g^{-1}(X_i)} & \xrightarrow{\ \coprod g_i\ } & \coprod h_{X_i} \\
\downarrow & & \downarrow \\
h_Y & \xrightarrow{\ g\ } & h_X
\end{array}
$$

($\Longleftarrow$) For every i and j, we get

$$
\begin{array}{ccccc}
h_{V_{ij}} & = & h_{X_i} \times_F h_{X_j} & \longrightarrow & h_{X_i} \\
& & \downarrow & & \downarrow{\scriptstyle \pi_i} \\
& & h_{X_j} & \xrightarrow{\ \pi_j\ } & F
\end{array}
$$

Since $\pi_j : h_{X_j} \to F$ is relatively representable, the fiber product is represented by some affine V_{ij}. Since π_j is an affine open immersion, V_{ij} is an open subset of X_i. So we have $\{X_i\}$, and for every i and j, we have $V_{ij} \subseteq X_i$, and we have isomorphisms $\varphi_{ij} : V_{ij} \xrightarrow{\sim} V_{ji}$. We need to have the cocycle condition $\varphi'_{ij}\varphi'_{jk} = \varphi'_{ik}$, where $\varphi'_{ij} = \varphi_{ij}|_{V_{ij} \cap V_{ik}} : V_{ij} \cap V_{ik} \to V_{ji} \cap V_{jk}$. The cocycle condition (and the fact that φ'_{ij} is well-defined) is obvious if you look at the functor of points: $\varphi'_{ij} : h_{X_i} \times_F h_{X_j} \times_F h_{X_k} \to h_{X_j} \times_F h_{X_i} \times_F h_{X_k}$ just switches factors around.

Thus, we have gluing data to define a scheme X. Let $G = \coprod_i h_{X_i}$. We have that $\sigma : G \to F$ is surjective, and we wish to show that $F \cong h_X$.

We get $G \times_F G \subseteq G \times G$, so for $U \in \mathsf{Aff}$, $G(U) \times_{F(U)} G(U) \subseteq G(U) \times G(U)$. This subset is an equivalence relation (it identifies points in the same fiber of $G(U) \to F(U)$).

Both h_X and F are isomorphic to the sheafification of $U \mapsto G(U)/\sim$.

[[★★★ why is h_X isomorphic to that? some argument follows, but I'm not yet happy.]] h_X is the sheaf associated to the presheaf $U \mapsto G(U)/\sim$. This induces a map of presheaves $(U \mapsto G(U)/\sim) \to h_X$, and $G \to F$ factors through it. Suppose we have $s_1 : U \to X_i$ and $s_2 : U \to X_j$ (through V_{ij}) which define the same element of $F(U)$. Then we have maps

$$
\begin{array}{ccc}
U & \xrightarrow{\ s_1\ } & X_i \\
{\scriptstyle s_2}\downarrow & & \downarrow{\scriptstyle h_i} \\
X_j & \xrightarrow{\ h_j\ } & X
\end{array}
$$

$\square$

Example 9.5. [[★★★ this example needs work]] $\mathbb{P}^n : \operatorname{Spec} R \mapsto \{$quotients $R^{n+1} \to L | L$ a projective R-module of rank $1\}/\simeq$, where the isomorphisms are "under R" (if an isomorphism exists, it is unique). (1) This is clearly a sheaf in Aff_{zar}; just look at

the kernel. (2) $\Delta : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ is a closed immersion. To see this, let

$$
\begin{array}{ccc}
P & \longrightarrow & h_{\operatorname{Spec} R} \\
\downarrow & \ulcorner & \downarrow {\scriptstyle (L_1, L_2)} \\
\mathbb{P}^n & \longrightarrow & \mathbb{P}^n \times \mathbb{P}^n
\end{array}
$$

we want to show that P is affine closed in $\operatorname{Spec} R$. As a functor on $\mathsf{Aff}/\operatorname{Spec} R$, we have

$$
P : (\operatorname{Spec} R' \xrightarrow{f} \operatorname{Spec} R) \mapsto \begin{cases} * & (R')^{n+1} \xrightarrow{\beta_i} f^* L_i \text{ equal for } i = 1, 2 \text{ in } \mathbb{P}^n(R') \\ \varnothing & \text{else} \end{cases}
$$

Using the proposition: (i) we know that P is a sheaf in the Zariski topology. (ii) choose an open cover $\operatorname{Spec} R = \bigcup \operatorname{Spec} R_i$ such that both L_1 and L_2 restricted to R_i are trivial modules (so $L_1 \cong L_2 \cong R_i$).

If L_1 and L_2 are trivial and $\ker \beta_i$ are free, then we claim that P is representable.

$$
\begin{array}{ccc}
R^{n+1} & \xrightarrow{\beta_1} & R \\
& {\scriptstyle \beta_2} \searrow & \downarrow \\
& & R
\end{array}
$$

means exactly that $\ker \beta_1 = \ker \beta_2$. We want that $\ker \beta_2 \subseteq R^{n+1} \xrightarrow{\beta_1} R$ is the zero map. $\ker \beta_2 \simeq R^n$, with $e_i \mapsto f_i$.

$$
h_P \times_{h_{\operatorname{Spec} R}} h_{\operatorname{Spec} R_i \cap \operatorname{Spec} R_j} \qquad h_{\operatorname{Spec} R_i} \times_{h_{\operatorname{Spec} R}} h_{\operatorname{Spec} R_j} = h_{\operatorname{Spec} R_i \cap \operatorname{Spec} R_j}
$$

$$
\begin{array}{ccc}
h_{Z_i} = P_i & \longrightarrow & h_{\operatorname{Spec} R_i} \\
\downarrow & & \uparrow \\
P & \longrightarrow & h_{\operatorname{Spec} R}
\end{array}
$$

So Z_i in $\operatorname{Spec} R_i$ is a closed subscheme, and these glue to give you a closed subscheme $Z \subseteq \operatorname{Spec} R$. By the same argument as before, the functor of points of Z is P.

Finally, we need an open cover. $U_i \subseteq \mathbb{P}^n$ is given by $R \mapsto \{R^{n+1} \twoheadrightarrow L | e_i \mapsto \text{basis for } L\}$.

Claim. *The inclusion $U_i \hookrightarrow \mathbb{P}^n$ is representable by affine open immersions.*

This is clear because this is the complement of the zero locus of something.

Claim. $U_i \cong h_{\mathbb{A}^n}$.

$$
\begin{array}{ccc}
R^{n+1} & \longrightarrow & L \\
& \searrow^{e_i} & \downarrow^{\wr\ e_i} \\
& \nearrow_{1} & R
\end{array}
$$

Something equivalent to $f_0, \ldots, \hat{f}_i, \ldots, f_n \in R$.

$$K_1 \to \mathcal{O}_X^{n+1} \to L_2 \to 0$$

should be zero, then

$$L_2^{-1} \otimes K_1 \to O_X$$

is zero.

    (3) $\coprod U_i \to \mathbb{P}^n$ is surjective (as sheaves).

$$\coprod U_i(\operatorname{Spec} R) \longrightarrow \mathbb{P}^n(\operatorname{Spec} R) \quad \ni \quad (R^{n+1} \twoheadrightarrow L)$$

after possibly replacing $\operatorname{Spec} R$ by a covering, this is in the image if and only if some e_i maps to a basis for L. We can write $\operatorname{Spec} R = \bigcup_j \operatorname{Spec} R_j$ such that for each j some e_i maps to a basis for $L|_{\operatorname{Spec} R_j}$. $\diamond$

10   Properties of Sheaves and Morphisms

The goal of this lecture is to extend properties of objects (resp. morephisms) of a site to sheaves (resp. morphisms of sheaves) on that site. We will let C be a subcanonical site (i.e. a site in which representable presheaves are sheaves).

Definition 10.1. A class of objects $S \subseteq \mathsf{C}$ is *stable* if for every covering $\{U_i \to U\}$, $U \in S$ if and only if $U_i \in S$ for each i. We call a property $\mathcal{P}$ of objects *stable* if the class of objects satisfying $\mathcal{P}$ is stable.   ◇

Example 10.2. Stable properties in Aff with the Zariski topology: locally noetherian, reduced, normal, regular, ...   ◇

Definition 10.3. Let $F : \mathsf{Aff}^{op} \to \mathsf{Set}$ be a separated scheme, and let $\mathcal{P}$ be a stable property of affine schemes. Then we say that F has property $\mathcal{P}$ if there exists a covering $\{h_{X_i} \to F\}$ (i.e. $\coprod h_i \to F$ is surjective as a map of sheaves, with $h_{X_i} \to F$ affine open immersions) with X_i affine such that each X_i has property $\mathcal{P}$.   ◇

Remark 10.4. Exercise: Equivalently, we could require that for *every* affine covering $\{h_{X_i} \to F\}$, the X_i have $\mathcal{P}$.   ◇

Definition 10.5. A subcategory $\mathsf{D} \subseteq \mathsf{C}$ is *closed*[1] if

1. D contains all isomorphisms, and

2. for all cartesian diagrams as below, $f \in \mathsf{D}$ implies that $f' \in \mathsf{D}$.

$$
\begin{array}{ccc}
U & \longrightarrow & V \\
f' \downarrow & \ulcorner & \downarrow f \\
X & \longrightarrow & Y
\end{array}
$$

◇

Definition 10.6. A subcategory $\mathsf{D} \subseteq \mathsf{C}$ is *local on the base* if for all morphisms $f : X \to Y$ in C and all coverings $\{Y_i \to Y\} \in Cov(Y)$, $f \in \mathsf{D}$ if and only if all the maps $f_i : X \times_Y Y_i \to Y_i$ are in D.   ◇

Definition 10.7. A subcategory $\mathsf{D} \subseteq \mathsf{C}$ is *stable* if it is closed and local on the base.   ◇

Definition 10.8. A subcategory $\mathsf{D} \subseteq \mathsf{C}$ is *local on domain* if for all $f : X \to Y$ in C and all $\{X_i \xrightarrow{\phi_i} X\} \in Cov(X)$, $f \in \mathsf{D}$ if and only if $f \circ \phi_i \in \mathsf{D}$ for all i.   ◇

Definition 10.9. If $\mathcal{P}$ is a property of morphisms in C is satisfied by isomorphisms and closed under composition, then we say that $\mathcal{P}$ is *closed* (resp. *local on the base*, *stable*, *local on domain*) if the category $\mathsf{C}_{\mathcal{P}} \subseteq \mathsf{C}$ (all objects and morphisms are morphisms with $\mathcal{P}$) is closed (resp. local on the base, stable, local on domain).   ◇

---

[1][[★★★ AFAIK, this has nothing to do with the other kind of closed categories (the ones with internal hom functors).]]

Example 10.10. Let $\mathsf{C} = \mathsf{Sch}$ with the Zariski topology.

Some stable properties: proper, separated, surjective, quasi-compact, ...

Some stable and local on domain properties: locally of finite type, locally of finite presentation, flat, étale, universally open, locally quasi-finite, smooth, ...   ◇

Definition 10.11. A relatively representable morphism of sheaves $f \colon F \to G$ is said to have a closed property of morphisms $\mathcal{P}$ if for every $X \in \mathsf{C}$ and every morphism $X \to G$, the pullback $F \times_G X \to X$ has $\mathcal{P}$.   ◇

[[★★★ say something about Knutson's S3 properties (being a cover is local on domain) and talk about properties of non-representable morphisms]]

Definition 10.12. Let $\mathcal{P}$ be a stable property of maps in Aff, and let $f : F \to G$ be a relatively representable morphism of sheaves. We say *f has property* $\mathcal{P}$ if for every affine $X \in \mathsf{Aff}$ and every map $h_X \to G$, the corresponding map $h_X \times_G F \to h_X$ has property $\mathcal{P}$.   ◇

Remark 10.13. Note that the affine notion of properness is not the right one ... for global notions you have to do things differently.   ◇

Definition 10.14. Let $f : F \to G$ be a morphism of separated schemes, and $\mathcal{P}$ a stable and local on domain property of maps in Aff. We say *f has property* $\mathcal{P}$ if for every pair of affine coverings $\{G_i \to G\}$ and $\{F_{ij} \to F \times_G G_i\}$, the compositions $F_{ij} \to F \times_G G_i \to G_i$ have property $\mathcal{P}$. (we have to do this because we didn't assume f was relatively representable by affines) [[★★★ why do we need to assume F and G are separated schemes? That is, why do we need Δ_F and Δ_G to be affine closed immersions?. . . maybe we need $F \times_G G_i$ to be schemes so that they have open affine covers, so we need F and G to be schemes.]]

$$
\begin{array}{ccc}
F_{ij} \xrightarrow[\forall]{} F \times_G G_i \xrightarrow{\quad} G_i \\
\downarrow \qquad\qquad\qquad \downarrow{\scriptstyle\forall} \\
F \xrightarrow{\qquad\qquad} G
\end{array}
$$

◇

As we can see, in order to define what it means for a sheaf or morphism of sheaves to have a property, representability of certian morphisms is important. In particular, we'll see that representability of the diagonal morphism of a sheaf is important. This is because of the following lemma.

Lemma 10.15. *Assume* C *has products and fiber products. A sheaf F on C has representable diagonal if and only if all morphisms $X \to F$ from objects of C are representable (i.e. $X_1 \times_F X_2$ is an object of C for all objects X_i of C)*

Proof. First assume that $\Delta_F\colon F \to F \times F$ is representable, and let $f_i\colon X_i \to F$ be morphisms, $i = 1, 2$. Verify that the diagram on the left is cartesian. Since $X_1 \times X_2$ is an object in C and Δ_F is representable, $X_1 \times_F X_2$ is an object in C.

$$
\begin{array}{ccc}
X_1 \times_F X_2 & \longrightarrow & X_1 \times X_2 \\
\downarrow & & \downarrow {\scriptstyle f_1 \times f_2} \\
F & \xrightarrow{\ \Delta_F\ } & F \times F
\end{array}
\qquad\qquad
\begin{array}{ccc}
\bullet & \longrightarrow & T \\
\downarrow & & \downarrow {\scriptstyle \Delta_T} \\
T \times_F T & \longrightarrow & T \times T \\
\downarrow & & \downarrow {\scriptstyle (p_1 \circ f) \times (p_2 \circ f)} \\
& & {\scriptstyle = g_1 \times g_2} \\
F & \xrightarrow{\ \Delta_F\ } & F \times F
\end{array}
$$

Conversely, suppose $X_1 \times_F X_2$ is in C for any morphisms $f_i\colon X_i \to F$ from objects in C, and let $f\colon T \to F \times F$ be a morphism from an object in C. Composing with the two projections, this f induces two morphisms $g_i = p_i \circ f\colon T \to F$. Note that $f = (g_1 \times g_2) \circ \Delta_T$. From the diagram on the right, we see that the $T \times_{f,F \times F,\Delta} F \cong \bullet \cong T \times_{T \times T} (T \times_{g_1,F,g_2} T)$, which is an object of C by the hypothesis that $T \times_{g_1,F,g_2} T$ is an object in C and because C has products and fiber products. $\qquad\square$

Definition 10.16. A stable property $\mathcal{P}$ of morphisms is an *effective descent class* if the following property holds. For any morphism from a sheaf to an object $F \to X$ and any covering $\{X_i \to X\}$, if $F \times_X X_i$ are objects of C and $F \times_X X_i \to X_i$ have $\mathcal{P}$, then F is an object of C (and therefore $F \to X$ has $\mathcal{P}$). $\qquad\qquad\diamond$

[[★★★ by Example 7.16, closed immersions are an effective descent class in the fppf topology. Similarly, affine morphisms are an effective descent class. Therefore, so are open immersions, immersions, quasi-affine maps]]

[[★★★ define what it means for a construction to be local on the base. Any construction local on the base which is in an effective descent class is effective. In particular, the following are effective constructions (in the fppf topology, I think): affine morphisms, reduced subscheme structure on a set, scheme-theoretic closed image, open complement of a closed subscheme, reduced complement of an open subscheme, reduction of a scheme]]

[[★★★ Add (sub?)section on (pre-)relations. If $R \to U \times U$ belongs to any effective descent class, then the quotient sheaf F has representable diagonal. If $R \rightrightarrows U$ are covers, then $U \to F$ is a cover.]]

Lemma 10.17 ([EGA, IV.8.14.2]). *A morphism of schemes $f : X \to \operatorname{Spec} A$ is locally of finite presentation if and only if for every filtering inductive system of A-algebras $\{B_i\}$, the canonical map $\varinjlim h_X(\operatorname{Spec} B_i) \to h_X\big(\operatorname{Spec}(\varinjlim B_i)\big)$ is bijective.*

In the case where $X = \operatorname{Spec} R$, finite presentation means that we have a surjection $\pi : A[x_1, \ldots, x_r] \twoheadrightarrow R$, and $\ker \pi$ is finitely generated by $f_1, \ldots, f_s$. We can choose some B_i which contains the x_i. The f_i may not be zero, but they are in the limit, so we can find some B_j and a map $R \to B_j$ so that $R \to B$ factors through B_j.

Example 10.18. Let B be any A-algebra, and write $B = \bigcup B_i$, where B_i are finitely generated over A. Then $B = \varinjlim B_i$. The lemma says that $\varinjlim h_X(\operatorname{Spec} B_i) \xrightarrow{\sim} h_X(\operatorname{Spec} B)$.

$$\operatorname{Spec} B \longrightarrow X$$
$$\searrow \qquad \nearrow \exists$$
$$\operatorname{Spec} B_i$$

$\diamond$

Definition 10.19. Let $f : X \to Y$ be a morphism of schemes. We call f *formally smooth* (resp. *formally unramified, formally étale*) if for every affine Y-scheme $Y' \to Y$ and every closed immersion $Y'_0 \hookrightarrow Y'$ defined by a nilpotent ideal, the map $h_X(Y') \to h_X(Y'_0)$ is surjective (resp. injective, bijective). If f is also locally of finite presentation, then it is *smooth* (resp. *unramified, étale*). $\diamond$

Proposition 10.20. *A map of rings $A \to B$ is étale if and only if B is isomorphic to $A[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ with $n \leq m$ such that the unit ideal in B is generated by the $n \times n$ minors of the matrix $(\partial f_i/\partial x_j)$.*

[[★★★ This proposition says that the definition of étale from [Har77] agrees with this one. You prove it by looking at some infinitesimal neighborhood of the diagonal.]]

Remark 10.21. The class of étale maps of schemes is the smallest class of maps in Sch which (i) includes all étale maps of affine schemes, and (ii) is stable and local on domain in the Zariski topology, and (iii) if $\{X_i \xrightarrow{f_i} Y\}$ is a collection of morphisms, then the map $X = \coprod_i X_i \to Y$ is étale if and only if each f_i is étale. [[★★★ follows from local on domain]] $\diamond$

11   Algebraic spaces

We replace Aff by Sch, and the Zariski topology is replaced by the étale topology.

Definition 11.1. An *algebraic space* over S is a functor $X : (\mathsf{Sch}/S)^{op} \to \mathsf{Set}$ such that

1. X is a sheaf on the big étale topology on S,

2. $\Delta : X \to X \times_S X$ is representable, and

3. there exists an S-scheme $U \to S$ and a surjective étale morphism $U \to X$ (surjective as a map of sheaves).[1]

$\diamond$

Remark 11.2. In the definition of a separated scheme, representablity of Δ follows from existence of the covering $\{h_{X_i} \to F\}$. You still need the closedness of Δ.

Let $U = \coprod X_i$, and let $R = U \times_F U \subseteq U \times U$. If you require something something[[★★★]], then you get condition 2. $\diamond$

Remark 11.3. In [Knu71], Δ is assumed to be quasi-compact. If X is an algebraic space, we have an étale covering $U \to X$. Let $R = U \times_X U$, then R defines an equivalence relation on U, so we can take the sheafification of $\big(T \to U(T)\big)/\sim$, and we get X back. If you assume Δ is quasi-compact, then you can start with some equivalence relation and form an algebraic space this way. $\diamond$

If you replace étale with Zariski, you get schemes back out of the definition (i.e. you don't get anything new). If you use flat topology instead of étale, you don't get a new notion. [[★★★ somewhere, you should mention that algebraic spaces are fppf sheaves, talk about Artin's fppf slice theorem, and that you don't get a more general notion by considering "algebraic spaces in the fppf topology"]]

Example 11.4. (Not necessarily separated) schemes are algebraic spaces. $\diamond$

Quotients by free actions of finite groups

Let X be a separated scheme over S, and let G be a finite group acting freely on X over S. The action is an S-morphism $G \times X \to X \times_S X$, given by $(g, x) \mapsto \big(g(x), x\big)$.[2] Note that the morphism $X \cong g \times X \to X \times X$ is the diagonal morphism followed by an isomorphism (g acting on the first factor). Since X is separated, the diagonal map is a closed immersion. We can encode the statement that the action is free by saying that the map $G \times X \to X \times_S X$ is a closed immersion (i.e. it is a union of closed immersions that don't interfere with each other).

---

[1]Étale and surjective are stable properties, so this means that for every scheme V and every morphism $V \to X$, the fiber product $V \times_X U$ is representable (by a scheme), and the morphism of schemes $V \times_X U \to V$ is étale and surjective.

[2]By $G \times X$, we may the disjoint union of $|G|$ copies of X, labeled by elements of G.

Definition 11.5. $[X/G]$ is the sheafification of the presheaf on $(\mathsf{Sch}/S)_{et}$ given by $Z \mapsto X(Z)/G$. $\diamond$

Let $F = [X/G]$. The quotient map $X \to F$ is a *G-bundle* in the following sense. If Y is some S-scheme, then given a morphism $y : Y \to F$, we can form the pullback.

$$
\begin{array}{ccc}
P_y = X \times_F Y & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y & \underset{y}{\longrightarrow} & F
\end{array}
$$

The pull-back P_y has a G-action induced by the G-action on X. Moreover, $P_y(Z)$ is either empty or it has a simple transitive G-action. [[★★★ It seems like the G action should be free *over Y* (not over S unless Y is separated), but I don't think it should be transitive ... maybe it's transitive on geometric fibers or something]] To see that the action *over Y* is free, observe that the map

$$
\begin{array}{ccc}
G \times P_y & \longrightarrow & P_y \times_Y P_y \\
\| \wr & & \| \wr \\
G \times X \times_F Y \longrightarrow X \times_F Y \times_Y X \times_F Y & \cong & (X \times_S X) \times_F Y
\end{array}
$$

is the product of a closed immersion and the identity map, so it is a closed immersion. [[★★★ presumably this works even though the product is over F, which may not be a scheme]][[★★★ If we were looking at the action over S, it would be the product of a closed immersion and Δ_Y, which may not be a closed immersion.]]

It is an exercise in descent to show that P_y is actually represented by a scheme. [[★★★ I don't see where there is a descent argument]]

Definition 11.6. A *G-torsor* over Y is a scheme P [[★★★ it must be a scheme, yes?]] with a finite étale covering $P \to Y$ together with a G-action on P over Y such that G acts simply transitively on geometric fibers. This last condition is equivalent to the statement that $G \times P \to P \times_Y P$, $(g, p) \mapsto \big(g(p), p\big)$ is an isomorphism.[[★★★ why are these equivalent?]][3] A morphism of G-torsors is a G-equivariant morphism over Y. The *trivial G-torsor* is $G \times Y$. $\diamond$

Lemma 11.7. *If Y is a scheme, and $P \to Y$ is a G-torsor, then the natural map $[P/G] \to Y$ is an isomorphism.*

So far, we have shown that a morphism $y : Y \to [G/X] = F$ produces a G-torsor $P_y \to Y$ and a G-equivariant morphism $P_y \to X$. Conversely, given a G-torsor $P \to Y$ and a G-equivariant morphism $P \to X$, we get an induced morphism $Y = [P/G] \to [X/G] = F$. These constructions are inverse, so $[X/G](Y)$ is in bijection with isomorphism classes of G-torsors over Y with G-equivariant maps to X.

---

[3][[★★★ If we weren't working in the étale topology, then could we allow G to not be discrete and get a more general definition?]]

By the way, given a G-torsor P over Y and a G-equivariant map to X, you get a closed immersion $P \hookrightarrow Y \times_S X$. This is a closed immersion because the action of G on X is free. [[★★★ I don't see why this is a closed immersion]] [[★★★ *This* is where we use freeness of the G-action on X; it is important in the proof of the upcoming lemma. Is this also where we use separatedness of X?]] Two such pairs (torsor, morphism to X) are isomorphic exactly when they produce the same closed subscheme of $Y \times_S X$.

Remark 11.8. There exist non-quasi-projective schemes and free actions such that the quotient is not representable by a scheme. However, the following theorem tells us that the quotient *is* always an algebraic space.   ◇

Lemma 11.9. $F = [X/G]$ *is an algebraic space.* [[★★★ *later, we'll prove that if we have finite stabilizers, then you get an algebraic space*]]

Proof. (1) F is defined as a sheafification, so it is a sheaf.

(2) We wish to show that the diagonal is representable. Given a map $g : Y \to F \times_S F$, we form the pull-back.

$$
\begin{array}{ccc}
Z & \longrightarrow & Y \\
\downarrow & {\ulcorner} & \downarrow g \\
F & \xrightarrow{\Delta} & F \times_S F
\end{array}
\qquad
Z(T \xrightarrow{f} Y) = \begin{cases} * & f^*P_1 = f^*P_2 \subseteq T \times X \\ \varnothing & \text{otherwise} \end{cases}
$$

We will show that Z is represented by a closed subscheme of Y.[4] Note that g is the same as a choice of two G-torsors over Y, call them P_1 and P_2. We get associated closed immersions $j_i : P_i \hookrightarrow Y \times_S X$. Choose an étale cover $Y' \to Y$ so that P_1 pulls back to the trivial G-torsor (for example, take $Y' = P_1$). Then we have a section $s : Y' \to P_1'$.

$$
\begin{array}{ccc}
Z' & \longrightarrow & Y' \\
\downarrow & {\ulcorner} & \downarrow \\
P_2' & \longhookrightarrow & Y' \times X
\end{array}
\qquad
\begin{array}{c}
y \\ \downarrow \\ j_1'\big(s(y)\big)
\end{array}
$$

Since $Z' \to Y'$ is a base extension of a closed immersion, it is a closed immersion, so Z' is associated to some quasi-coherent sheaf of ideals $\mathcal{I}_{Z'}$ on Y'. For some reason [[★★★]], $\mathcal{I}_{Z'}$ comes with descent data, so by descent for quasi-coherent sheaves of ideals,[5] it is the pull-back of some quasi-coherent sheaf of ideals $\mathcal{I}_Z$ on Y. For some reason, this is the sheaf of ideals associated to Z [[★★★ why should Z' have anything to do with Z?]]. Thus, Z is a closed subscheme of Y; in particular, it is a scheme.

(3) Observe that $X \to F$ is étale surjective. This is just the fact that for every $y : Y \to F$, P_y is a scheme and $P_y \to Y$ is étale and surjective (noting also that an étale surjection of schemes translates into a surjection of sheaves).   □

Remark 11.10. If you drop the assumption that X is separated, Z is still represented by a scheme, but the descent is trickier.   ◇

---

[4] We are actually showing that Δ_F is a closed immersion, so F will be a *separated* algebraic space.

[5] Note that the étale topology is coarser than the fppf topology. i.e. every étale cover is an fppf cover, so the descent theorem applies.

A quotient by a non-free action

Let $\mathbb{P} = \mathbb{P}^3_{\mathbb{C}}$, with coordinates Y_0, Y_1, Y_2, and Y_3. [[★★★ where do we use that we are working over $\mathbb{C}$?]] Consider the conics

$$C_0 : Y_0 Y_1 + Y_1 Y_2 + Y_2 Y_0 = Y_3 = 0$$
$$C_1 : Y_0 Y_1 + Y_1 Y_3 + Y_3 y_0 = Y_2 = 0$$

The points of intersection $C_0 \cap C_1$ are $P_0 = [1, 0, 0, 0]$ and $P_1 = [0, 1, 0, 0]$. The involution $\sigma : (Y_0, Y_1, Y_2, Y_3) \mapsto (Y_1, Y_0, Y_3, Y_2)$ switches the conics and the two intersection points.

Let X_i be the scheme given by blowing up C_i and the blowing up C_{1-i}. Over $\mathbb{P} \smallsetminus \{P_0, P_1\}$, C_0 and C_1 don't intersect, so it doesn't matter in which order you blow up. Thus, we get open subsets $U_i \subseteq X_i$, with $U_0 \cong U_1$. Obtain Z by gluing X_0 and X_1 along $U_0 \cong U_1$. Observe that the action of σ lifts to Z. One can check that Z is not quasi-projective.

Define $[Z/\sigma]$ as the sheafification of the presheaf on $(\mathsf{Sch}/\operatorname{Spec}\mathbb{C})_{et}$ given by $Y \mapsto Z(Y)/\sigma$. One can check that this sheaf is is not a scheme, but is an algebraic space. One can also check that $[Z/\sigma]$ can be obtained in the following way. [[★★★ As far as I can tell, we don't check these things.]]

The fixed points of σ lie over points of the form $[0, 0, 1, -1]$, $[0, 0, 1, 1]$, $[1, 1, \alpha, \alpha]$, or $[1, -1, \alpha, -\alpha]$. We have an open cover $Z = Z_1 \cup Z_2$, where Z_1 is obtained by deleting points lying over C_1 and C_2, and let Z_2 is the open subset where σ acts freely. One can show that Z_1 is a quasi-projective scheme. By some theorem from invariant theory, the quotient $[Z_1/\sigma]$ is a scheme. One checks that $[Z_2/\sigma]$ is an algebraic space. Then $[Z/\sigma]$ can be obtained by gluing $[Z_1/\sigma]$ to $[Z_2/\sigma]$ along $[Z_1 \cap Z_2/\sigma]$. [[★★★ Note that gluing algebraic spaces along isomorphic open sub-algebraic spaces produces an algebraic space: the sheaf condition is trivial, the representability probably follows from gluing schemes, and the disjoint union of the two étale covers forms an étale cover.]]

Quotients by relations

Let U and R be schemes, and let $R \to U \times U$ be a morphism such that the image of $R(Y) \to U(Y) \times U(Y)$ is an equivalence relation for every scheme Y. Then define U/R as the sheafification of the presheaf on Sch_{et} given by $Y \mapsto U(Y)/\sim_R (Y)$. This will sometimes be an algebraic space, as we will see in the next lecture (Proposition 12.11).

Example 11.11. Let $U = \operatorname{Spec} k[x, y]/(xy)$, the union of the axes in the affine plane. There is an involution $\sigma : U \to U$ induced by swapping x and y. Let R be the closed subscheme of $U \times U$ which is the union of the diagonal and the graph of σ. We'll show later (in the proof of Theorem 13.3) that U/R is $\operatorname{Spec}\big(k[x, y]/(xy)\big)^{\sigma} = \operatorname{Spec} k[x + y]$, which is the affine line $\mathbb{A}^1_k$.   ◇

Example 11.12. Let U and σ be as in the previous example. Let $R = U \sqcup (U \smallsetminus \{0\})$, and consider $R \xrightarrow{\Delta \sqcup \Gamma(\sigma)} U \times U$, where $\Gamma(\sigma)$ is the graph of σ (minus the point at

the origin). Then U/R is an algebraic space, and $U \to [U/R]$ is an étale cover by Proposition 12.11. Note that the k points of $[U/R]$ are the same as those of $\mathbb{A}^1_R$, but U/R cannot be $\mathbb{A}^1_k$ because there is no étale cover $U \to \mathbb{A}^1_k$. In particular, note that the dimension of the tangent space of U/R at the origin is 2. $\diamond$

12 Properties of Algebraic Spaces. Étale Relations.

Today we'll make precise some definitions of things we saw in the previous lecture. We're always working over some base scheme S.

Definition 12.1. Let $\mathcal{P}$ be a property of schemes which is stable in the étale topology. Then an algebraic space X *has property* $\mathcal{P}$ if there exists an étale surjection $U \to X$ where U is a scheme with property $\mathcal{P}$.   ◇

For example, we can talk about algebraic spaces being locally noetherian, reduced, regular, purely n-dimensional, normal, or pretty much any property of schemes which is not a global property.

Definition 12.2. Let $\mathcal{P}$ be a property of morphisms of schemes which is stable in the étale topology, and let $f : X \to Y$ be a representable morphism of algebraic spaces. Then f *has* $\mathcal{P}$ if there is an étale cover $V \to Y$ such that $V \times_Y X \to V$ has property $\mathcal{P}$.   ◇

For example, we can talk about representable morphisms of algebraic spaces being closed immersions, open immersions, proper, etc.

Definition 12.3. Let $\mathcal{P}$ be a property of morphisms of schemes which is stable and local on domain in the étale topology, and let $f : X \to Y$ be a morphism of algebraic spaces. Then f *has* $\mathcal{P}$ if there exist étale covers $v : V \to Y$ and $u : U \to X$ such that the projection $U \times_Y V \to V$ has $\mathcal{P}$ (note that $U \times_Y V$ is a scheme by Lemma 10.15).

$$
\begin{array}{ccc}
U \times_Y V & \xrightarrow{\ \ \mathcal{P}\ \ } & V \\
\downarrow & & \downarrow{\scriptstyle v} \\
U & \xrightarrow{\ u\ } X \xrightarrow{\ f\ } & Y
\end{array}
$$

◇

For example, we can talk about morphisms of algebraic spaces being étale, flat, smooth, surjective, etc.

Remark 12.4. If $f : X \to Y$ is representable and $\mathcal{P}$ is stable and local on domain, then Definitions 12.2 and 12.3 will produce the same notion. To see this, take $U = X \times_Y V$ and apply the following remark.   ◇

Remark 12.5. In the above three definitions, we always say that there exists some étale cover(s) so that something happens. In fact, this is equivalent to the (apparently

stronger) statement that for *every* morphism(s), that thing happens.
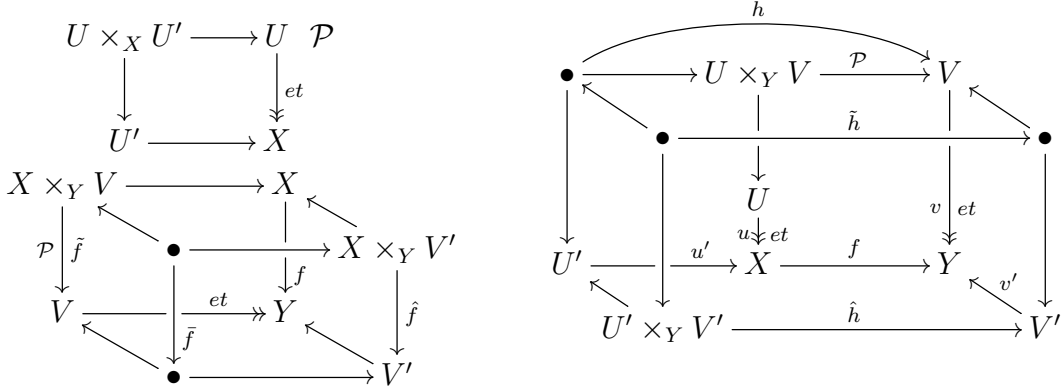
In the top left diagram, U has $\mathcal{P}$ (a stable property of schemes). Note that $U \times_X U' \to U'$ is an étale surjection. By stability of $\mathcal{P}$ (the "stable under base extension" part of stability), $U \times_X U'$ has $\mathcal{P}$. Again by stability (the "descends along covers" part), U' has $\mathcal{P}$.

In the bottom left diagram $\tilde{f}$ is $\mathcal{P}$ (a stable property of morphisms of schemes). Every square in sight is cartesian (it isn't important exactly what the bullets are), all the horizontal maps are étale and surjective. Since $\mathcal{P}$ is stable under pull-backs, $\bar{f}$ is $\mathcal{P}$. Since $\mathcal{P}$ descends along covers, $\hat{f}$ is also $\mathcal{P}$.

In the right diagram, $\mathcal{P}$ is stable and local on domain. All the squares are cartesian (never mind what the bullets are), all the long vertical maps are étale and surjective. Since $\mathcal{P}$ is local on domain, h is $\mathcal{P}$. Since $\mathcal{P}$ is stable under pull-backs, $\tilde{h}$ is $\mathcal{P}$. Since $\mathcal{P}$ descends along covers, $\hat{h}$ is $\mathcal{P}$.   ◇

Corollary 12.6 (to Remark 12.5). *If $\mathcal{P}$ is a stable property of objects (resp. stable property of morphisms, resp. stable local on domain property of morphisms) in the étale topology on the category of schemes, then $\mathcal{P}$ is a stable property of objects (resp. stable property of morphisms, resp. stable local on domain property of morphisms) in the étale topology in the category of algebraic spaces.*

Proof. This is a straightforward exercise given the remark.   □

Remark 12.7. In fact, we can make sense of an algebraic space or a morphism of algebraic spaces having property $\mathcal{P}$ even if $\mathcal{P}$ is not stable. It is enough for $\mathcal{P}$ to descend along étale covers. In that case, $\mathcal{P}$ will descend along étale covers of algebraic spaces as well.

This definition allows us to talk about an algebraic space being quasi-compact or a morphism of algebraic spaces being dominant.   ◇

Proposition 12.8. *The subcategory $\mathsf{AlgSp}/S \subseteq (\mathsf{Sch}/S)_{et}$ is closed under finite projective limits.*

Proof. By Lemma 3.9, it is enough to check that products and fiber products are representable.

$$\begin{array}{cc}
\text{(Products)} & \text{(Fiber Products)} \\
\begin{array}{ccc} P & \longrightarrow & X_1 \\ \downarrow & & \\ X_2 & & \end{array} &
\begin{array}{ccc} P & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_3 \end{array}
\end{array}$$

Products: Fix étale covers $U_1 \to X_1$ and $U_2 \to X_2$. (1) Since products of sheaves are sheaves, P is a sheaf.

(2) We must check representability of the diagonal. Let Z be a scheme.

$$\begin{array}{ccc}
(Z \times_{X_1 \times X_1} X_1) \times_Z (Z \times_{X_2 \times X_2} X_2) & \longrightarrow & Z \\
\downarrow & & \downarrow \\
(X_1 \times X_2) \longrightarrow (X_1 \times X_2) \times (X_1 \times X_2) & \cong & (X_1 \times X_1) \times (X_2 \times X_2)
\end{array}$$

Since $Z \times_{X_1 \times X_1} X_1$ and $Z \times_{X_2 \times X_2} X_2$ are schemes (since Δ_{X_1} and Δ_{X_2} are representable), so is the product.

(3) $U_1 \times U_2 \to X_1 \times X_2$ is an étale surjection.

Fiber Products: (1) Since the product of sheaves over another sheaf is a sheaf, P is a sheaf. (2) We must check that the diagonal is representable. Let Z be a scheme, then we wish to show that W in the diagram below is a scheme.

$$\begin{array}{ccc}
W & \longrightarrow & Z \\
\downarrow & & \downarrow \\
(X_1 \times_{X_3} X_2) \xrightarrow{\Delta} (X_1 \times_{X_3} X_2) \times (X_1 \times_{X_3} X_2) & & \\
\downarrow & & \downarrow \\
(X_1 \times X_2) \xrightarrow{\Delta} (X_1 \times X_2) \times (X_1 \times X_2) & &
\end{array}$$

Verify that the bottom square is cartesian; then the outer square is cartesian. By the case of products, $\Delta_{X_1 \times X_2}$ is representable, so W is a scheme.

(3) We have that $U_1 \times_{X_3} U_2 \to X_1 \times_{X_3} X_2$ is an étale cover, and $U_1 \times_{X_3} U_2$ is a scheme by Lemma 10.15.   □

Definition 12.9. If U and R are S-schemes, we say that a morphism $R \to U \times_S U$ is an *equivalence pre-relation* if for every S-scheme T, the image of $R(T)$ in $U(T) \times U(T)$ is an equivalence relation.[1] We say that an equivalence pre-relation *has a property $\mathcal{P}$* of morphisms if the two projection maps $R \to U \times U \rightrightarrows U$ have property $\mathcal{P}$. Furthermore, we say that an equivalence pre-relation is an *equivalence relation* if for every S-scheme T, $R(T) \to U(T) \times U(T)$ is injective. We say that "$R \rightrightarrows U$ is a (pre-)equivalence relation (with property $\mathcal{P}$)".   ◇

---

[1][[★★★ Is this acceptable terminology?]]

Let U/R denote the sheafification of the presheaf on $(\mathsf{Sch}/S)_{et}$ given by $T \mapsto U(T)/\sim_{R(T)}$.

Remark 12.10. Every algebraic space is isomorphic to U/R for some scheme U and some étale equivalence relation $R \rightrightarrows U$. To see this, let $U \to X$ be some étale cover, and consider $R = U \times_X U$. This is a scheme by Lemma 10.15. Then the projections are étale because $U \to X$ is étale (the projections are obtained by base extension). One can check that $X \cong U/R$. [[★★★ check this]]   ◇

Proposition 12.11. *Let $R \rightrightarrows U$ be an étale equivalence relation, and assume that $R \to U \times U$ is either quasi-compact or an immersion (closed immersion followed by an open immersion). Then the quotient $X = U/R$ is an algebraic space.*

Proof. (1) As a sheafification, X is a sheaf.

(2) We need to check representability of Δ_X. Let Z be a scheme and let $z : Z \to X \times X$ be a morphism, and let $P = Z \times_{X \times X} X$. We wish to show that P is a scheme.

Base Case: Assume z factors through $U \times U$. The bottom square in the left diagram is cartesian (basically by definition of X), and the outer square is cartesian (by assumption), so the top square is cartesian. Thus, P is a product of schemes over a scheme, so it is a scheme.
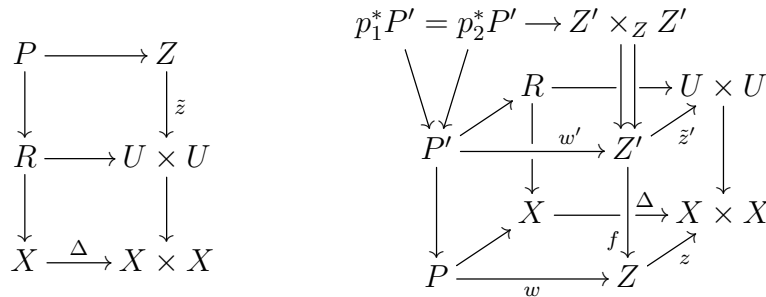
General Case: We have that $U \to X$ is a surjection of étale sheaves, so $U \times U \to X \times X$ is a surjection of étale sheaves. This means that for any element $z \in (X \times X)(Z)$, there is some scheme Z' with an étale cover $f : Z' \to Z$ so that the image $\tilde{z} \in (X \times X)(Z')$ is in the image of $(U \times U)(Z')$. In the diagram on the right, all squares are cartesian except the "left" and the "right" squares. By the base case, P' is a scheme.

Note that any stable property of $R \to U \times U$ is shared by w'. Note also that $w' : P' \to Z'$ comes with descent data relative to the étale surjection $Z' \to Z$.

(w' closed immersion) P' is given by a quasi-coherent sheaf of ideals on Z' which descends to Z. Thus, P is a closed subscheme of Z, so it is a scheme.

(w' open immersion) Since $P' \subseteq Z'$ is open and $f : Z' \to Z$ is étale (and so open), $f(P') \subseteq Z$ is open. To show that $P = f(P')$, it is enough to show that $P'' = f^{-1}\big(f(P')\big)$. Let $p' \in P$ and $y \in Z'$ have the same image in Z, then we want to show that $y \in P''$. But we have that $\big(f(y), (p', y)\big) \in p_2^* P' = P \times_Z (Z' \times_Z Z')$, so $p_2(p', y) = y \in P''$. [[★★★ how can this be made clearer?]]

(w' immersion) [[★★★ I don't know... somehow P is a scheme]]

(w' quasi-compact) [[★★★ I don't know... somehow P is a scheme]]

(3) We claim that $U \to X$ is an étale surjection. By Lemma 10.15, $U \to X$ is representable. To verify that it is an étale surjection, we need to take a morphism from a scheme $T \to X$ and check that $T \times_X U \to T$ is an étale surjection. Since $U \to X$ is a surjection of étale sheaves, there is some étale cover $T' \to T$ so that $T' \to X$ factors through U.

$$
\begin{array}{ccc}
\bullet & \longrightarrow & T \times_X U \\
& R \longrightarrow U & \\
et & T' \xrightarrow{\ et\ } T & \\
U & \longrightarrow X &
\end{array}
$$

In the diagram above, all squares are cartesian except the "top" and "bottom" squares. Since $R \to U$ is étale surjective (surjective because R contains the diagonal), and étale surjective is stable, we get that $T \times_X U \to T$ is étale surjective, as desired.   □

13 Affine/(Finite Étale Relation) = Affine, Part I

For the next two lectures, we will fix the following setup. Let U be an affine scheme, and let $R \rightrightarrows U$ be a finite étale equivalence pre-relation.[1] In particular, since R is finite over an affine scheme, it is affine. Let $X = U/R$ be the quotient sheaf (X is an algebraic space by Proposition 12.11). We define $R' = R \underset{U}{{}_{p_2}\times_{p_1}} R$, noting that R' is affine as well.

Warning 13.1. If the morphism $R \to U \times U$ is not a monomorphism (i.e. if R is only a *pre*-relation), then it is not true that $R \cong U \times_X U$. However, there is still a canonical morphism $r : R \to U \times_X U$. Let $\sigma : U \times_X U \to U \times_X U$ be the morphism which switches the two factors. We will assume that the pre-relation R comes with a morphism $\bar{\sigma} : R \to R$ making the diagram on the left cartesian. In general, such a map need not exist, but in most cases we care about, it is given to us. Note that pre-composing with $\bar{\sigma}$ swaps the two projections $p_1, p_2 : R \to U$.

$$
\begin{array}{ccc}
R & \xrightarrow{\ \bar{\sigma}\ } & R \\
\ \downarrow{\scriptstyle r} & {\scriptstyle \ulcorner} & \downarrow{\scriptstyle r} \\
U \times_X U & \xrightarrow[\ \sim\ ]{\ \sigma\ } & U \times_X U
\end{array}
$$

$$
\begin{array}{ccccc}
R' & \xrightarrow{\qquad\ \ `p_{23}'\qquad\ \ } & & & R \\
{\scriptstyle `p_{12}'}\downarrow & \searrow & & & \downarrow{\scriptstyle r} \\
 & U \times_X U \times_X U & \xrightarrow{\ p_{23}\ } & U \times_X U & \\
 & p_{12}\downarrow \quad {\scriptstyle \ulcorner} & & \downarrow{\scriptstyle p_1} & \\
R & \xrightarrow{\ \ r\ \ } & U \times_X U & \xrightarrow{\ \ p_2\ \ } & U
\end{array}
$$
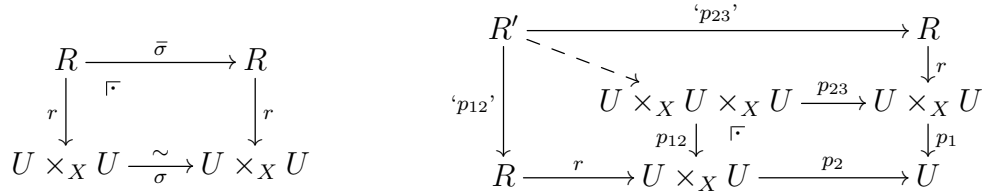
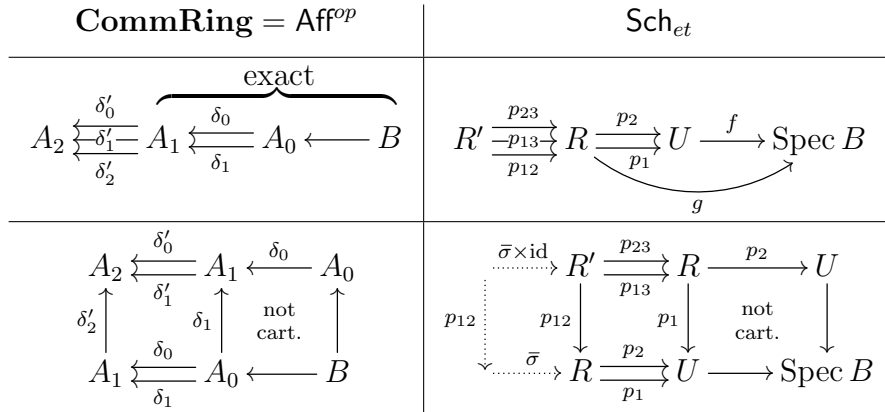Furthermore, we can define $p_{12}, p_{23} : R' \to R$ as shown in the diagram on the right. Then we define $p_{13} = p_{23} \circ (\bar{\sigma} \times \mathrm{id}) : R \times_U R = R' \to R$. [[★★★ Something is fishy here, $\bar{\sigma} \times \mathrm{id} : R \times_{p_2, U, p_1} R \to R \times_{p_1, U, p_1} R$]]

Let $U = \operatorname{Spec} A_0$, $R = \operatorname{Spec} A_1$, and $R' = \operatorname{Spec} A_2$. Let $\delta_i : A_1 \to A_0$ (resp. $\delta_i' : A_2 \to A_1$) be the ring morphism corresponding to "projecting out the $(i+1)$-th component". Define the equalizer $B = Eq(\, A_0 \underset{\delta_1}{\overset{\delta_0}{\rightrightarrows}} A_1 \,)$. Below we have our picture in the category of commutative rings on the left and the corresponding picture in the category of étale sheaves on Sch on the right.

CommRing $=$ Aff^{op}	Sch$_{et}$
exact	
$A_2 \underset{\delta_2'}{\overset{\delta_0'}{\underset{\rightleftarrows}{\overset{\delta_1'}{\rightleftarrows}}}} A_1 \underset{\delta_1}{\overset{\delta_0}{\leftleftarrows}} A_0 \longleftarrow B$	$R' \underset{p_{12}}{\overset{p_{23}}{\underset{\rightrightarrows}{\overset{p_{13}}{\rightrightarrows}}}} R \underset{p_1}{\overset{p_2}{\rightrightarrows}} U \xrightarrow{\ f\ } \operatorname{Spec} B$; g
$\begin{array}{ccc} A_2 & \overset{\delta_0'}{\underset{\delta_1'}{\leftleftarrows}} & A_1 \overset{\delta_0}{\leftarrow} A_0 \\ {\scriptstyle \delta_2'}\uparrow & {\scriptstyle \delta_1}\uparrow & {\scriptstyle\text{not cart.}}\uparrow \\ A_1 & \overset{\delta_0}{\underset{\delta_1}{\leftleftarrows}} & A_0 \longleftarrow B \end{array}$	$\begin{array}{ccc} \dashrightarrow R' & \overset{p_{23}}{\underset{p_{13}}{\rightrightarrows}} R \xrightarrow{p_2} U \\ {\scriptstyle p_{12}} \quad {\scriptstyle p_{12}}\downarrow & {\scriptstyle p_1}\downarrow \quad {\scriptstyle\text{not cart.}}\downarrow \\ \dashrightarrow R & \overset{p_2}{\underset{p_1}{\rightrightarrows}} U \longrightarrow \operatorname{Spec} B \end{array}$; $\bar{\sigma}\times\mathrm{id}$; $\bar{\sigma}$

---

[1]Actually, we really care about the case where R is a relation. The pre-relation approach is developed because it is interesting that Lemma 13.8 holds in that case. See Example 14.1.

In the bottom right-hand diagram, we check that the top row is exact (by definition of R' and p_{13})[[★★★ not clear at all when R is a pre-relation]], and that the two squares on the left (obtained by removing p_{13} and p_1, or p_{23} and p_2) are cartesian: the "top" square is cartesian by definition of R', and the "bottom" square can be obtained from the top square sticking on the cartesian square of dotted arrows in the diagram. Since Aff is a full subcategory of Sch_{et}, it follows that the top row is exact in the diagram on the left, and the two left-most squares are co-cartesian. Moreover, by the definition of B, the bottom row of the left-hand diagram is exact.

> **Warning 13.2.** Sch_{et} is a strictly larger than Aff, so exactness of the bottom row of the diagram on the left *does not* imply exactness of the bottom row in the diagram on the right. If that bottom row were exact, then we would have that $\operatorname{Spec} B$ is isomorphic to the quotient sheaf $X = U/R$. The main theorem of this lecture says that this happens if R is a relation.   ⌐

Theorem 13.3. *If U is affine, and $R \rightrightarrows U$ is a finite étale equivalence relation, then the algebraic space $X = U/R$ is isomorphic to the affine scheme $\operatorname{Spec} B$.*

Proof. By Lemmas 13.9 and 13.5, A_0 is flat and integral over B. By the going up theorem, it follows that $\operatorname{Spec} A_0 \to \operatorname{Spec} B$ is surjective, so A_0 is faithfully flat over B.

$$R = \operatorname{Spec} A_1 \xrightarrow{\ \sim\ } \operatorname{Spec} A_0 \otimes_B A_0 \xrightarrow{\ et\ } \operatorname{Spec} A_0 = U$$

$$\operatorname{Spec} A_0 \longrightarrow \operatorname{Spec} B$$

(with vertical maps, the right one labeled "f. flat")

By Lemma 13.11, $\delta_0 \otimes \delta_1 : A_0 \otimes_B A_0 \to A_1$ is an isomorphism. Since the projection maps $R \to U$ are étale, we have that $\operatorname{Spec} A_0 \otimes_B A_0 \to \operatorname{Spec} A_0$ is étale. By what [Vis05] claims is [EGA, IV.2.7.1] [[★★★ but isn't quite, as far as I can tell]], étaleness descends along faithfully flat base extension. Thus, we now know that $\operatorname{Spec} A_0 \to \operatorname{Spec} B$ is an étale surjection.

By Lemma 13.8, the sequence $\operatorname{Spec} A_1 \rightrightarrows \operatorname{Spec} A_0 \to \operatorname{Spec} B$ is exact as a sequence of schemes. We'd like to show that it is exact as a sequence of étale sheaves. Let $\mathcal{F}$ be an étale sheaf, and let $f : \operatorname{Spec} A_0 \to \mathcal{F}$ be a morphism of étale sheaves which coequalizes p_1 and p_2. Then we would like to show that f factors uniquely through $\operatorname{Spec} B$.

$$\operatorname{Spec} A_1 \underset{p_1}{\overset{p_2}{\rightrightarrows}} \operatorname{Spec} A_0 \longrightarrow \operatorname{Spec} B \qquad \mathcal{F}(\operatorname{Spec} A_1) \underset{\mathcal{F}p_1}{\overset{\mathcal{F}p_2}{\leftleftarrows}} \mathcal{F}(\operatorname{Spec} A_0) \longleftarrow \mathcal{F}(\operatorname{Spec} B)$$

(with f down to $\mathcal{F}$, and "∃ unique?")

By Yoneda's lemma, we may think of f as an element of $\mathcal{F}(\operatorname{Spec} A_0)$. Since $\operatorname{Spec} A_0 \to \operatorname{Spec} B$ is an étale cover and $\operatorname{Spec} A_1 \cong \operatorname{Spec} A_0 \otimes_B A_0 = \operatorname{Spec} A_0 \times_{\operatorname{Spec} B} \operatorname{Spec} A_0$, the sequence on the right is exact by the sheaf axiom! This says exactly that f factors uniquely through $\operatorname{Spec} B$. Thus, $R \rightrightarrows U \to \operatorname{Spec} B$ is exact as a sequence of étale sheaves, so $B \cong U/R$.   □

Lemma 13.4. *Let $R \rightrightarrows U$ be a finite étale equivalence pre-relation. Then the ranks of the two projections are equal and locally constant.*

Proof. [[★★★ I don't see what's happening here. "The rank of an étale map is locally constant, so it is constant on connected components of U"?]]

Let $W^{(n)} \subseteq U$ be the largest open where $p_1 : R \to U$ has rank n. Let $R' = U \times_X U \times_X U = (U \times_X U) \times_U (U \times_X U) = R_{p_2} \times_U {}_{p_1} R$.

$$
\begin{array}{ccc}
R' & \xrightarrow{p_{23}} & R \\
{\scriptstyle p_{12}}\downarrow & {\scriptstyle \ulcorner} & \downarrow{\scriptstyle p_1} \\
R & \xrightarrow{p_2} & U
\end{array}
\qquad\qquad
\begin{array}{ccc}
R' & \xrightarrow{p_{13}} & R \\
{\scriptstyle p_{12}}\downarrow & {\scriptstyle \ulcorner} & \downarrow{\scriptstyle p_1} \\
R & \xrightarrow{p_1} & U
\end{array}
$$

$p_2^{-1}(W^{(n)})$ is the locus where p_{12} has rank n (by the diagram on the left). By the diagram on the right, this is also $p_1^{-1}(W^{(n)})$. This shows that $X = \coprod W^{(n)}/R_{W^{(n)}}$.   □

Lemma 13.5. *Let A_0 and B be as in the setup. Then A_0 is integral over B.*

Proof. [[★★★ go through this]] Let $a \in A_0$. Let $P_{\delta_1}(T, \delta_0(a)) = T^n - \sigma_1 T^{n-1} + \cdots + (-1)^n \sigma_n$ be the characteristic polynomial of $\times \delta_0(a) : A_1 \to A_1$, where A_1 is viewed as an A_0-module via δ_1.

$\delta_0(P_{\delta_1}(T, \delta_0(a))) = P_{\delta_2'}(T, \delta_0' \delta_0(a))$ by the bottom square being cartesian. Also, we have $\delta_1(P_{\delta_1}(T, \delta_0(a))) = P_{\delta_2'}(T, \delta_1' \delta_0(a))$. Since the top row is an equalizer, these two things are actually equal. Thus, $\delta_0(\sigma_i) = \delta_1(\sigma_i)$ for every i. Thus, $\sigma_i \in B$. On the other hand, by the Cayley-Hamilton Theorem tells us that $\delta_0(a)^n - \delta_1(\sigma_1)\delta_0(a)^{n-1} + \cdots + (-1)^n \delta_1(\sigma_n) = 0$ in A_1. Since the σ_i live in B, we get $\delta_0\big(a^n - \sigma_1 a^{n-1} + \cdots + \sigma_n\big) = 0$. since δ_0 is étale surjective, it is faithfully flat, so it is injective by Lemma 6.5, so we get that $a^n - \sigma_1 a^{n-1} + \cdots + \sigma_n = 0$, so a is integral, as desired.   □

Lemma 13.6. *Let all notation be as in the setup of this lecture. Let $x, y \in U$ be points with the same image in $\operatorname{Spec} B$. Then there exists a $z \in R$ with $p_2(z) = x$ and $p_1(z) = y$. That is $\operatorname{Spec} B$ is set-theoretically the quotient $\operatorname{Spec} A_0/R$.*

Proof. [[★★★ go over this]] Suppose not. Well, x and y are prime ideals in A_0. Then we know that x is distinct from $\delta_0^{-1}(t)$ for every $t \in A_1$ with $\delta_1^{-1}(t) = y$. For such a prime t, $\delta_0^{-1}(t) \cap B = \delta_1^{-1}(t) \cap B = y \cap B = x \cap B$ because B is the equalizer.

Since we have an integral morphism of rings, we can apply Cohen-Seidenberg (going up), which implies that x is not contained in $\delta_0^{-1}(t)$ for every prime $t \subseteq A_1$ with $\delta_1^{-1}(t) = y$. By prime avoidance, there is an $a \in x$ such that a is not in any $\delta_0^{-1}(t)$ (there are finitely many t over y because the morphisms $A_0 \to A_1$ are finite). Then $\times \delta_0(a) : A_1 \to A_1$, and we get $N_{\delta_1}(\delta_0(a)) = \sigma_n$. Then $\delta_0(a)$ is not contained in any of the t's, so $N_{\delta_1}(\delta_0(a)) \notin y$. On the other hand,

$$
\begin{array}{ccc}
A_1 & \xrightarrow{\Delta} & A_1 \\
{\scriptstyle \times \delta_0(a)}\downarrow & & \downarrow{\scriptstyle \times a} \\
A_1 & \xrightarrow{\Delta} & A_0
\end{array}
$$

$U \hookrightarrow R \subseteq U \times U$ contains the diagonal. The diagram commutes as a diagram of A_0-modules (A_1 via δ_1). This implies that $\sigma_n \in B \cap x$.

So $\sigma_n \in B \cap x$ and $\sigma_n \notin B \cap y$, which is a contradiction because $B \cap y = B \cap x$.

$$\square_{\text{Lemma}}$$

Corollary 13.7. $\operatorname{Spec} B$ *is* topologically *the quotient* $\operatorname{Spec} A_0 / R$.

Proof. By Lemma 13.5, A_0 is integral over B. It follows from the going up theorem that $f : \operatorname{Spec} A_0 \to \operatorname{Spec} B$ is a closed surjective map.[2] Therefore, the topology on $\operatorname{Spec} B$ is induced from the topology on $\operatorname{Spec} A_0$. Together with Lemma 13.6, we get the result. $\qquad\square$

Lemma 13.8. $\operatorname{Spec} B$ *is* scheme theoretically *the quotient* $\operatorname{Spec} A_0 / R$. *That is, for any morphism* $\rho : \operatorname{Spec} A_0 \to T$ *(where T is a scheme) such that $\rho \circ p_1 = \rho \circ p_2$, there exists a unique morphism of schemes* $\bar{\rho} : \operatorname{Spec} B \to T$ *such that $\rho = \bar{\rho} \circ f$.*

Proof. Let $g = f \circ p_1 = f \circ p_2$. By the Corollary 13.7, we get a unique continuous map $\bar{\rho} : \operatorname{Spec} B \to T$. In order to make it into a morphism of schemes, we need a map

$$\bar{\rho}^{-1}\mathcal{O}_T \dashrightarrow^{\exists} \mathcal{O}_{\operatorname{Spec} B}$$
$$\searrow \qquad \downarrow$$
$$f_*\mathcal{O}_{\operatorname{Spec} A_0}$$
$$\rightrightarrows$$
$$g_*\mathcal{O}_{\operatorname{Spec} A_1}$$

The map exists and is unique because $B \to A_0 \rightrightarrows A_1$ is exact.[[★★★ explain]] Finally, check that it is local, which we can check on $\operatorname{Spec} A_0$.[[★★★ explain]] $\qquad\square$

Lemma 13.9. *Let A_0 and B be as in the setup. Then A_0 is flat over B.*

Proof. We can check flatness locally on B. Thus, we can assume B is local. By what [Vis05] claims is [EGA, IV.2.7.1] [[★★★]], it is enough to check flatness after making a flat (and therefore faithfully flat, since B is local) base change.

$$
\begin{array}{ccc}
\operatorname{Spec} A_1' \rightrightarrows \operatorname{Spec} A_0' \longrightarrow \operatorname{Spec} B' \\
\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow {\scriptstyle [[\bigstar\bigstar\bigstar \text{ faithfully?}]]\ \text{flat}} \\
\operatorname{Spec} A_1 \rightrightarrows \operatorname{Spec} A_0 \longrightarrow \operatorname{Spec} B
\end{array}
$$

---

[2] Let $\phi : B \to A_0$ be the ring map corresponding to f. Going up directly tells us that $\operatorname{Spec} A_0 \to \operatorname{Spec} B$ is surjective. Let $V(I) \subseteq \operatorname{Spec} A_0$ is a closed set (the set of all primes containing some ideal I), then observe that A_0/I is integral over $B/\phi^{-1}(I)$. Applying going up, we see that every prime containing $\phi^{-1}(I)$ is in the image of $V(I)$. On the other hand, the pullback of any prime containing I contains $\phi^{-1}(I)$. Thus, $f\big(V(I)\big) = V\big(\phi^{-1}(I)\big)$, which is closed.

Thus, we can assume B is local with infinite residue field [[★★★ if the residue field is $k(p)$, take the separable closure and this extends to an extension of the ring ... strict hensilization?]]. This implies that A is semi-local. [[★★★ Lemma 13.6 implies that A_0 is finite over B because A_1 is finite over A_0 and multiple things over a point in B differ by things in A_1]] □

Lemma 13.10. *Let B be a local ring with infinite residue field, $i : B \to A$ a homomorphism of semi-local rings sending $\mathfrak{m}_B$ to the Jacobson radical[3] of A. Let M be a free A-module of rank n, and $N \subseteq M$ a B-submodule which generates M as an A-module. Then N contains a basis for M as an A-module.*

Proof. By Nakayama, replace B by $B/\mathfrak{m}$, A by $A/\mathfrak{r}$, M by $M/\mathfrak{r}M$, and N by $N/(N \cap \mathfrak{r}M)$. Now the result is an exercise (it's a Chinese remainder argument). [[★★★ do this proof]] □

Lemma 13.11. *Let B, A_0, and A_1 be as in the setup. Then $\delta_0 \otimes \delta_1 : A_0 \otimes_B A_0 \to A_1$ is an isomorphism.*

Proof. [[★★★ do this proof]] Apply Lemma 13.10 to $B = B$, $A = A_1$, $M = A_1$ (viewed as an A_0-module via δ_1), and $N = \delta_0(A_0)$. We need to check that N generates M as an A_0-module, which is the same as checking that $A_0 \otimes_B A_0 \xrightarrow{\delta_0 \otimes \delta_1} A_1$ is surjective. This is true because we have an equivalence relation. We have that $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A_0 \otimes_B A_0$

$$
\begin{array}{ccc}
A_0 \otimes_B A_0 & \xrightarrow{\delta_0 \otimes \delta_1} & A_1 \\
{\scriptstyle a \mapsto a \otimes 1}\Big\uparrow & \nearrow{\scriptstyle \delta_0} & \\
A_0 & &
\end{array}
$$

$\operatorname{Spec} A_0 \to \operatorname{Spec} A_1$ is proper because it is finite ... we get that $\operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A_0 \otimes_B A_0$ is a closed immersion.

Choose $a_1, \ldots, a_n \in A_0$ so that $\delta_0(a_i)$ are a basis for A_1 (as a module over A_0 via δ_1), with $\varepsilon : \mathbb{Z}^n \to A_0$. $i : B \hookrightarrow A_0$.

$$
\begin{array}{ccccc}
A_2 & \underset{\delta_1'}{\overset{\delta_0'}{\rightleftarrows}} & A_1 & \xleftarrow{\ \delta_0\ } & A_0 \\
{\scriptstyle \wr}\Big\uparrow{\scriptstyle u_2} & & {\scriptstyle \wr}\Big\uparrow{\scriptstyle u_1} & & {\scriptstyle \wr}\Big\uparrow{\scriptstyle u_0} \\
A_1 \otimes_{\mathbb{Z}} \mathbb{Z}^n & \underset{\delta_0 \otimes 1}{\overset{\delta_1 \otimes 1}{\rightleftarrows}} & A_0 \otimes_{\mathbb{Z}} \mathbb{Z}^n & \xleftarrow{\ i \otimes 1\ } & B \otimes_{\mathbb{Z}} \mathbb{Z}^n \\
& & \Big\uparrow & & \Big\uparrow \\
& & A_0 & \xleftarrow{\qquad} & B
\end{array}
$$

The u_i are what you think they are: $u_0 = i \otimes \varepsilon$, $u_1 = \delta_1 \otimes \delta_0 \varepsilon$, $u_2 = \delta_2' \otimes \delta_0' \delta_0 \varepsilon$. By construction, u_1 is an isomorphism. We threw in a $\mathbb{Z}^n$ in the bottom, which doesn't

---

[3]The intersection of all maximal ideals.

affect the fact that the squares are co-cartesian. Thus, u_2 is an isomorphism. Both rows are exact, so it follows that u_0 is also an isomorphism. [[★★★ whoa]] $A_0 \otimes_B A_0 \to A_1$ is a surjective map of free modules over A_0 of the same rank, so it is an isomorphism. The bottom square is co-cartesian.   □

14   Affine/(Finite Étale Relation) = Affine, Part II

Example 14.1. If $U = \operatorname{Spec} A$ and G is a finite group which acts freely on on the right of U (we need the action to be free to get a relation rather than a pre-relation). Then $R_G = U \times G \to U \times U$ is given by $(u, g) \mapsto \big(g(u), u\big)$. The right action of G on U induces a left action of G on A: for $g \in G$, the isomorphism $- \cdot g^{-1} : U \xrightarrow{\sim} U$ corresponds to some isomorphism $g \cdot - : A \xleftarrow{\sim} A$. The two projections $R \to U$ correspond to the two morphisms $A \to \prod_{g \in G} A$ given by $a \mapsto (ga)_{g \in G}$ [[★★★ or maybe $(g^{-1}a)_{g \in G}$]] and $a \mapsto (a)_{g \in G}$. The equalizer (which we called B) is the ring A^G of invariants of A under the G-action. By the theorem, $\operatorname{Spec} A^G$ is the sheaf quotient of U by the G-action.

Even if the action of G on U is not free, $R_G = U \times G \to U \times U$ is a pre-relation, so Lemma [[★★★ exact as schemes]] tells us that $\operatorname{Spec} A^G$ is the *categorical quotient* of U by G. That is, $\operatorname{Spec} A^G$ satisfies the right universal property in the category of schemes (that any G-invariant morphism from $U = \operatorname{Spec} A$ factors through $\operatorname{Spec} A^G$). The interesting thing in the case of a free action is that the morphism $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is étale.   ◇

Corollary 14.2. *Let X be an algebraic space with quasi-compact diagonal morphism. Then there exists a scheme V and a dense open immersion $j : V \hookrightarrow X$ (we make sense of this even though "dominant" is not stable; for any étale morphism from a scheme $Z \to X$, $Z \times_X V \to Z$ must be a dominant open immersion).*

Proof. Choose an étale cover $U \to X$ and set $R = U \times_X U$, so $X = U/R$. If U_i are quasi-compact open subschemes which cover U, then we define $R_i = R \times_{U \times U} (U_i \times U_i)$ and $X_i = U_i/R_i$. Check that $X_i \to X$ are open immersions.[1] If we could find dominant open immersions of schemes $V_i \to X_i$, then they would glue to give a dominant open immersion $V \to X$. Thus, we have reduced to the case where X is quasi-compact (quasi-compact is not stable; we just mean that it is étale covered by something quasi-compact).

We may assume U is quasi-compact. Choose a dense open affine subset $W \subseteq U$ (note that you can always do this [[★★★]]). Then we can get the obvious restriction $R_W = R \times_{U \times U} (W \times W)$. Since Δ_X is quasi-compact and $U \times U \to X \times X$ is étale [[★★★ is Δ_X q-compact enough?]], $R \to U \times U$ is quasi-compact. Since U is quasi-compact, the projections $U \times U \to U$ are quasi-compact. Thus, the projections $R \to U$ are quasi-compact as well as étale, so $R_W \to W$ are quasi-compact and étale. A quasi-compact étale map is quasi-finite,[2] and there is some kind of semi-continuity result [[★★★]] which tells us that there is a dense open subset $W' \subseteq W$ where $R_{W'} \to W$ are finite. We may choose W' to be affine, so then $R_{W'}$ will be affine since it is finite

---

[1]Check that $X_i(T) = \{f \in X(T) | T \times_X U \to T \text{ surjective}\}$. In general, $T \times_X U \to T$ is étale, so its image is open in T; this open subset is $T \times_X X_i$. This shows that $X_i \to X$ is an open immersion.

[2]Quasi-finite means that geometric fibers are finite. Quasi-compact and étale imply finite because the only étale extensions of an algebraically closed field Ω are disjoints unions of copies of Ω; quasi-compactness implies that they must be finite disjoint unions.

over affine. By the theorem, $W'/R_{W'}$ is an affine scheme. Now we have a dominant open immersion $W'/R_{W'} \to X$.[[★★★]]   □

Example 14.3. Consider $\mathbb{C}[x,y]$ with an action of $\mathbb{Z}/2$, given by $x, y \mapsto -x, -y$. If you take invariants, the invariant ring is generated by $x^2 = z, y^2 = w, xy = u$, so it is $\mathbb{C}[z,w,u]/(zw = u^2)$. So we get a map $U = \operatorname{Spec} \mathbb{C}[x,y] \to \operatorname{Spec} \mathbb{C}[z,w,u]/(zw = u^2)$ [[★★★ which is not étale]], so that for any morphism $\operatorname{Spec} \mathbb{C}[x,y] \to T$ which is $\mathbb{Z}/2$-invariant, it factors uniquely.

Here $\mathbb{Z}/2 \times U \to U \times U$ is not a monomorphism. Something is still étale, but not an equivalence relation. What we've constructed here is the quotient in the category of ringed spaces.   ◇

Remark 14.4. For non-free group actions, the quotient construction does not commute with base change. In the free case, we have

$$
\begin{array}{ccc}
R & & \\
\downdownarrows & & \\
U & & \\
\downarrow & & \\
U/R & \longleftarrow & U'/R' \\
\downarrow & & \downarrow \\
S & \longleftarrow & S'
\end{array}
$$

If your group has order prime to the residue characteristic, then it still commutes with base change because invariants is an exact functor and the representation category is semi-simple.   ◇

15 Quasi-coherent Sheaves on Algebraic Spaces

I cheated you a while back about descent for quasi-compact mono-morphisms, so I made it an exercise. Part (c) of that exercise should refer to this class. It is étale descent for separated quasi-finite morphisms of finite presentation.

The remaining goals for algebraic spaces:

– an algebraic space which is quasi-finite over a scheme is a scheme.

– Chow's lemma for algebraic spaces.

First we need some theory about quasi-coherent sheaves.

Definition 15.1 (Small étale site on an algebraic space)**.** For an algebraic space X, we define $Et(X)$ to be the site whose objects are étale morphisms $X' \to X$, where X' is a scheme, whose morphisms are X-morphisms, and where a covering of $X' \to X$ is a collection $\{X'_i \to X'\}$ such that $\coprod X'_i \to X'$ is surjective.

We denote the topos associated to this site by X_{et}, and we define the sheaf $\mathcal{O}_X(X' \to X) = \Gamma(X', \mathcal{O}_{X'})$ to be the *structure sheaf* on the site.   ⋄

Remark 15.2. Consider the site $\widetilde{Et}(X)$, whose objects are étale morphisms $X' \to X$, where X' is just some algebraic space, whose morphisms are X-morphisms, and where a covering of $X' \to X$ is a collection $\{X'_i \to X'\}$ such that $\coprod X'_i \to X'$ is surjective. Note that $Et(X)$ is a full subsite of $\widetilde{Et}(X)$. Since every algebraic space can be étale covered by a scheme, a sheaf on $\widetilde{Et}(X)$ is determined by its values on $Et(X)$ (in particular, we can extend the structure sheaf to all of $\widetilde{Et}(X)$). Thus, the two sites yield the same topos, X_{et}.   ⋄

Remark 15.3. Any morphism in $Et(X)$ is étale. Let X' and X'' be schemes étale over X, and let $U \to X$ be an étale surjection from a scheme. Let U' and U'' be obtained by base extension. Then U' and U'' are étale schemes over a scheme U, so any morphism $U'' \to U'$ over U must be étale. Since étale-ness is stable in the étale topology, $X'' \to X'$ is étale.

⋄
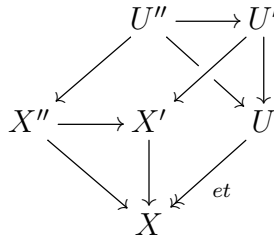
Definition 15.4. An $\mathcal{O}_X$-module $\mathcal{F}$ is *quasi-coherent* if

1. for every $X' \to X$ étale, with X' a scheme, the sheaf $\mathcal{F}_{X'}$ (obtained by restricting to the small Zariski site on X') is quasi-coherent, and

2. for every X-morphism $f : X'' \to X'$ of schemes étale over X, the natural map $f^\star \mathcal{F}_{X'} \to \mathcal{F}_{X''}$ is an isomorphism.

A quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is said to be *coherent* if $\mathcal{F}_{X'}$ is coherent for each X'.   ◇

Remark 15.5. The kernel, cokernel, and image of any morphism of quasi-coherent sheaves on X is quasi-coherent. Any extension of quasi-coherent sheaves is quasi-coherent. If X is locally noetherian, then the same is true for coherent sheaves. [[★★★ check this]]   ◇

Lemma 15.6. *If X is a scheme, then the category* $\mathsf{Qcoh}(X_{et})$ *of quasi-coherent* $\mathcal{O}_{X_{et}}$-*modules is equivalent to the category* $\mathsf{Qcoh}(X_{zar})$ *of quasi-coherent* $\mathcal{O}_{X_{zar}}$-*modules.*

Proof. Given $G \in \mathsf{Qcoh}(X_{zar})$, we get $G_{fppf} \in \mathsf{Qcoh}(X_{fppf})$ as in Proposition 7.12. Note that every quasi-coherent $\mathcal{O}_{X_{fppf}}$-module is a quasi-coherent $\mathcal{O}_{X_{et}}$-module (we're just restricting to a smaller category and a coarser topology), so we get that $G_{fppf} \in \mathsf{Qcoh}(X_{et})$ (when we restrict to the étale topos, we'll call the sheaf G_{et}). Conversely, given $\mathcal{G} \in \mathsf{Qcoh}(X_{et})$, we have that $\mathcal{G}_X \in \mathsf{Qcoh}(X_{zar})$.

For a quasi-coherent Zariski sheaf G, Proposition 7.12 tells us that $G \cong (G_{et})_X$. For a quasi-coherent étale sheaf $\mathcal{G}$, one verifies that $\mathcal{G} \cong (\mathcal{G}_X)_{et}$: given an étale map $f : Y \to X$, $(\mathcal{G}_X)_{et}(Y \to X) = \Gamma(Y, f^\star \mathcal{G}_X) \cong \Gamma(Y, \mathcal{G}_Y) = \mathcal{G}(Y \to X)$, where the isomorphism in the middle follows from the fact that $f^\star \mathcal{G}_X \cong \mathcal{G}_Y$, which follows from quasi-coherence of $\mathcal{G}$.   □

Lemma 15.7 ("Quasi-coherence is étale-local")**.** *Let X be an algebraic space and let $U \twoheadrightarrow X$ be an étale cover by a scheme U. Then a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is quasi-coherent if and only if the restriction to the small étale site on U, $\mathcal{F}|_U$, is quasi-coherent.*

Proof. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module, then $\mathcal{F}|_U$ is quasi-coherent because for any étale morphism $Y \to U$, $(\mathcal{F}|_U)_Y = \mathcal{F}_Y$.

If $\mathcal{F}|_U$ is quasi-coherent and $X' \to X$ is étale, with X' a scheme, then $\mathcal{F}|_{X' \times_X U}$ is a quasi-coherent sheaf which is the restriction of $\mathcal{F}|_{X'}$ along the projection map. Since $X' \times_X U \to X'$ is an étale surjection, and because of Lemma 15.6, Theorem 7.13 tells us that $\mathcal{F}|_{X'}$ is a quasi-coherent sheaf. In particular, $\mathcal{F}_{X'} = (\mathcal{F}|_{X'})_{X'}$ is a quasi-coherent sheaf, and for any X-morphisms of schemes $f : X'' \to X'$ (which is necessarily étale by Remark 15.3) $f^\star \mathcal{F}_{X'} \to \mathcal{F}_{X''}$ is an isomorphism.   □

Pullbacks of quasi-coherent sheaves

Recall the discussion from Lecture 7 in the section about descent for sheaves of modules. A morphism of algebraic spaces $f : X \to Y$ induces a morphism of topoi $(f_*, f^*) : X_{et} \to Y_{et}$ and an adjoint pair $(f_*, f^\star) : \mathcal{O}_X$-$\mathsf{mod} \to \mathcal{O}_Y$-$\mathsf{mod}$. If f is étale, then f^* and $f^\star$ are given by restricting to the small étale site on X (Remark 7.7).

Lemma 15.8. *Let $f : X \to Y$ be a morphism of algebraic spaces. If $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module, then $f^\star \mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module.*

Proof. By Lemma 15.7 we may replace X by an étale cover. Thus, we may assume X is a scheme. Moreover, if $g : Y' \to Y$ is an étale cover of Y by a scheme, then we have

$$
\begin{array}{ccc}
X \times_Y Y' & \xrightarrow{\tilde{f}} & Y' \\
{\scriptstyle \tilde{g}}\downarrow & & {\scriptstyle et}\downarrow{\scriptstyle g} \\
X & \xrightarrow{f} & Y
\end{array}
$$

Since g is étale, $g^\star \mathcal{F}$ is just the restriction of $\mathcal{F}$ to the small étale site on Y', so it is quasi-coherent. If we could prove that $\tilde{f}^\star$ of a quasi-coherent sheaf is quasi-coherent, then we would have that $\tilde{f}^\star g^\star \mathcal{F} \cong \tilde{g}^\star f^\star \mathcal{F}$ is quasi-coherent. Since $\tilde{g}$ is an étale cover, Lemma 15.7 tells us that $\tilde{g}^\star f^\star F$ is quasi-coherent if and only if $f^\star \mathcal{F}$ is quasi-coherent. Thus, we've reduced to the case where X and Y are schemes.

$$
\begin{array}{ccc}
Et(X) & \longleftarrow & Et(Y) \\
\updownarrow & & \updownarrow \\
Op(X) & \longleftarrow & Op(Y)
\end{array}
\qquad
\begin{array}{ccc}
X_{et} & \overset{f_{et}^*}{\underset{f_{et*}}{\rightleftarrows}} & Y_{et} \\
{\scriptstyle \varepsilon_X^*}\upuparrows{\scriptstyle \varepsilon_{X*}} & & {\scriptstyle \varepsilon_{Y*}}\upuparrows{\scriptstyle \varepsilon_Y^*} \\
X_{zar} & \overset{f_{zar*}}{\underset{f_{zar}^*}{\rightleftarrows}} & Y_{zar}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{O}_{X_{et}}\text{-mod} & \overset{f_{et}^\star}{\underset{f_{et*}}{\rightleftarrows}} & \mathcal{O}_{Y_{et}}\text{-mod} \\
{\scriptstyle \varepsilon_X^\star}\upuparrows{\scriptstyle \varepsilon_{X*}} & & {\scriptstyle \varepsilon_{Y*}}\upuparrows{\scriptstyle \varepsilon_Y^\star} \\
\mathcal{O}_{X_{zar}}\text{-mod} & \overset{f_{zar*}}{\underset{f_{zar}^\star}{\rightleftarrows}} & \mathcal{O}_{Y_{zar}}\text{-mod}
\end{array}
$$

The diagram on the left is an obviously commutative diagram of continuous[1] functors of sites (all of which have finite projective limits). These functors induce the commutative diagram of morphisms of topoi in the middle (the inner and outer squares commute). This gives us the inner commutative square on the right, and the outer commutative square is obtained by looking at the left adjoints.

So we have $\varepsilon_X^\star \circ f_{zar}^\star \cong f_{et}^\star \circ \varepsilon_Y^\star$. Note that the construction $F \mapsto F_{et}$ is exactly given by $F \mapsto \varepsilon^* F$. By the equivalence of categories in Lemma 15.6, it follows that an étale sheaf $\mathcal{F}$ on Y is quasi-coherent if and only if $\mathcal{F}_Y$ is quasi-coherent and $\mathcal{F} \cong \varepsilon_Y^\star \mathcal{F}_Y$. Thus, we have $f_{et}^\star \mathcal{F} \cong f_{et}^\star \varepsilon_Y^\star \mathcal{F}_Y \cong \varepsilon_X^\star (f_{zar}^\star \mathcal{F}_Y)$, which is quasi-coherent because we're taking the usual scheme pullback followed by $\varepsilon_X^\star$, which preserves quasi-coherence. $\square$

Remark 15.9. The last part of the proof gives us a recipe for how to compute $f^\star \mathcal{F}$. Let $V \to Y$ and $U \to X$ be étale, and assume we have a factorization

$$
\begin{array}{ccc}
U & \xrightarrow{\tilde{f}} & V \\
{\scriptstyle et}\downarrow & & \downarrow{\scriptstyle et} \\
X & \xrightarrow{f} & Y
\end{array}
$$

The proof shows us that $f_{et}^\star \mathcal{F} \cong \varepsilon_Y^\star (f_{zar}^\star \mathcal{F}_Y)$. It follows that $(f^\star \mathcal{F})_U = \tilde{f}_{zar}^\star (\mathcal{F}_V)$. [[★★★ somehow]]

You can always get a factorization like this (for some V) after replacing U by an étale cover; then something about quasi-coherent sheaves behaving nicely with respect to étale covers gives you $(f^\star \mathcal{F})_U$. [[★★★]]   ◇

---

[1]It is clear that these functors take coverings to coverings and commute with fiber products.

Proposition 15.10. *Let $f : X \to Y$ be a quasi-compact and quasi-separated[2] morphism of algebraic spaces. Then for every quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, the sheaf of $\mathcal{O}_Y$-modules $f_*\mathcal{F}$ is quasi-coherent.*

Proof. It is étale local on Y, so we may assume Y is an affine scheme. Cover X with some scheme V so that $V \to Y$ is quasi-compact, then choose an affine étale cover U of V. Then let $R = U \times_X U$; this is also quasi-compact because we have the following diagram, and we know that Δ is quasi-compact (so $R \to U \times_Y U$ also is) and $U \times_Y U$ is affine.

$$
\begin{array}{ccc}
R & \longrightarrow & U \times_Y U \\
\downarrow{\scriptstyle\ulcorner} & & \downarrow \\
X & \xrightarrow{\Delta} & X \times_Y X
\end{array}
\qquad\qquad
R \rightrightarrows U \xrightarrow{\ \ } X \underset{g}{\overset{f}{\rightrightarrows}} Y \quad \overset{h}{\frown}
$$

Let $g : U \to Y$ and $h : R \to Y$ [[★★★ for some reason, these are both quasi-compact and quasi-separated ... it is easy for g, but what about h?]]. Then $f_*\mathcal{F} = Eq\big(g_*(\mathcal{F}|_U) \rightrightarrows h_*(\mathcal{F}|_R)\big)$ because $U \twoheadrightarrow X$ is a covering (this is the sheaf axiom when you uncoil it. This is similar to Display 7.6). Thus, it is enough to show that $g_*(\mathcal{F}|_U)$ and $h_*(\mathcal{F}|_R)$ are quasi-coherent because an equalizer of quasi-coherent sheaves is quasi-coherent. So it is enough to consider the case where both X and Y are schemes. In that case, the result follows from the usual result for schemes, and the following claim, which says roughly that f_{et*} agrees with f_{zar*} for quasi-coherent sheaves.

    <u>Claim</u>: if $\mathcal{F}$ is a quasi-coherent sheaf on X, then $f_{et*}\mathcal{F} \cong \varepsilon_Y^\star(f_{zar*}\mathcal{F}_X)$. To see this, recall that $\varepsilon_Y^\star$ and ε_{Y*} are inverses when restricted to the subcategories of *quasi-coherent* sheaves, with $\varepsilon_{X*}\mathcal{F} = \mathcal{F}_X$ and $\varepsilon_X^\star F = F_{et}$. Then we compute that $f_{et*}\mathcal{F} \cong \varepsilon_Y^\star \varepsilon_{Y*} f_{et*}\mathcal{F} \cong \varepsilon_Y^\star(f_{zar*}\varepsilon_{X*}\mathcal{F}) = \varepsilon_Y^\star(f_{zar*}\mathcal{F}_X)$. $\qquad\square$

---

[2]This means that the diagonal map over Y, $X \to X \times_Y X$ is quasi-compact.

16   Relative Spec

Definition 16.1. Let X be an algebraic space (over some scheme S), and let $\mathcal{A}$ be a quasi-coherent sheaf of algebras on X. Then we define $\mathcal{S}pec_X \mathcal{A} : (\mathsf{Sch}/S)^{op} \to \mathsf{Set}$ by $T \mapsto \{(f, \iota) | f \in X(T), \iota \in \mathrm{Hom}_{\mathcal{O}_T\text{-alg}}(f^\star \mathcal{A}, \mathcal{O}_T)\}$. Since $f^\star \mathcal{A}$ and $\mathcal{O}_T$ are quasi-coherent, it doesn't matter if we think of them as Zariski sheaves or étale sheaves when we talk about ι. Note that we have a "projection" $\mathcal{S}pec_X \mathcal{A} \to X$ given on the level of functors of points by forgetting ι. $\diamond$

The projection $\mathcal{S}pec_X \mathcal{A} \to X$ is an affine morphism, and every affine morphism is canonically of this form. [[★★★ I think this should be true]]

Proposition 16.2. $\mathcal{S}pec_X \mathcal{A}$ *is an algebraic space.*

Proof. (1) $\mathcal{S}pec_X \mathcal{A}$ is an étale sheaf because X is an étale sheaf and morphisms like ι can be constructed locally in the étale topology (this is descent for quasi-coherent sheaves of algebras in the étale topology).

(2) Let's check that the diagonal is representable. Let T be a scheme, and let $(h, \varepsilon) \times (h', \varepsilon')$ be a morphism from T to $\mathcal{S}pec_X \mathcal{A} \times \mathcal{S}pec_X \mathcal{A}$. In the diagram below, we define P', T', and P so that all squares are cartesian, and we choose the morphism $\mathcal{S}pec_X \mathcal{A} \to P'$ so that $\mathcal{S}pec_X \mathcal{A} \to X$ is the canonical projection and $\mathcal{S}pec_X \mathcal{A} \to \mathcal{S}pec_X \mathcal{A} \times \mathcal{S}pec_X \mathcal{A}$ is the diagonal. We wish to show that P is a scheme. Note that T' is a scheme because Δ_X is representable. We will show that P is a closed subscheme of T'.

$$
\begin{array}{ccccc}
W & & & & \\
& \searrow^{r} \quad \searrow^{g} & & & \\
& P \xrightarrow{\quad} T' \xrightarrow{\ s\ } T & & \\
(f,\iota)\downarrow & \downarrow \qquad \downarrow \qquad \downarrow (h,\varepsilon)\times(h',\varepsilon') & & \\
\mathcal{S}pec_X(\mathcal{A}) \longrightarrow P' \longrightarrow \mathcal{S}pec_X \mathcal{A} \times \mathcal{S}pec_X \mathcal{A} & & \\
\downarrow \qquad \downarrow & & \\
X \xrightarrow{\ \Delta\ } X \times X & &
\end{array}
$$

As functors, we have that

$$
\begin{aligned}
P' : & W \mapsto \{(f, \iota_1, \iota_2) | f \in X(W), \iota_i : f^\star \mathcal{A} \to \mathcal{O}_W\} & (f_1 = f_2 = f) \\
T' : & W \mapsto \{g \in T(W) | hg = h'g\} & (f = hg = h'g, \iota_1 = g^\star \varepsilon, \iota_2 = g^\star \varepsilon') \\
P : & W \mapsto \{r \in T'(W) | r^\star s^\star \varepsilon = r^\star s^\star \varepsilon'\} & (g = sr, f = hg = h'g, \iota = g^\star \varepsilon = g^\star \varepsilon')
\end{aligned}
$$

[[★★★ do these need more explanation?]]

Let $t = hs = h's : T' \to X$. The coequalizer of the two maps $t^\star \mathcal{A} \underset{s^\star \varepsilon'}{\overset{s^\star \varepsilon}{\rightrightarrows}} s^\star \mathcal{O}_T \cong \mathcal{O}_{T'}$ is of the form $\mathcal{O}_{T'}/\mathcal{I}$ for some quasi-coherent sheaf of ideals $\mathcal{I}$. The closed subscheme of T' corresponding to $\mathcal{I}$ has functor of points

$$
W \mapsto \{r \in T'(W) | \mathcal{O}_{T'} \to r_* \mathcal{O}_W \text{ factors through } \mathcal{O}_{T'}/\mathcal{I}\}.
$$

Since $\mathcal{O}_{T'}/\mathcal{I}$ was defined as the coequalizer of $s^\star\varepsilon$ and $s^\star\varepsilon'$, this is exactly the functor of points of P. Thus, P is the closed subscheme of (the scheme) T' defined by the quasi-coherent sheaf of ideals $\mathcal{I}$, so it is a scheme.

(3) Let $\pi : U \to X$ be an étale cover of X. The calculation on the left verifies that the diagram on the right is cartesian.

$$
\begin{aligned}
(\mathcal{S}pec_U \, \pi^\star\mathcal{A})(W) &= \left\{(g,\iota) \mid g \in U(W), \iota : g^\star\pi^\star\mathcal{A} \to \mathcal{O}_W\right\} \\
&= \left\{(f,\iota,g) \mid g \in U(W), (f,\iota) \in (\mathcal{S}pec_X \, \mathcal{A})(W), f = \pi g\right\} \\
&= (U \times_X \mathcal{S}pec_X \, \mathcal{A})(W)
\end{aligned}
$$

$$
\begin{array}{ccc}
\mathcal{S}pec_U \, \pi^\star\mathcal{A} & \longrightarrow & U \\
{\scriptstyle\tilde{\pi}}\downarrow & {\scriptstyle\ulcorner} & \downarrow{\scriptstyle\pi} \\
\mathcal{S}pec_X \, \mathcal{A} & \longrightarrow & X
\end{array}
$$

On the other hand, $\mathcal{S}pec_U \, \pi^\star\mathcal{A}$ is the usual relative Spec of the sheaf of algebras $\pi^\star\mathcal{A}$ on U, so it is a scheme. Since π is an étale surjection, so is $\tilde{\pi}$. □

Lemma 16.3. *If $g : X \to Y$ is a morphism of algebraic spaces, and $\mathcal{A}$ is a quasi-coherent sheaf of algebras on Y, then $X \times_Y \mathcal{S}pec_Y \, \mathcal{A} \cong \mathcal{S}pec_X(g^\star\mathcal{A})$.*

Proof. For a test scheme T, we can compute

$$
\begin{aligned}
(X \times_Y \mathcal{S}pec_Y \, \mathcal{A})(T) &= \left\{(x,f,\iota) \mid x \in X(T), f \in Y(T), f = gx, \iota : f^\star\mathcal{A} \cong x^\star g^\star\mathcal{A} \to \mathcal{O}_T\right\} \\
&= \left\{(x,\iota) \mid x \in X(T), \iota : x^\star(g^\star\mathcal{A}) \to \mathcal{O}_T\right\} \\
&= (\mathcal{S}pec_X \, g^\star\mathcal{A})(T) \hspace{3cm} \square
\end{aligned}
$$

Example 16.4 (Stein factorization)**.** Let $f : X \to Y$ be a separated quasi-compact morphism of algebraic spaces. We say that f is *Stein* if the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism. In general, for any quasi-separated quasi-compact morphism $X \to Y$, there is a factorization $X \to Z \to Y$, where $X \to Z$ is Stein and $Z \to Y$ is affine. Namely, we can take $Z = \mathcal{S}pec_Y(f_*\mathcal{O}_X)$ (note that $f_*\mathcal{O}_X$ is quasi-coherent by Proposition 15.10).

$$
\begin{array}{ccccc}
& & Z' & & \\
& \nearrow & | & \searrow^{\text{affine}} & \\
X & & |\, \exists! & & Y \\
& \searrow & \downarrow & \nearrow & \\
& & \mathcal{S}pec_Y(f_*\mathcal{O}_X) & &
\end{array}
$$

[[★★★ I think]] Stein factorization has the following universal property. If $X \to Z' \to Y$ is any factorization such that $Z' \to Y$ is affine, then there is a unique arrow $Z' \to \mathcal{S}pec_Y(f_*\mathcal{O}_X)$ making the diagram above commute. In particular, Stein factorization is unique up to unique isomorphism. ◇

Remark 16.5. Stein factorization behaves nicely with respect to flat base change. Let $X \to Z \to Y$ be the Stein factorization of $f : X \to Y$, let $g : Y' \to Y$ be a flat

morphism, and let all the squares in the diagram below be cartesian.

$$
\begin{array}{ccc}
X' \xrightarrow{\ \ } Z' \xrightarrow{\ \ } Y' \\
\end{array}
$$

By [Har77, III.9.3], we have an isomorphism $f'_* \tilde{\mathcal{O}}_{X'} = f'_* \tilde{g}^* \mathcal{O}_X \cong g^* f_* \mathcal{O}_X$. By Lemma 16.3, we have that $\mathcal{Spec}(g^* f_* \mathcal{O}_X) \cong \mathcal{Spec}_Y f_* \mathcal{O}_X \times_Y Y' = Z \times_Y Y'$. Thus, we have that Z' is the Stein factorization of $X' \to Y'$.   ◇

Example 16.6 (Scheme-theoretic image)**.** Let $f : X \to Y$ be a quasi-compact immersion of algebraic spaces. Let $\mathcal{I} = \ker(\mathcal{O}_Y \to f_* \mathcal{O}_X)$. Then we define the *scheme-theoretic image* of X in Y to be $\mathcal{Spec}_Y(\mathcal{O}_Y/\mathcal{I})$.   ◇

Example 16.7 ($X \mapsto X_{red}$)**.** If X is an algebraic space, then we can define $\mathcal{N}_X \subseteq \mathcal{O}_X$ to be the sheaf of locally nilpotent elements. Define $X_{red} = \mathcal{Spec}_X(\mathcal{O}_X/\mathcal{N}_X)$.

The functor $X \mapsto X_{red}$ is right adjoint to the inclusion of reduced algebraic spaces into algebraic spaces.   ◇

Example 16.8 (Support of a sheaf)**.** Let $\mathcal{F}$ be a [[★★★ quasi-?]]coherent sheaf on X, and let $\mathcal{K} = \ker\bigl(\mathcal{O}_X \to \mathcal{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F})\bigr)$. Then we define the *support* of $\mathcal{F}$ to be $\mathrm{Supp}(\mathcal{F}) = \mathcal{Spec}_X(\mathcal{O}_X/K)$ (note that $\mathcal{K}$ and $\mathcal{O}_X/\mathcal{K}$ are quasi-coherent by Remark 15.5).   ◇

Remark 16.9. [[★★★ should this remark be scrapped?]] Alternatively, if $\mathcal{F}$ is coherent, we can define support as follows.

$$
\begin{array}{ccc}
\mathrm{Supp}(\mathcal{F}_R) & \longrightarrow & R \\
\Downarrow & & \Downarrow \\
\mathrm{Supp}(\mathcal{F}_U) & \hookrightarrow & U \\
& & \downarrow \\
& & X
\end{array}
$$

$U = \operatorname{Spec} A$, $M = \Gamma(U, \mathcal{F}_U)$, then $\mathrm{Supp}(\mathcal{F}_U) = V(\mathrm{ann}(M))$. To specify a closed subspace, it is enough to specify a closed subscheme of each morphism from an affine U to X in a compatible way. [[★★★ You have to check compatibility ... maybe this doesn't behave well with respect to localization if we only have quasi-coherent ... this works for coherent]] If $A \to A'$ is étale, then there is a natural map $\mathrm{ann}(M) \otimes_A A' \to$

$\operatorname{ann}(M \otimes_A A')$, which we wish to check is an isomorphism.

$$
\begin{array}{ccccc}
\operatorname{ann}(M) & \longrightarrow & A & \longrightarrow & \operatorname{Hom}_A(M, M) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{ann}(M \otimes_A A') & \longrightarrow & A' & \longrightarrow & \operatorname{Hom}_{A'}(M \otimes_A A', M \otimes_A A') \\
\wr \uparrow & & \wr \uparrow & & \wr \uparrow \\
\operatorname{ann}(M) \otimes_A A' & \longrightarrow & A' & \longrightarrow & \operatorname{Hom}_A(M, M) \otimes_A A'
\end{array}
$$

The bottom sequence is obtained from the top by tensoring with A'.   ⋄

Any construction that can be done locally in the étale topology [[★★★ something]]. For example $\mathcal{P}\textit{roj}$, which we won't do.

17 Separated, quasi-finite, locally finite type $\implies$ quasi-affine

Definition 17.1. A morphism $f : X \to Y$ is of *finite presentation* if it is locally of finite presentation, quasi-compact and quasi-separated (i.e. $X \to X \times_Y X$ is quasi-compact). This is local on Y. [[★★★ given the corrected statement of the theorem, should this definition still be here, or does it just get in the way?]] ◇

Definition 17.2. A morphism $f : X \to Y$ is *quasi-finite* if it is locally quasi-finite and quasi-compact. ◇

Definition 17.3. A morphism $f : X \to Y$ is *quasi-affine* if there exists a factorization $X \hookrightarrow W \to Y$, where $X \hookrightarrow W$ is a quasi-compact open immersion and $W \to Y$ is affine. ◇

Theorem 17.4. *Let $f : X \to Y$ be a separated, quasi-finite, locally of finite type morphism of algebraic spaces, and let let $X \xrightarrow{g} Z = \mathcal{S}pec_Y(f_*\mathcal{O}_X) \xrightarrow{h} Y$ be the Stein factorization. Then g is an open immersion. In particular, f is quasi-affine.*

Remark 17.5. (David:) It is immediate that g is at least injective, since by (Hartshorne III.11) a Stein morphism has connected fibers and f is quasi-finite. ◇

For X and Y schemes, this theorem is [EGA, IV.18.12.12]. This is Deligne's version of Zariski's main theorem (the non-noetherian version). There is also some theorem by Peskine and Szpirro that goes into the non-noetherian case.

In the course of proving the theorem, we will need to use the following result.

Proposition 17.6 ([EGA, IV.18.12.3]). *Let $f : U \to Y$ be a morphism of schemes which is locally of finite type and separated, and let $y \in Y$ be a point such that $f^{-1}(y)$ is discrete and finite. Then there exists an étale morphism $Y' \to Y$ and a point $y' \in Y'$ mapping to y such that $U' = U \times_Y Y' = U_1' \sqcup U_2'$ such that $f'|_{U_1'} : U_1' \to Y'$ is finite and $U_2' \cap f'^{-1}(y') = \varnothing$.*

In some sense, this is Zariski's main theorem [[★★★ in what sense?]]. In the analytic topology, this is probably clear. [[★★★ finite in the analytic topology means quasi-finite and proper]]

Remark 17.7 (Aside). If Y is noetherian and $y \in Y$, then consider $\hat{\mathcal{O}}_{Y,y}$.

$$
\begin{array}{ccc}
\hat{U}_y & \longrightarrow & \operatorname{Spec} \hat{\mathcal{O}}_{Y,y} \\
\uparrow & & \uparrow \\
U_{y,n} & \longrightarrow & \operatorname{Spec}(\mathcal{O}_{Y,y}, \mathfrak{m}_Y^n)
\end{array}
$$

This is how you break of the U_1' ... it is the projective limit of the $U_{y,n}$s. You check that it is open and closed. This is some fancy version of Hensel's lemma. If you haven't seen this kind of thing before, just think about the complete case. ◇

Proof of Theorem 17.4. The proof is done in a series of steps.

Step 1: Reduce to Y affine. Let $Y' \to Y$ be an étale cover of Y by a scheme, and let $Y'' \to Y'$ be an inclusion of an affine open subscheme. Then we get the following diagram (all squares are cartesian).

$$
\begin{array}{ccccc}
X'' & \xrightarrow{g''} & Z'' & \longrightarrow & Y'' \\
\downarrow & & \downarrow & & \downarrow \\
X' & \xrightarrow{g'} & Z' & \longrightarrow & Y' \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle et} \\
X & \xrightarrow{g} & Z & \longrightarrow & Y
\end{array}
$$

By Remark 16.5, Z' and Z'' are the Stein factorizations of $X' \to Y'$ and $X'' \to Y''$ respectively. Assume the theorem is true when Y is an affine scheme. Then we have that g'' is an open immersion for every open affine Y'' in Y'. Since being an open immersion is stable (local on the base) in the Zariski topology, we have that g' is an open immersion. Finally, open immersions descend along étale surjections [[★★★ where do we prove this?]], so g is an open immersion.

Step 2a: Invoke [EGA, IV.18.12.3]. Choose an étale cover $U \to X$ with U a scheme so that $U \to Y$ is separated, quasi-finite, and locally of finite type. Then in particular, U is quasi-compact, so we can cover it by a finite number of affine open sets. Thus, we may assume U itself is affine. In particular, $U \to Y$ is of finite type. Applying [EGA, IV.18.12.3], for any point $y \in Y$, we get an étale morphism $Y' \to Y$ and a point $y' \in Y$ mapping to y such that $U' = U \times_Y Y' = U'_1 \sqcup U'_2$, with U'_1 finite over Y' and U'_2 not intersecting the fiber of y'. Note that by taking an affine open neighborhood of y' in Y', we can replace Y' by an affine scheme, in which case U' is also an affine scheme.

Define X'_1 to be $U'_1/(U'_1 \times_X U'_1)$ and define X'_2 to be $U'_2/(U'_2 \times_X U'_2)$.

Step 2b: X'_1 is a scheme and $X'_1 \to X'$ is a closed immersion. Note that the projections $U'_1 \times_{Y'} U'_1 \rightrightarrows U'_1$ are finite because $U'_1 \to Y'$ is finite, and the projections are pullbacks of this morphism.

$$
\begin{array}{ccccc}
U' \times_{X'} U' & \longrightarrow & U'_1 \times_{Y'} U'_1 & \rightrightarrows & U'_1 \\
\downarrow & & \downarrow & & \\
X' & \xrightarrow{\ \Delta\ } & X' \times_{Y'} X' & &
\end{array}
$$

Since $X \to Y$ is separated, $X \to X \times_Y X$ is a closed immersion, so $X' \to X' \times_{Y'} X'$ is a closed immersion, so it is finite. It follows that $U'_1 \times_{X'} U'_1 \to U'_1 \times_{Y'} U'_1$ is finite, so the two projections $U'_1 \times_{X'} U'_1 \rightrightarrows U'_1$ are finite. [[★★★ for some reason, this is an étale equivalence relation]] By Theorem 13.3, we have that X'_1 is an affine scheme.

To see that $X'_1 \to X'$ is a closed immersion, let T be a scheme with a morphism to X'. We are trying to show that $T \times_{X'} X'_1 \to T$ is a closed immersion. But $T \times_{X'} X'_1$ is the image of the morphism g in the diagram on the left, so it is enough to show that

g is closed.

$$
\begin{array}{ccc}
U_1' \times_{X'} T \longrightarrow T \times_{X'} X_1' \xrightarrow{\quad g \quad} T & \qquad & U_1' \times_{X'} T \hookrightarrow U_1' \times_{Y'} T \xrightarrow[\text{finite}]{g} T \\
\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow & & \downarrow \qquad\qquad \downarrow \\
U_1' \longrightarrow X_1' \longrightarrow X' & & X' \hookrightarrow X' \times_{Y'} X'
\end{array}
$$

Looking at the diagram on the right, we see that $U_1' \times_{X'} T \to U_1' \times_{Y'} T$ is a closed immersion because $X' \to X' \times_{Y'} X'$ is, and we see that $U_1' \times_{Y'} T \to T$ is finite (and therefore closed) because $U_1' \to Y'$ is finite. Thus, g is closed, so $T \times_{X'} X_1' \to T$ is a closed immersion.

Thus, we have shown that $X' = X_1' \sqcup X_2'$, with $X_2' \times_{Y'} y' = \varnothing$. Since Stein factorization is stable under flat base change, we have that $Z' = Z \times_Y Y' = Z_1' \sqcup Z_2'$, where Z_i' is the stein factorization of $X_i' \to Y'$.

Step 3: $X \to Z$ is étale. Being étale is local on Z in the étale topology. So for each $z \in Z$, it is enough to find an étale morphism $Z' \to Z$ (covering a neighborhood of z) so that $X \times_Z Z' \to Z'$ is étale at every point in Z' lying over z. For $z \in Z$, let $y \in Y$ be the image of z. Applying Step 2, we find an étale morphism $Y' \to Y$ such that $X' = X_1' \sqcup X_2'$, with X_1' affine and $X_2' \times_{Y'} y' = \varnothing$. Thus, it is enough to show that $X_1' \to Z_1'$ is étale. But $X_1' \to Y'$ is a separated, quasi-finite, locally of finite type morphism of schemes, so [EGA, IV.18.12.12] tells us that $X_1' \to Z_1'$ is an open immersion, so it is étale.

Step 4: Reduce to $Y = Z$. It is straightforward to check that $X \to Z$ is separated, quasi-finite, and of finite type, so we'll skip that.[1] [[★★★ I haven't checked it]]

Step 5: $X \to Z$ is a monomorphism. Since $X \to Z$ is Stein, it is its own Stein factorization. Applying Step 2, we see that for any point $z \in Z$, there is an étale cover of a neighborhood of z, $Z' \to Z$, which can be taken to be affine, so that we get the diagram on the left below. Replacing Z' by the connected component containing z', we get that $X_2' = \varnothing$ (otherwise Z' would be disconnected, as we saw at the end of Step 2b), so $X' = X_1'$. Thus, we have that $X' \to Z'$ is finite and an open immersion, so it is an isomorphism.

$$
\begin{array}{ccc}
X_1' \sqcup X_1' \longrightarrow Z' \qquad z' & \qquad & T' \overset{a'}{\underset{b'}{\rightrightarrows}} X' \xrightarrow{\sim} Z' \\
\downarrow \qquad\qquad \downarrow \quad\; \downarrow & & \downarrow \qquad \downarrow \qquad \downarrow \\
X \longrightarrow Z \qquad z & & T \overset{a}{\underset{b}{\rightrightarrows}} X \longrightarrow Z
\end{array}
$$

Next we check that $X \to Z$ is a monomorphism. To do this, let $a, b : T \to X$ be two morphisms from some scheme T so that the compositions with $X \to Z$ are equal. For any point $z \in Z$, we can find a $Z' \to Z$ so that $X' \xrightarrow{\sim} Z'$. It follows that $a' = b'$. Since being equal is local in the étale topology, we get that $a = b$, as desired.

---

[1]By the way, if we had started with the assumption that $X \to Y$ is locally of finite presentation, we would *not* get that $X \to Z$ is locally of finite presentation. Finite type is really the right hypothesis.

Now we have that $X \to Z$ is an étale monomorphism. Let $W \subseteq Z$ be the (open) image of X, then we claim that $X \cong W$ (which would prove that $X \to Z$ is an open immersion). To see this, observe that X and W have isomorphic presentations as algebraic spaces.

$$
\begin{array}{ccc}
U \times_W U \rightrightarrows U \longrightarrow W \\
\| \wr \qquad\qquad \| \\
U \times_X U \rightrightarrows U \longrightarrow X
\end{array}
$$

The vertical isomorphism on the left follows from the fact that $X \to W$ is a monomorphism. $\qquad\square$

Corollary 17.8. *Let* $f : X \to Y$ *be a separated locally quasi-finite, locally of finite type morphism, with* Y *a scheme. Then* X *is a scheme.*

Proof. Let $\pi : U \to X$ be an étale cover by a scheme, and let $R = U \times_X U$. Let $Y = \bigcup Y_i$ be an affine open cover of Y, and let $(f \circ \pi)^{-1}(Y_i) = \bigcup_j U_{ij}$ be an affine open cover of the pre-image of Y_i. Now we have that $X_{ij} = U_{ij}/R_{ij}$ is an open subspace of X.

$$
\begin{array}{ccc}
R_{ij} & \longrightarrow & R \\
\big\downarrow\big\downarrow & & \big\downarrow\big\downarrow \\
U_{ij} & \overset{open}{\hookrightarrow} & U \\
\big\downarrow & & \big\downarrow \\
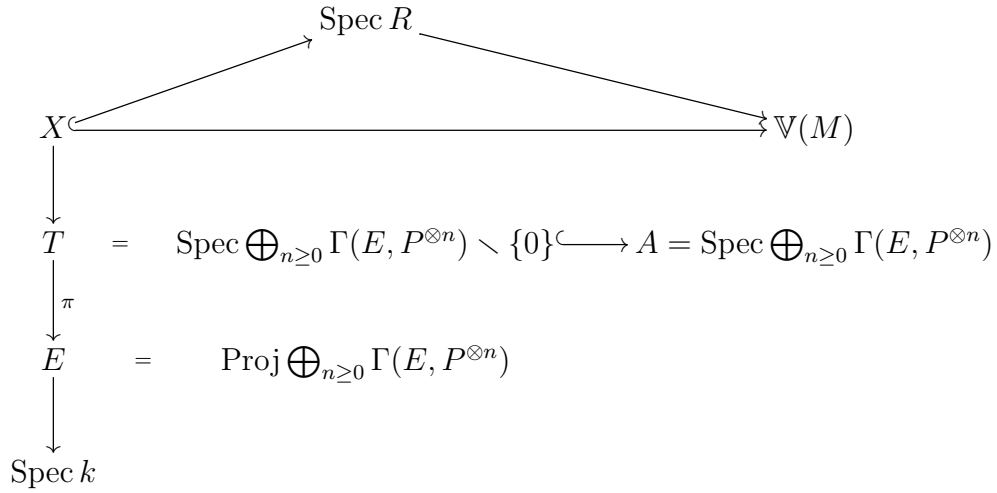X_{ij} := U_{ij}/R_{ij} & \longrightarrow & X
\end{array}
$$

Note that $X_{ij} \to Y$ is quasi-compact for each i and j, and it is still separated, locally quasi-finite, and locally of finite type. In particular, it is quasi-finite, so Theorem 17.4 tells us that the morphisms $X_{ij} \to Y$ are quasi-affine. That is, we have factorizations $X_{ij} \to W_{ij} \to Y$, where $X_{ij} \to W_{ij}$ is are open immersions and $W_{ij} \to Y$ are affine. Since the W_{ij} are affine over a scheme, they are schemes. Since the X_{ij} are open subspaces of schemes, they are schemes. Thus, X has an open cover by schemes, so it is a scheme. $\qquad\square$

⚠ **Warning 17.9** (Conrad, de Jong, Osserman, and Vakil). Even if X and Y are noetherian, Z may not be. Let k be a field, and let E to be a genus 1 curve over k. Let N be a degree zero line bundle on E which is not torsion, and let P be a line bundle of degree at least 3. [[★★★ $\mathbb{V}$ means some symmetric algebra]]

$$
\begin{array}{ccc}
X = T \times_E \mathbb{V}(N) & \hookrightarrow & \mathbb{V}(P) \times_E \mathbb{V}(N) = \mathbb{V}(P \oplus N) \\
\big\downarrow & & \swarrow \\
E & &
\end{array}
$$

where $T = \mathbb{V}(P) - \{0\}$. $X = \mathcal{S}pec_E(\bigoplus_{n \in \mathbb{Z}, m \geq 0} N^m \otimes P^n)$. First we claim that $\Gamma(X, \mathcal{O}_X)$ is not noetherian. $\Gamma(X, \mathcal{O}_X) = \bigoplus_{n \in \mathbb{Z}, m \geq 0} \Gamma(E, N^m \otimes P^n) = R$. $H^0(E, N^m \otimes P^n) \neq 0$

if and only if (1) $n > 0$ or (2) $m = n = 0$. R has generators like in the picture[[★★★]]. It has a maximal ideal $\mathfrak{m}$ generated by the stuff above the line. This ideal is not finitely generated! But we claim that this ring is quasi-affine.

$$
\begin{array}{ccc}
 & \mathrm{Spec}\,R & \\
X \hookrightarrow & & \mathbb{V}(M) \\
\downarrow & & \\
T \quad = \quad \mathrm{Spec}\bigoplus_{n\geq 0}\Gamma(E, P^{\otimes n})\smallsetminus\{0\} \hookrightarrow A = \mathrm{Spec}\bigoplus_{n\geq 0}\Gamma(E, P^{\otimes n}) \\
\downarrow{\scriptstyle \pi} & & \\
E \quad = \quad \mathrm{Proj}\bigoplus_{n\geq 0}\Gamma(E, P^{\otimes n}) \\
\downarrow & & \\
\mathrm{Spec}\,k & &
\end{array}
$$

Let M on A be a coherent sheaf restricting to $\pi^* N$. Some things here are open immersions. X is open in $\mathbb{V}(M)$, and $\mathbb{V}(M)$ is affine. The Stein factorization goes through $\mathrm{Spec}\,R$.   ⌟

18   Chow's Lemma.

Today we'll prove Chow's lemma for algebraic spaces. In the interest of time, we'll cheat by quoting the following big theorem from scheme theory, and showing that it works for algebraic spaces.

Theorem 18.1 ([RG71, 5.2.2])**.** *Let X be a quasi-compact and quasi-separated scheme, $U \subseteq P$ a quasi-compact open subscheme of a scheme P, and $f : X \to P$ a morphism of schemes of finite presentation which is flat over U. Then there exists a blow-up $P' \to P$ supported on $P \smallsetminus U$ such that the strict transform of X, $X' \to P'$, is flat.*

The *strict transform X'* is the scheme-theoretic closure of the pre-image of U (i.e. of the morphism $X \times_P U \hookrightarrow X \times_P P'$). Note that strict transform makes sense for algebraic spaces as well.

Proof when X is an algebraic space. Let $\tilde{X} \to X$ be an étale cover, with $\tilde{X}$ a quasi-compact separated P-scheme. Let $P' \to P$ be a blow-up supported on $P \smallsetminus U$, and let $\tilde{X}'$ be the strict transform of $\tilde{X}$.

$$
\begin{array}{ccc}
\tilde{X} \times_P U & \longrightarrow & \tilde{X} \times_P P' \\
\downarrow & & \downarrow{\scriptstyle et} \\
X \times_P U & \longrightarrow & X \times_P P'
\end{array}
\qquad\qquad
\begin{array}{ccccc}
\tilde{X}' & \longrightarrow & X' & \longrightarrow & P' \\
\downarrow & {}^{\ulcorner} & \downarrow & & \downarrow \\
\tilde{X} & \longrightarrow & X & \longrightarrow & P
\end{array}
$$

The diagram on the left is cartesian (simply by abstract non-sense). Scheme-theoretic closure commutes with étale base change [[★★★ basically because quasi-coherent sheaves of ideals pull back, right?]], so the strict transform $\tilde{X}$ is isomorphic to the product of the strict transform X' with $\tilde{X}$ over X. That is, the left square of the right diagram is cartesian. Now applying the theorem in the case of schemes, we get that $\tilde{X}' \to P'$ is flat. But flatness is étale local and $\tilde{X}' \to X'$ is an étale cover, so $X' \to P'$ is flat, as desired. $\qquad\square$

Theorem 18.2 (Chow's Lemma)**.** *Let S be a quasi-compact scheme and X a separated algebraic space of finite presentation over S (in particular, X is quasi-compact). Then there exists a proper birational S-morphism from a scheme $X' \to X$ with X' quasi-projective over S.*

Note that proper is a stable property morphisms (by what [Vis05] claims is [EGA, IV.2.7.1]), so it makes sense for a morphism from a scheme to an algebraic space to be proper. The map is birational in the sense of Remark 12.7 (birationality descends along étale covers).

Proof. By Corollary 14.2, we may choose a dense open subspace $U \subseteq X$ with U a scheme. By Chow's lemma for schemes, we may assume U is quasi-projective over

S (maybe by shrinking) [[★★★ why is $U \to S$ quasi-compact?]]. Thus, we have immersions $U \hookrightarrow X$ and $U \hookrightarrow \mathbb{P}_S^n$, so we get an immersion $U \hookrightarrow \mathbb{P}_S^n \times_S X$.

$$
\begin{array}{ccccccccc}
& & & & U & & & & \\
& & & & \downarrow \ \ \ \searrow & & & & \\
X' & \longrightarrow & X_1 & \xrightarrow[\text{trans}]{\text{strict}} & X_0 & \xrightarrow{cl} & \mathbb{P}_S^n \times_S X & \longrightarrow & X \\
& & \text{flat} \downarrow \pi' & & \pi \downarrow & & \downarrow & & \downarrow \\
& & P' & \xrightarrow[\text{up}]{\text{blow}} & P & \xrightarrow{cl} & \mathbb{P}_S^n & \longrightarrow & S
\end{array}
$$

Let X_0 be the scheme-theoretic closure of U in $X \times_S \mathbb{P}_S^n$, and let P be the scheme-theoretic closure of U in $\mathbb{P}_S^n$. Since $\mathbb{P}_S^n \times_S X \to X$ is proper and closed immersions are proper, $X_0 \to X$ is a proper birational map (birational because X_0 and X are both birational to U). Moreover, note that $P \to S$ is projective. Both X_0 and P are birationally equivalent to U, so π is birational. Since π is an isomorphism on the image of U, it is flat there, so we may apply Theorem 18.1 to get a blow-up $P' \to P$ supported on $P \smallsetminus U$ so that the strict transform X_1 of X_0 is flat over P'. Note that $X_1 \to X_0 \times_P P'$ is a closed immersion, so it is proper, and $X_0 \times_P P' \to X_0$ is proper (because $P' \to P$ is proper), so $X_1 \to X_0$ is proper, so $X_1 \to X$ is proper and birational.

On the pre-image of U, π' is the same as π (an isomorphism). Since π' is flat, the fibers of X_1 over P' are all zero-dimensional, so they are all finite (since $X_1 \to P'$ is of finite presentation[[★★★]]). By Corollary 17.8, X_1 is a scheme. By Chow's lemma for schemes, [[★★★ have we verified the hypotheses?]] There is some scheme X' quasi-projective over S, with $X' \to X_1$ proper birational. Then note that $X' \to X$ is also proper birational. □

Remark 18.3. You can get away without applying Chow's lemma for schemes a second time. Blow-ups are projective (since they are defined as Proj of a sheaf of graded algebras), so $P' \to S$ is projective. We have that $\pi' : X_1 \to P'$ is flat and birational. One can check that this implies that π' is an open immersion, so it is quasi-projective. Thus, we could have actually taken $X' = X_1$. ◇

What you can prove for schemes which are not quasi-projective always involves using Chow's lemma to reduce to the quasi-projective case. This kind of shows that everything you can do for schemes, you can do for algebraic spaces.

19   Sheaf Cohomology

The next results will take a couple of lectures.

Theorem 19.1. *Let $f : X \to Y$ be a proper morphism of locally noetherian algebraic spaces, and let F be a coherent sheaf on X. Then $R^q f_* F$ is a coherent $\mathcal{O}_Y$-module for all $q \geq 0$.*

Remark 19.2. Working man's definition of proper: Let $Y = \operatorname{Spec} A$ with A noetherian. Then $X \to \operatorname{Spec} A$ is proper if the following. By Chow's lemma, we can choose $X' \to X$ proper birational surjection so that $X' \to \operatorname{Spec} A$ is projective [[★★★ proper + birational implies surjective?]]. [[★★★ something for every choice of X']] Then the theorem roughly says that $H^q(X, F)$ are finitely generated A-modules.   ◇

Something about pullback commuting with finite projective limits wasn't justified. here is a main point. [[★★★ what?]]

Lemma 19.3. *Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories which admits an exact left adjoint. Then F takes injective objects to injective objects.*

Proof. Let $I \in \mathcal{A}$ be injective, let $0 \to M \to N$ be exact in $\mathcal{A}$, and let $G : \mathcal{B} \to \mathcal{A}$ be the left adjoint of F. To prove that $F(I)$ is injective, we must show that given a morphism $M \to F(I)$, we can find the dashed arrow in the diagram.

$$
\begin{array}{ccc}
0 \longrightarrow M \longrightarrow N & & 0 \longrightarrow G(M) \longrightarrow G(N) \\
\quad\quad\downarrow \diagdown & \xleftrightarrow{\text{adjunction}} & \quad\quad\downarrow \diagdown \\
\quad\quad F(I) & & \quad\quad I
\end{array}
$$

Since G is exact, the top row of the diagram on the right is exact. Since I is injective, the dashed arrow exists.   □

Let X be an algebraic space, and let $U \to X$ be an étale surjection, with U a scheme. For every point $u \in U$, we choose a separable closure of the residue field $k(u) \hookrightarrow \overline{k(u)}$, and let $\bar{u} : (\operatorname{Spec} \overline{k(u)})_{et} \to X_{et}$ be the corresponding point. Since all étale covers of a separably closed field are trivial, the topos $(\operatorname{Spec} \overline{k(u)})_{et}$ is the point topos (as a category, it is Set), so $\bar{u}$ is a point of the topos X_{et}. The following lemma roughly says that a sheaf injects into the product of its stalks at these separable points.

Lemma 19.4. *Let X be an algebraic space, and let $U \to X$ be an étale cover by a scheme. For any sheaf $F \in X_{et}$, one gets an injective map of sheaves $F \to \prod_{u \in U} \bar{u}_* \bar{u}^* F$.*

Proof. Let $(V \to X) \in Et(X)$ and $\alpha_1, \alpha_2 \in F(V)$ which map the same element in the product $\prod_u \bar{u}_* \bar{u}^* F$. Then we want to show that they are equal.

It is enough to show that if $\alpha_1, \alpha_2 \in F(X)$, then stuff ..., that is, we may assume $V = X$ for some reason. Moreover, we can assume X is a scheme.[[★★★ how do we make these reductions?]]

$\overline{u}_*\overline{u}^*F(V) = \overline{u}^*F(V \times_X \operatorname{Spec}\overline{k(u)})$. This is the sheaf associated to the presheaf where we compute $\varinjlim_W F(W)$, where the limit is over W such that

$$V \times_X \operatorname{Spec}\overline{k(u)} \quad = \quad \operatorname{Spec}\overline{k(u)} \longrightarrow W$$

 So for every $u \in U$, there exists an étale morphism $W_u \to X$ whose image contains the image of u such that $\alpha_1|_{W_u} = \alpha_2|_{W_u}$. That means that $\coprod W_u \to X$ is an étale surjection and $\alpha_1 = \alpha_2$ when restricted to the cover $\coprod W_u$. That means that they are equal because F is a sheaf.   □

Lemma 19.5. *Let X be an algebraic space, and let Λ be a sheaf of rings in X_{et}. Then the category Λ-mod has enough injectives.*

This is really true much more generally by the way. There will be enough injectives whenever there are "enough points" in the sense of Lemma 19.4.

Proof. If F is a Λ-module, then $\overline{u}^*F$ is a $\overline{u}^*\Lambda$-module. We know that the category of modules over a ring has enough injectives, so choose an embedding $\overline{u}^*F \hookrightarrow I_{\overline{u}}$ where $I_{\overline{u}}$ is an injective $\overline{u}^*\Lambda$-module. Then we have $F \hookrightarrow \prod_{\overline{u}} \overline{u}_*\overline{u}^*F \hookrightarrow \prod_{\overline{u}} \overline{u}_*I_{\overline{u}}$. Built into the definition of a morphism of topoi is that u^* is exact, so Lemma 19.3 tells us that $\overline{u}_*I_{\overline{u}}$ is injective, and an arbitrary product of injective objects is injective.   □

The key point to the next theorem will be that if X is an affine scheme, and F is a quasi-coherent sheaf in X_{et}, then $H^q(X_{et}, F) = 0$ for $q > 0$. This is good to prove if you're bored with the lectures.

Higher Direct Images of Proper Maps

Lemma 19.6. *Let $f : X \to Y$ be a morphism of topoi and assume that Ab_X and Ab_Y have enough injectives (this always holds, actually). Then there is a spectral sequence $E_2^{p,q} = H^p(Y, R^qf_*F) \Rightarrow H^{p+q}(X, F)$ for all $F \in \mathsf{Ab}_X$.*

Proof. R^qf_* is the q-th derived functor of $f_* : \mathsf{Ab}_X \to \mathsf{Ab}_Y$, and H^p is the p-th derived functor of the global sections functor. Observe that we have a factorization as shown, and f_* takes injective objects to injective objects (Lemma 19.3).

$$\mathsf{Ab}_X \xrightarrow{f_*} \mathsf{Ab}_Y$$
$$\Gamma(X,-) \searrow \quad \downarrow \Gamma(Y,-)$$
$$\mathsf{Ab}$$

There is a very general theorem (the *Grothendieck spectral sequence*, [Lan02, XX.9.6]) that says that whenever you have a factorization $G \circ F$ of a functor so that F takes injective objects to G-acyclic objects, then you get a spectral sequence $R^pG(R^qF) \Rightarrow R^{p+q}(G \circ F)$.   □

Lemma 19.7. *Let $f : X \to Y$ be an affine morphism of algebraic spaces. Then for any quasi-coherent sheaf F on X and $q > 0$ we have that $R^q f_* F = 0$, where we think of $f_* : \mathsf{Ab}_{X_{et}} \to \mathsf{Ab}_{Y_{et}}$.*

Proof. Given $q > 0$, consider the following statments.

$(*)_q$ For every affine morphism $f : X \to Y$ of algebraic spaces and F quasi-coherent on X, $R^s f_* F = 0$ for all $1 \leq s \leq q$.

$(*)'_q$ For every affine scheme X and every F quasi-coherent on X, $H^s(X_{et}, F) = 0$ for $1 \leq s \leq q$.

We claim that $(*)'_q$ implies $(*)_q$. This is because $R^s f_* F$ is the sheaf associated to the presheaf $(Y' \to Y) \mapsto H^s(X \times_Y Y', F|_{X \times_Y Y'})$ [[★★★ exercise]]. Thus it is enough to consider the case when Y is affine, so $(*)'_q \Rightarrow (*)_q$.

Now we prove that $(*)'_q$ holds by induction on q. For $q = 1$, we are looking at $H^1(X_{et}, F) = Ext^1(\mathcal{O}_X, F)$ with F quasi-coherent and X an affine scheme. You can think of the elements as isomorphism classes of extensions of $\mathcal{O}_X$-modules

$$0 \to F \to E \to \mathcal{O}_X \to 0.$$

The point is that $\mathrm{Hom}(\mathcal{O}_X, -) \cong \Gamma(X, -)$, so they give the same derived functors (whether you compute in $\mathcal{O}_X\text{-mod}$ or in Ab). Now the result is clear because F and $\mathcal{O}_X$ are quasi-coherent, so E is quasi-coherent. By the equivalence of categories $\mathsf{Qcoh}(X_{zar}) \cong \mathsf{Qcoh}(X_{et})$, we know that all such sequences are split because we can compute the Zariski cohomology to be zero.

So we've proven $(*)'_1$, and therefore $(*)_1$. Now assume $(*)_q$. This implies that for every affine morphism $f : X \to Y$ and every F quasi-coherent on X, we have an inclusion $H^{q+1}(Y, f_* F) \hookrightarrow H^{q+1}(X, F)$. To see this, look at the spectral sequence $E_2^{s,t} = H^s(Y, R^t f_* F) \Rightarrow H^{s+t}(X, F)$.
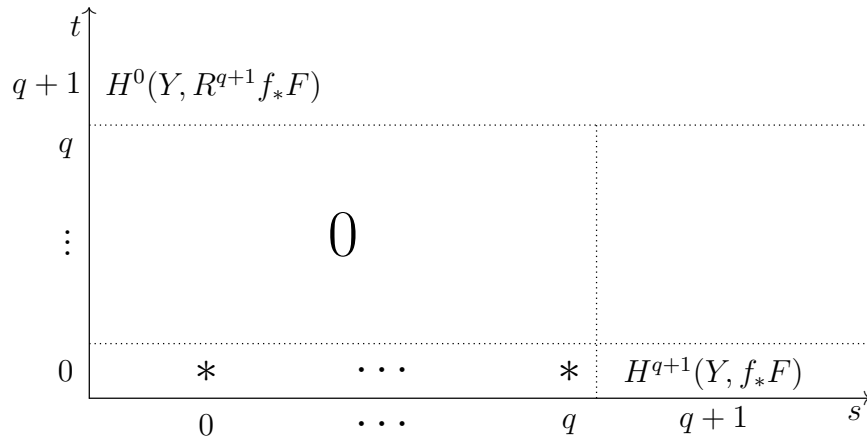
There is a natural map $H^{q+1}(Y, f_* F) \to H^{q+1}(x, F)$, and the kernel is the stuff killed of as you run the spectral sequence. But you can see that you aren't killing anything off because of all the zeros.

Now we'd like to show that $(*)_q \Rightarrow (*)'_{q+1}$. Take a class $\alpha \in H^{q+1}(X_{et}, F)$. We'd like to say that it is zero. To compute, we take an injective resolution, take global sections and then take cohomology. There exists an étale surjection $g : U \to X$ with U an affine scheme such that $\alpha \mapsto 0 \in H^{q+1}(U_{et}, F)$ [[★★★ This is because we have a resolution of sheaves!]]. $F \mapsto I^\bullet$. $\tilde\alpha \in \Gamma(X, I^{q+1})$ locally because this is a resolution.

Now consider the sequence

$$0 \to F \to g_* g^* F \to Q \to 0.$$

Then we get a long exact sequence

$$\underbrace{H^q(X, Q)}_{=0 \text{ by } (*)'_q} \to H^{q+1}(X, F) \to H^{q+1}(X, g_* g^* F) \hookrightarrow H^{q+1}(U, g^* F)$$

and $\alpha \mapsto 0$ in $H^{q+1}(U, g^* F)$, so it is zero.   □

Lemma 19.8. *Let X be a scheme, and let $\varepsilon : X_{et} \to X_{zar}$ be the natural morphism of topoi (coming from the inclusion of sites). Then for every quasi-coherent sheaf F on X_{et} we have $R^q \varepsilon_* F = 0$ for all $q > 0$.*

Proof. $R^q \varepsilon_* F$ is the étale sheaf associated to the presheaf on X_{zar} given by $U \mapsto H^q(U_{et}, F)$. It is enough to check on affine open subsets, and there they are zero.   □

Lemma 19.9. *Let X be a quasi-compact separated algebraic space. Then $\mathsf{Qcoh}(X)$ has enough injectives and for every injective $I \in \mathsf{Qcoh}(X)$, we have $H^q(X, I) = 0$ for $q > 0$.*

Proof. Let $\pi : U \to X$ be a quasi-compact étale surjection with U a disjoint union of affine schemes. By the case of schemes, $\mathsf{Qcoh}(U)$ (we can be ambiguous about the topology because they are equivalent) has enough injectives, and for injective $I_U \in \mathsf{Qcoh}(U)$, $\pi_* I_U$ is injective in $\mathsf{Qcoh}(X)$ (since π_* has an exact left adjoint).

So for any $F \in \mathsf{Qcoh}(X)$, choose an injection $\pi^* F \hookrightarrow I_U$, with $I_U \in \mathsf{Qcoh}(U)$ injective. Then you get

$$F \hookrightarrow \pi_* \pi^* \hookrightarrow \pi_* I_U$$

In fact, by taking F to be injective, we see that any injective in $\mathsf{Qcoh}(X)$ is a direct summand of $\pi_* I_U$ for some injective $I_U \in \mathsf{Qcoh}(U)$

So to prove the second statement, it is enough to note that for an injective $I \in \mathsf{Qcoh}(U)$, $H^q(X, \pi_* I) = H^q(U_{et}, I) = H^q(U_{zar}, I) = 0$. The first equality is because we have a spectral sequence $E_2^{p,q} = H^p(X, R^q \pi_* I) \Rightarrow H^{p+q}(U, I)$. All of the $R^q \pi_* I = 0$ for $q > 0$ are zero because X is separated, so $\pi : U \to X$ is an affine morphism.   □

Lemma 19.10. *Let $f : X \to Y$ be a quasi-compact and separated morphism of algebraic spaces. Then for every quasi-coherent F on X, the sheaves $R^q f_* F$ are quasi-coherent on Y.*

Proof. We're looking at the sheaf associated to the presheaf $(Y' \to Y) \mapsto H^q(X \times_Y Y', F)$. To check quasi-coherence, we can look étale locally on Y, so we can assume Y is an affine scheme. So $\mathsf{Qcoh}(X)$ has enough injectives. If F happens to be injective in $\mathsf{Qcoh}(X)$, the $H^q(X \times_Y Y', F)$ are zero. Thus, to compute the cohomology, we can choose an injective resolution over X and push it forward. But we know that push-forwards of quasi-coherent sheaves are quasi-coherent. Choose an injective resolution $F \to I^\bullet$ in $\mathsf{Qcoh}(X)$. Then $R^q f_* F = \mathcal{H}^q(f_* I^0 \to f_* I^1 \to \cdots)$. $\square$

Theorem 19.11. *If $f : X \to Y$ is a proper morphism of locally noetherian algebraic spaces and F is coherent on X, then $R^q f_* F$ are coherent on Y.*

Recall that for now proper just means that $[[\bigstar\bigstar\bigstar\,]]$

Proof. Coherence is local on Y, so we can assume Y is an affine scheme. Thus, there exists a proper birational morphism $X' \to X$ so that $X' \to Y$ is projective.

Next we need that for every integral closed subspace of X, there is a coherent sheaf supported on it. $\square$

Coherence of higher direct images

Last time I said that you can think of H^1 as extensions. That's true, but it is better to start the induction at $q = 0$, where the statement is vacuous.

$(*)_0 \Rightarrow (*)_1'$. Choose $\alpha \in H^1(X, F)$. By the same reason as last time, there exists an affine étale morphism $p : U \to X$ such that $\alpha \mapsto 0$ in $H^1(U, p^* F)$. Now consider

$$0 \to F \to p_* p^* F \to Q \to 0$$

which gives us

$$H^0(X, p_* p^* F) \twoheadrightarrow H^0(X, Q) \to H^1(X, F) \xrightarrow{\alpha \mapsto 0} \underbrace{H^1(X, p_* p^* F)}_{\hookrightarrow H^1(U, p^* F)}$$

The descent is hidden in the (obvious) statement that global sections are the same in the Zariski and étale topologies.

Theorem 19.12. *Let $f : X \to Y$ be a proper morphism of locally noetherian algebraic spaces and F a coherent sheaf on X. Then $R^q f_* F$ is a coherent $\mathcal{O}_Y$-module for every $q \geq 0$.*

We use the working definition of properness.

Proof. Reduce to the case Y is affine.

Devissage is on the homework: X a noetherian locally separated algebraic space, and let $K = \mathsf{Coh}\mathcal{O}_X$. Let $K' \subseteq K$ be a subcategory such that

   1. $0 \in K'$

2. for every exact sequence $0 \to A' \to A \to A'' \to 0$ in K, if two out of three are in K', then so is the third.

3. If $A_1 \oplus A_2 \in K'$, then $A_1, A_2 \in K'$.

4. for every reduced and irreducible closed subspace $Z \subseteq X$, there exists a $G \in K'$ sith $\operatorname{Supp} G = Z$.

Then $K' = K$. Integral means reduced and irreducible [[★★★ Ogus: well, we had that weird example where we have two tangent directions on the line ... that is integral, but doesn't feel like it should be]]

Take $K' \subseteq \mathsf{Coh}X$ to be the subcategory of coherent sheaves for which the theorem holds. (1) zero is in there, (2) use the long exact sequence, (3) cohomology is the direct sum, and we must check (4).

Remark 19.13. [[★★★ strengthening of the theorem as follows]] X separated of finite type algebraic space over a noetherian affine scheme Y [[★★★ let's change it to over a field]], then there exists d such that for *every sheaf* F on X, $H^q(X, F) = 0$ for $q \geq 0$. Do this by taking a dense open which is a scheme and do some kind of devissage. The d is not the dimension in general. ◇

Check (4). Let $Z \subseteq X$. We can choose π birational proper surjective by Chow's lemma.
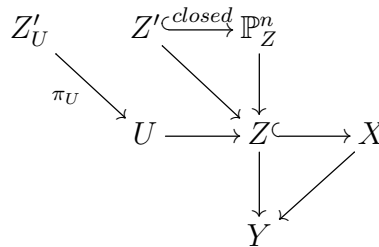
Let $L = \mathcal{O}_{Z'}(1)$. For $m \gg 0$, $\pi^* \pi_* L^{\otimes m} \to L^{\otimes m}$ is surjective [[★★★ you don't need this surjectivity]] and $T^q \pi_* L^{\otimes m} = 0$ for $q > 0$. To check these, it is enough to find $U \to Z$, with $\pi_U : Z'_U \to U$ [[★★★ use Serre's vanishing theorem]]. This implies that $H^p(Z, \pi_* L^{\otimes m}) = H^p(Z', L^{\otimes m})$ because we have the spectral sequence $E_2^{p,q} = H^p(Z, R^q \pi_* L^{\otimes m}) \Rightarrow H^{p+q}(Z', L^{\otimes m})$, and we have that $E_2^{p,q} = 0$ for $q > 0$. This proves that $H^p(Z, \pi_* L^{\otimes m})$ are finite type $\Gamma(Y, \mathcal{O}_Y)$-modules. So $\pi_* L^{\otimes m} \in K'$. We still need to show that $\operatorname{Supp} \pi_* L^{\otimes m} = Z$.

The support is defined by the ideal of sections which annihilate that module. Suppose $f \in \mathcal{O}_Z(U)$ annihilates $\pi_* L^{\otimes m}$. Then we claim that $f = 0$. To see this, choose a dense open $U^0 \subseteq U$ such that $Z'_{U^0} \to U^0$ is an isomorphism. The restriction of $\pi_* L^{\otimes m}$ to U^0 is a locally free sheaf of non-zero rank, so $f \mapsto 0 \in \mathcal{O}_Z(U^0)$. But then $f = 0$ because U is reduced.

Note that we produced for any reduced subspace a sheaf with that support. We didn't need irreducible. □

Next week, we'll define stacks. Now we'll define a proper morphism.

Underlying topological space of an algebraic space

There is a topological space associated to an algebraic space. Let X be an algebraic space over a scheme S. A point of X is a monomorphism $i : \operatorname{Spec} k \to X$ with k a field. You shouldn't confuse this with a geometric point (where we take k to be a separably closed field), this is just a monomorphism of sheaves. If we take k to be separably closed field, it usually won't be a monomorphism of sheaves! You could drop the monomorphism assumption if you change the equivalence relation to say there is a third thing they map to. [[★★★ this definition would make some things easier]]

Example 40.14. Let X be a scheme. If we have any morphism $\operatorname{Spec} k \to X$, this gives a point $x \in X$ and embedding $k(x) \hookrightarrow k$. That is, the morphism always factors through $\operatorname{Spec} k(x)$. Given an inclusion of fields $k \hookrightarrow k'$, when is $\operatorname{Spec} k' \to \operatorname{Spec} k$ a monomorphism? If and only if $k \to k'$ is an isomorphism [[★★★ exercise in Galois theory]] What about inseparable extensions? To be a monomorphism, it must be a monomorphism after base change. Look at $k' \to k' \otimes_k k'$.

Consider $k(t)$ and $k(t^{1/p})$, then we have $k(x) \to k(t)[x,y]/(x^p = y^p = 1) = k(x)[u]/(u^p = 0)$ with $u = x - y$. We get two maps $k(x)[u]/(u^p = 0) \to k(x)[u]/(u^p = 0)$, given by $u \mapsto u$ and $u \mapsto 0$, which induce the same map. Therefore, it isn't a monomorphism. ◇

We say that $i_1 : \operatorname{Spec} k_1 \to X$ is equivalent to $i_2 : \operatorname{Spec} k_2 \to X$ if there exists a diagram

$$\operatorname{Spec} k_1 \xrightarrow[\sim]{\sigma} \operatorname{Spec} k_2$$
$$i_1 \downarrow \quad\swarrow i_2$$
$$X$$

Then we take $|X|$ to be the set of points modulo equivalence. If $Y \subseteq X$ is a closed subspace, then you get an inclusion $|Y| \subseteq |X|$, and we define the topology on $|X|$ by declaring these subsets to be closed.

Key point: $|X|$ is functorial in X. This is not clear. If we have $X \to Y$ and $\operatorname{Spec} k \to X$ a monomorphism, there is no reason $\operatorname{Spec} k \to Y$ should be a monomorphism. The following lemma gives it to us. One then checks that the map is continuous.

Lemma 40.15. *Let $f : \operatorname{Spec} k \to Y$ be a map of algebraic spaces with k a field. Then there exists a unique (up the equivalence relation) point $i : \operatorname{Spec} k' \to Y$ and factorization*

$$f : \operatorname{Spec} k \xrightarrow{g} \operatorname{Spec} k' \xrightarrow{i} Y$$

Proof. This is annoyingly subtle. How do you know there are any points in Y at all. You know there is a dense open which has points, but blah.

First note that we can assume Y is quasi-compact: Take any étale cover $U \to Y$, then take a point in U_k and see where it goes in U, and take a quasi-compact open subset of U and quotient by the relation. Choose $U \to Y$ an étale surjection with U a quasi-compact scheme. Let Z_p be the disjoint union of spectra of residue fields of

images of $\operatorname{Spec} k \times_Y U \to U$. Since U is quasi-compact and étale, the first is a finite union of points. Similarly, let R_p be the disjoint union of spectra of residue fields of images of $\operatorname{Spec} k \times_Y (U \times_Y U) \rightrightarrows U \times_Y U$. Then we see that $R_p = Z_1 \times_{U,p_1} (U \times_Y U)$. So R_p is a finite étale equivalence relation on Z_p (in fact, it is the induced relation). Thus, the quotient Z_p/R_p, which is a scheme. We get a monomorphism $Z_p/R_p \hookrightarrow Y$, and my construction we get a flat surjection $\operatorname{Spec} k \to Z_p/R_p$, so Z_p/R_p is a spectrum of a field. $\qquad\square$

$f : X \to Y$ is proper if it is of finite type, separated, and universally closed (i.e. for every $Y' \to Y$, the map $|Y \times_Y Y'| \to |Y'|$ is a closed map). Check from the definition that this agrees with the working definition:

Lemma 40.16. *If $X \to Y$ is separated of finite presentation and $X' \to X$ is a proper representable surjection, then $X \to Y$ is proper if and only if $X' \to Y$ is proper.*

If X is given to you as a functor, the only way to check properness is with the valuative criterion.

21   Fibered categories

A very good reference for this upcoming stuff is [Vis05]. Vistoli does more than we will.

The impression is that a lot of people got lost in the details of algebraic spaces. That's ok, because we're about to do it all over again (in some sense).

algebraic spaces	Artin stacks
a sheaf X	a stack in groupoids $\mathcal{X}$
Δ_X representable by schemes	$\Delta_{\mathcal{X}}$ representable by algebraic spaces
étale surjection from a scheme	smooth surjection from an algebraic space

Note that the smooth surjection from an algebraic space may as well be from a scheme since any algebraic space is smoothly covered by a scheme.

Let's say we want to study genus g curves. Then we'd like to say there is a functor $\mathcal{M}_g : \mathsf{Sch}^{op} \to \mathsf{Cat}$, given by $S \mapsto \mathcal{M}_g(S)$, whose objects are proper smooth morphisms $B \to S$ whose geometric fibers are connected genus g curves and whose morphisms are S-isomorphisms. However, the pullback functors $f^* : \mathcal{M}_g(S) \to \mathcal{M}_g(S')$ associated to morphisms $f : S' \to S$ do not behave nicely with respect to composition. We have that $(fg)^*$ is canonically isomorphic to g^*f^*, but not *equal* to it. That is, $\mathcal{M}_g : \mathsf{Sch}^{op} \to \mathsf{Cat}$ is a lax 2-functor. To get a better understanding of such things, we need to build up some machinery. [[★★★ maybe this is worth shortening since it gets repeated]] The problem is that we do not have a canonical choice for the pullback of a curve along a morphism.

Sometimes people say that a stack is is a "category-valued functor", but that isn't right because it is really a category-valued lax 2-functor. [[★★★ or something like that]]

Definition 21.1. Let C be any category. Then a *category over* C is a category $\mathcal{F}$ with a functor $p : \mathcal{F} \to \mathsf{C}$. ◇

Example 21.2. Let $\mathsf{C} = \mathsf{Sch}$. Then we define $\mathcal{M}_g$ to be the category whose objects are pairs $(S, B/S)$ where S is a scheme and $B \to S$ is a proper smooth morphism whose geometric fibers are connected genus g curves. The morphisms $(S', B'/S') \to (S, B/S)$ are cartesian diagrams

$$\begin{array}{ccc} B' & \xrightarrow{\tilde{f}} & B \\ \downarrow & \ulcorner & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

The functor $p : \mathcal{M}_g \to \mathsf{Sch}$ is given by $(S, B/S) \mapsto S$. ◇

Definition 21.3. Let $p : \mathcal{F} \to \mathsf{C}$ be a category over C. An arrow $\phi : \xi \to \eta$ in $\mathcal{F}$ is called *cartesian* if for any $\psi : \zeta \to \eta$ and for any $h : p(\zeta) \to p(\xi)$ such that

$p(\psi) = p(\phi) \circ h$, there exists a unique $\theta : \zeta \to \xi$ so that $\psi = \phi \circ \theta$.
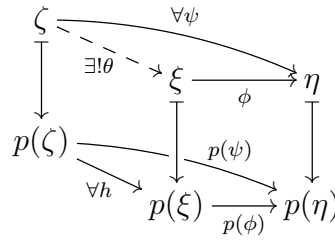
In this case, ξ is called a *pullback of η to $p(\xi)$*.   ◇

**Definition 21.4.** let $p : \mathcal{F} \to \mathsf{C}$ be a category over C, and let $U \in \mathsf{C}$. We define the *fiber $\mathcal{F}(U)$* to be the sub-category of $\mathcal{F}$ whose objects are $\xi \in \mathcal{F}$ such that $p(\xi) = U$ and whose morphisms are $\phi : \xi' \to \xi$ in $\mathcal{F}$ such that $p(\phi) = \mathrm{id}_U$. That is, $\mathcal{F}(U)$ is the subcategory of $\mathcal{F}$ whose objects lie over U and whose morphisms lie over id_U.   ◇

Definition 21.5. A *fibered category* over C is a category over C $p : \mathcal{F} \to \mathsf{C}$ such that for every arrow $f : U \to V$ in C and every $\xi \in \mathcal{F}(V)$, there exists $\eta \in \mathcal{F}(U)$ and a cartesian arrow $\phi : \eta \to \xi$ with $p(\phi) = f$.   ◇

Remark 21.6. A fibered category over C is a category over C in which pullbacks always exist. It is an easy exercise to check that different pullbacks are unique up to unique isomorphism.   ◇

Example 21.7. $\mathcal{M}_g$ is a fibered category over Sch. In fact, every arrow in $\mathcal{M}_g$ is cartesian. [[★★★ expand?]]   ◇

Example 21.8 ("representable fibered categories"). Let C be a category and let $X \in \mathsf{C}$. Then C/X is a fibered category over C in which every arrow is cartesian. To see this, let Y'', Y', and Y be objects over X, let ϕ and ψ be X-morphisms, and let h be a morphism such that $\psi = \phi \circ h$.
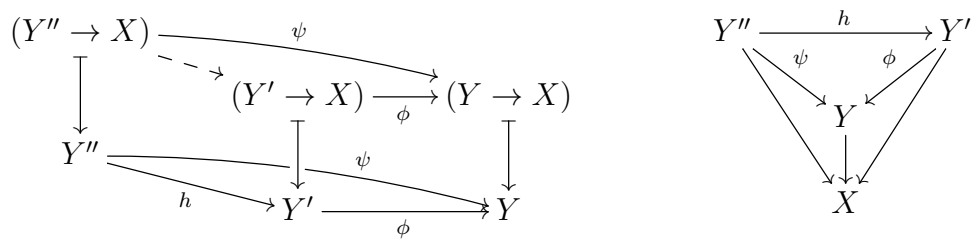
Then h is an X-morphism (the outer triangle on the right is composed of three commutative triangles, so it is commutative), and the only X-morphism from Y'' to Y' which "projects" to h when you forget about the maps to X is h itself.   ◇

In the next lecture, we'll prove an analogue of Yoneda's lemma for fibered categories, which will say roughly that the construction above gives a fully faithful embedding of C into the category of fibered categories over C.

Definition 21.9. Let $\mathcal{F}$ and $\mathcal{G}$ be fibered categories over C. Then a *morphism of fibered categories* $f : \mathcal{F} \to \mathcal{G}$ is a functor such that

1. $p_{\mathcal{G}} \circ f = p_{\mathcal{F}}$ (actual equality!)

2. f sends cartesian arrows to cartesian arrows.   ◇

Remark 21.10. One can restrict a morphism of fibered categories to the fibers. If $\xi \in \mathcal{F}$, with $p_{\mathcal{F}}(\xi) = U$, then $p_{\mathcal{G}}(f(\xi)) = p_{\mathcal{F}}(U) = U$, and if $\phi : \xi' \to \xi$ is a morphism in $\mathcal{F}(U)$, then $p_{\mathcal{G}}(f(\phi)) = p_{\mathcal{F}}(\phi) = \mathrm{id}_U$.   ◇

   Why can you actually demand equality in the first condition? Let $\mathcal{M}_{1,1}$ be the category of pairs $(S, E/S)$, where E is an elliptic curve over S. Then we have a functor $J : \mathcal{M}_{1,1} \to \mathcal{M}_{1,1}$ given by $(S, E/S) \mapsto (S, Jac(E)/S)$. Note that the equality is strict. It just so happens that J is isomorphic to the identity functor. [[★★★ I don't understand what this paragraph is doing.]]

Definition 21.11. If $g, g' : \mathcal{F} \to \mathcal{G}$ are two morphisms of fibered categories, then a *base-preserving natural transformation* $\alpha : g \to g'$ is a natural transformation such that for every $\xi \in \mathcal{F}$, the arrow $\alpha_\xi : g(\xi) \to g'(\xi)$ projects to the identity in C. That is, a base-preserving natural transformation is one which restricts to a natural transformation on each fiber. Define the category $\mathrm{HOM}_{\mathsf{C}}(\mathcal{F}, \mathcal{G})$, whose objects are morphisms of fibered categories, and whose morphisms are base-preserving natural transformations.   ◇

Remark 21.12. The composition of two morphisms of fibered categories is a morphism of fibered categories, and the composition of two base-preserving natural transformations (in either way)[1] is a base-preserving natural transformation. Thus, we have *the 2-category of fibered categories over* C.   ◇

Definition 21.13. A morphism of fibered categories $f : \mathcal{F} \to \mathcal{G}$ is an *equivalence* if there exists a morphisms $g : \mathcal{G} \to \mathcal{F}$ such that $f \circ g \cong \mathrm{id}_{\mathcal{G}}$ in $\mathrm{HOM}_{\mathsf{C}}(\mathcal{G}, \mathcal{G})$ and $g \circ f \cong \mathrm{id}_{\mathcal{F}}$ in $\mathrm{HOM}_{\mathsf{C}}(\mathcal{F}, \mathcal{F})$. That is, both compositions should be identities up to 2-morphisms.   ◇

   Next we will prove that equivalences of fibered categories "can be checked on fibers" (Proposition 21.15). First we'll need to prove the following lemma.

Lemma 21.14. *Let* $f : \mathcal{F} \to \mathcal{G}$ *be a morphism of fibered categories such that for every* $U \in \mathsf{C}$, *the functor* $f_U : \mathcal{F}(U) \to G(U)$ *is fully faithful. Then* f *is fully faithful.*

---

[1] There are two ways to compose natural transformations: (1) if $F, G, H : \mathsf{C} \to \mathsf{D}$ are functors and $\eta : F \to G$ and $\tau : G \to H$ are natural transformations, then $\tau \cdot \eta : F \to H$ is a natural transformation, and (2) if $F, G : \mathsf{C} \to \mathsf{D}$ and $F', G' : \mathsf{D} \to \mathcal{E}$ are functors, and $\eta : F \to G$ and $\eta' : F' \to G'$ are natural transformations, then $\eta' \circ \eta : F' \circ F \to G' \circ G$ is a natural transformation. A 2-category must have two such flavors of composition of 2-morphisms. See Appendix A5.

Proof. Given $x, y \in \mathcal{F}$ and an arrow $\phi : f(x) \to f(y)$, we wish to show that there exists a unique arrow $\psi : x \to y$ such that $f(\psi) = \phi$. Let $U = p_\mathcal{F}(x) = p_\mathcal{G}(f(x))$, $V = p_\mathcal{F}(y) = p_\mathcal{G}(f(y))$, and let $\bar{\phi} = p_\mathcal{F}(\phi) : U \to V$.

$$
\begin{array}{ccc}
x & & f(x) \\
\chi\downarrow\ \ \searrow^{\psi} & & \exists!\varepsilon\ \downarrow\ \ \searrow^{\phi} \\
\tilde{y} \xrightarrow[\exists h]{\text{cart}} y & \qquad & f(\tilde{y}) \xrightarrow[f(h)]{} f(y) \\
\downarrow\qquad\downarrow & & \downarrow\qquad\downarrow \\
\mathrm{id}_U\ \circlearrowleft\ U \xrightarrow{\bar{\phi}} V & & \mathrm{id}_U\ \circlearrowleft\ U \xrightarrow{\bar{\phi}} V
\end{array}
$$

Since $\mathcal{F}$ is a fibered category, there is some cartesian arrow $h : \tilde{y} \to y$ lying over $\bar{\phi}$. Since f is a morphism of fibered categories, we have that $f(h)$ is cartesian. Thus, there is a unique ε with $p_\mathcal{G}(\varepsilon) = \mathrm{id}_U$ which makes the top triangle on the right commute. Since f_U is fully faithful, $\varepsilon = f(\chi)$ for a unique morphism $\chi : x \to \tilde{y}$, with $p_\mathcal{F}(\chi) = \mathrm{id}_U$. Then we can take $\psi = h \circ \chi$.

Finally, one checks uniqueness. Let $\psi' : x \to y$, with $f(\psi') = \phi$. Then by cartesian-ness of h, we get some map $\chi' : x \to \tilde{y}$ so that $p_\mathcal{F}(\chi') = \mathrm{id}_U$ and $\psi' = h \circ \chi'$. Since $\phi = f(\psi') = f(h) \circ f(\chi')$, cartesian-ness of $f(h)$ tells us that $f(\chi') = \varepsilon$. Since f_U is fully faithful, this implies that $\chi' = \chi$, so $\psi' = \psi$. $\qquad\square$

Proposition 21.15. *Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of fibered categories. Then f is an equivalence of fibered categories if and only if for every object $U \in \mathsf{C}$, the restriction $f_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an equivalence of categories in the usual sense.*

Proof. ($\Rightarrow$) Let $g : \mathcal{G} \to \mathcal{F}$ be an inverse for f, so we have base-preserving natural isomorphisms $f \circ g \cong \mathrm{id}_\mathcal{G}$ and $g \circ f \cong \mathrm{id}_\mathcal{F}$. These restrict to isomorphisms $f_U \circ g_U \cong \mathrm{id}_{\mathcal{G}(U)}$ and $g_U \circ f_U \cong \mathrm{id}_{\mathcal{F}(U)}$.

($\Leftarrow$) We need to find some $g : \mathcal{G} \to \mathcal{F}$ such that $f \circ g \cong \mathrm{id}_\mathcal{G}$ and $g \circ f \cong \mathrm{id}_\mathcal{F}$. For every $U \in \mathsf{C}$, we have a functor g_U and natural isomorphisms $\alpha_U : \mathrm{id}_\mathcal{G} \to f_U g_U$ and $\beta_U : g_U f_U \to \mathrm{id}_\mathcal{F}$.

For an object $y \in \mathcal{G}$, define $g(y) := g_{p_\mathcal{G}(y)}(y)$. By Lemma 21.14, f is fully faithful, so for any arrow $\phi : y \to y'$ in $\mathcal{G}$, there exists a unique arrow $g(\phi) : g(y) \to g(y')$ such that the following diagram commutes. That is, there is a unique arrow $g(\phi)$ so that $f(g(\phi)) = \alpha(y') \circ \phi \circ \alpha(y)^{-1}$.

$$
\begin{array}{ccc}
y & \xrightarrow{\phi} & y' \\
\alpha(y)\downarrow\wr & & \wr\downarrow\alpha(y') \\
f(g(y)) & \xrightarrow{f(g(\phi))} & f(g(y'))
\end{array}
$$

Note that g respects identity arrows and composition, so it is a functor (though we still don't know that it sends cartesian arrows to cartesian arrows). By the way we have defined g, the α_U glue together to give us a (base-preserving) natural isomorphism

$\alpha : \mathrm{id}_{\mathcal{G}} \to f \circ g$. Given any $x \in \mathcal{F}$, this gives us an isomorphism $\alpha(f(x)) : f(x) \to f\big(g(f(x))\big)$ with $p_{\mathcal{G}}\big(\alpha(f(x))\big) = \mathrm{id}_{p_{\mathcal{G}}(f(x))}$. By full faithfulness of f, $\alpha(f(x)) = f(\beta(x))$ for some unique isomorphism $\beta(x) : x \to g(f(x))$ with $p_{\mathcal{F}}(\beta(x)) = \mathrm{id}_{p_{\mathcal{F}}(x)}$. Since $f(\beta)$ is a natural transformation, $\beta : \mathrm{id}_{\mathcal{F}} \to g \circ f$ is a natural transformation (base-preserving by construction). Thus, g is an inverse to f.

Finally, we must check that g is a morphism of fibered categories (i.e. that it takes cartesian arrows to cartesian arrows). Let $\phi : y \to y'$ be a cartesian arrow in $\mathcal{G}$, let $\psi : z \to g(y')$, and let $h : p_{\mathcal{F}}(z) \to p_{\mathcal{F}}(g(y))$ such that $p_{\mathcal{F}}(\psi) = p_{\mathcal{F}}(g(\phi)) \circ h$. We'd like to show that we can fill in the dashed arrow on the left uniquely.

Applying f the the diagram on the left, we get the diagram on the right. Since ϕ is cartesian, there is a unique way to fill in the dashed arrow in the diagram on the right. Since f is fully faithful, there is a unique way to fill in the dashed arrow in the diagram on the left. $\qquad\square$

22   The 2-Yoneda lemma

Theorem 22.1 (2-Yoneda Lemma). *Let $p : \mathcal{F} \to \mathsf{C}$ be a fibered category, and let $X \in \mathsf{C}$. Then the "evaluation" functor $e_X : \mathrm{HOM}_{\mathsf{C}}(\mathsf{C}/X, \mathcal{F}) \to \mathcal{F}(X)$, given by $(f : \mathsf{C}/X \to \mathcal{F}) \mapsto f(\mathrm{id}_X)$, is an equivalence of categories.*

Proof. We need to find a quasi-inverse $\eta : \mathcal{F}(X) \to \mathrm{HOM}_{\mathsf{C}}(\mathsf{C}/X, \mathcal{F})$. Given $x \in \mathcal{F}(X)$, we need to define a morphism of fibered categories $\eta_x : \mathsf{C}/X \to \mathcal{F}$. Given $\phi : Y \to X$, choose a cartesian arrow $\phi^* x \to x$ over ϕ. Then we define $\eta_x(\phi)$ to be $\phi^* x$. This defines η_x on objects. Given an X-morphism $\varepsilon : Y' \to Y$, we should get $\eta_x(\varepsilon) : \phi'^* x \to \phi^* x$, and we do.

The uniqueness of the dashed arrow implies that η_x respects composition and identities. Note that η_x is a morphism of fibered categories (it respects the projections and sends cartesian arrows to cartesian arrows [[★★★ we didn't really check this]]). [[★★★ we still didn't say what η does to arrows.]]

Now we'll check that these functors are inverses. Consider

$$\mathcal{F}(X) \xrightarrow{\ \eta\ } \mathrm{HOM}_{\mathsf{C}}(\mathsf{C}/X, \mathcal{F}) \xrightarrow{\ e_X\ } \mathcal{F}(X)$$

$$x \longmapsto \eta_x \longmapsto \eta_x(\mathrm{id}_X) \ \ = \ \ (\mathrm{id}_X)^* x \cong x$$

The unique isomorphism $\mathrm{id}^* x \cong x$ exists because $x \xrightarrow{\mathrm{id}} x$ is a pullback along id_X. Since the behavior of η on morphisms was determined using a universal property, it is easy to check that $e_X \circ \eta$ is isomorphic to the identity on morphisms as well.

Now the other direction.

$$\mathrm{HOM}_{\mathsf{C}}(\mathsf{C}/X, \mathcal{F}) \xrightarrow{\ e_X\ } \mathcal{F}(X) \xrightarrow{\ \eta\ } \mathrm{HOM}_{\mathsf{C}}(\mathsf{C}/X, \mathcal{F})$$

$$f \longmapsto f(\mathrm{id}_X) \longmapsto \eta(f(\mathrm{id}_X))$$

The morphism $\eta(f(\mathrm{id}_X))$ is given by $(\phi : Y \to X) \mapsto \phi^* f(\mathrm{id}_X)$. So we want a natural isomorphism $f(\phi) \cong \phi^* f(\mathrm{id}_X)$. Since $\mathrm{id}_X : X \to X$ is the initial object in C/X, there exists a unique morphism $\phi \to \mathrm{id}_X$, and this is a cartesian arrow (just like every other arrow in C/X). We know that f takes cartesian arrows to cartesian arrows, so $f(\phi) \to f(\mathrm{id}_X)$ is cartesian, and we also have that $\phi^* f(\mathrm{id}_X) \to f(\mathrm{id}_X)$ is also a cartesian arrow (by construction), so there is a unique isomorphism $f(\phi) \to \phi^* f(\mathrm{id}_X)$. □

Presheaves and categories fibered in sets

Definition 22.2. Let $F : \mathsf{C}^{op} \to \mathsf{Set}$ be a functor. Then define $p : F^{fib} \to \mathsf{C}$ to be a fibered category whose objects are pairs (U, x) with $U \in \mathsf{C}$ and $x \in F(U)$. The morphisms $(U', x') \to (U, x)$ are morphisms $f : U' \to U$ such that $Ff(x) = x'$ in $F(U')$. The projection is $(U, x) \mapsto U$. $\diamond$

We can check that $F^{fib} \to \mathsf{C}$ is a fibered category.

If $Ff(x) = x'$ and $\big(F(f \circ h)\big)(x) = x''$, then $Fh(x') = x''$ by the axioms of a presheaf.

One can picture the fibered category F^{fib} as an object for every section of F, with morphisms given by reversing the restriction maps. In particular, the pullback of an object along an arrow is unique (it is given by restriction).

Example 22.3. Let C be the category $x \xrightarrow{\alpha} y \xrightarrow{\beta} z \,\circlearrowleft\, \gamma$ (identity morphisms not drawn), and let $F : \mathsf{C}^{op} \to \mathsf{Set}$ be given by $F(x) = \{a, b, c\}$, $F(y) = \{d, e\}$, $F(z) = \{f, g, h\}$, with $F\alpha(d, e) = (b, c)$, $F\beta(f, g, h) = (d, d, e)$, and $F(f, g, h) = (g, h, h)$. Then F^{fib} looks like this.

Note that every arrow is cartesian, cartesian arrows are unique, and the only morphisms in the fibers are identity morphisms (remember that morphisms in the fiber must lie over the identity morphism). $\diamond$

Example 22.4. $h_X^{fib} = \mathsf{C}/X$. $\diamond$

Definition 22.5. A category is called a *set* if it is a small category in which all morphisms are identity morphisms. Note that any set can be interpreted as a category in this way. $\diamond$

Remark 22.6. A category C is equivalent to a set if and only if for every pair of objects $x, y \in \mathsf{C}$, $\mathrm{Hom}_{\mathsf{C}}(x, y)$ is either empty or consists of a single isomorphism. Such a category is called a *discrete groupoid*. To show that a groupoid is discrete, it is enough to show that no objects have non-identity automorphisms. $\diamond$

Definition 22.7. A *category fibered in sets* over C is a fibered category $p : F \to \mathsf{C}$ such that for every $U \in \mathsf{C}$, the category $F(U)$ is a set. $\diamond$

Note that for a presheaf F, F^{fib} is fibered in sets (its fiber over U is the set $F(U)$).

Lemma 22.8. *Let $G \to \mathsf{C}$ be a fibered category and $F \to \mathsf{C}$ a category fibered in sets. Then $\mathrm{HOM}_{\mathsf{C}}(G, F)$ is a set.*

Proof. We have to check that the only morphisms are identity morphisms. Let $f, g : G \to F$ be two morphisms of fibered categories and let $\alpha : f \to g$ be a base preserving natural transformation. That means that for every $y \in G$, we have a map $\alpha_y : f(y) \to g(y)$ in $F(p_G(y))$, which only has identity morphisms. This implies that $f(y) = g(y)$ and $\alpha_y = \mathrm{id}$. $\square$

Corollary 22.9. *If $X, Y \in \mathsf{C}$, then $\mathrm{HOM}_{\mathsf{C}}(\mathsf{C}/X, \mathsf{C}/Y) \xrightarrow{\sim} (\mathsf{C}/Y)(X)$, which has objects $X \to Y$ and Y-morphisms which project to the identity arrow (so they must be identity maps), so it is just the set $\mathrm{Hom}_{\mathsf{C}}(X, Y)$.*

Proposition 22.10. *The functor* (presheaves) $\to$ (categories fibered in sets) *given by $F \mapsto F^{fib}$ is an equivalence of categories.*

Proof. This is obvious. Here is the inverse:

Given a category fibered in sets $\mathcal{F} \to \mathsf{C}$, define a presheaf $\mathcal{F}_* : \mathsf{C}^{op} \to \mathsf{Set}$ by $U \mapsto \mathrm{HOM}_{\mathsf{C}}(\mathsf{C}/U, \mathcal{F}) \cong \mathcal{F}(U)$ (this is actually a bijection of sets). Given $w : V \to U$ and $\rho : \mathsf{C}/U \to \mathcal{F}$, define $\mathcal{F}_* w(\rho)$ to be the composite $\mathsf{C}/V \xrightarrow{w} \mathsf{C}/U \xrightarrow{\rho} \mathcal{F}$. $\square$

23   Split fibered categories

People like to think of fibered categories $\mathcal{F} \to \mathsf{C}$ as functors $\mathsf{C}^{op} \to \mathsf{Cat}$, with $U \mapsto \mathcal{F}(U)$, but this is really not correct. If $\mathcal{F}$ were a functor, then a morphism $f : V \to U$ would induce a functor $\mathcal{F}f : \mathcal{F}(U) \to \mathcal{F}(V)$, and it doesn't. We'd like to say that for an object $u \in \mathcal{F}(U)$, we should define $\mathcal{F}f(u)$ as "the" pullback f^*u along f, but this requires us to choose a cartesian arrow lying over f, and there may be many of them.

   To fix this problem, we can add some information. A *cleavage* of a fibered category is a choice of cartesian arrow for each arrow in C and each object in the fiber of its target. Given a fibered category and a cleavage, a morphism $f : V \to U$ induces a functor $\mathcal{F}f = f^* : \mathcal{F}(U) \to \mathcal{F}(V)$, but all is not well. The identity morphism $\mathrm{id}_U : U \to U$ does not necessarily produce the identity functor (though this can be fixed by insisting that the cartesian arrow over an identity arrow is an identity arrow), and the functor $(f \circ g)^*$ does not necessarily equal the composite $g^* \circ f^*$ (i.e. your choices of cartesian arrows don't compose well). A *splitting* is a cleavage which doesn't have these problems.

Definition 23.1. Let $\mathcal{F} \to \mathsf{C}$ be a fibered category. A *splitting* of $\mathcal{F}$ is a subcategory $\mathcal{K} \subseteq \mathcal{F}$ such that the following conditions hold.

   1. every arrow in $\mathcal{K}$ is cartesian,

   2. $\mathcal{K}$ contains all objects, and

   3. for every $f : V \to U$ in C and $x \in F(U)$, there is a unique $y \to x$ over f in $\mathcal{K}$.

A *split fibered category* is a fibered category together with a splitting.   ◇

Lemma 23.2. *The category of split fibered categories over C (in which morphisms respect splittings) is equivalent to the category $\mathsf{Fun}(\mathsf{C}^{op}, \mathsf{Cat})$.*

Proof. By the discussion preceding the definition, a fibered category with a splitting defines a functor $\mathsf{C}^{op} \to \mathsf{Cat}$. Conversely, if $F : \mathsf{C}^{op} \to \mathsf{Cat}$ is a functor, then we can define F^{fib} to be the fibered category whose objects are pairs (U, γ) where $U \in \mathsf{C}$ and $\gamma \in F(U)$, in which a morphism $(V, \delta) \to (U, \gamma)$ is a pair (g, α), where $g : V \to U$ is a morphism in C and $\alpha : \delta \to Fg(\gamma)$ is a morphism in $F(V)$. The composition $(W, \varepsilon) \xrightarrow{(h, \beta)} (V, \delta) \xrightarrow{(g, \alpha)} (U, \gamma)$ is taken to be $\big(g \circ h, Fh(\alpha) \circ \beta\big)$.

   Let $\mathcal{K}$ be the subcategory of arrows of the form (g, id), where g is a morphism in C. Let's check that the arrows of $\mathcal{K}$ are cartesian. Let $W \xrightarrow{h} V \xrightarrow{g} U$ be an arrow in C, let $\gamma \in F(U)$, and let $\varepsilon \in F(W)$, with $(g \circ h, \alpha) : (W, \varepsilon) \to (U, \gamma)$ (the first part must be $g \circ h$ if the arrow is to lie over $g \circ h$). Then we wish to show that there is an unique arrow $(W, \varepsilon) \to \big(V, Fg(\gamma)\big)$ lying over h which composes with $(g, \mathrm{id}_{Fg(\gamma)})$ to give $(g \circ h, \alpha)$. Well, since the arrow must lie over h, it must be of the form (h, β) for some $\beta : F(g \circ h)(\gamma) \to \varepsilon$ in $F(W)$. But then we have $(g \circ h, \alpha) = (g, \mathrm{id}) \circ (h, \beta) = (g \circ h, Fh(\mathrm{id}) \circ \beta) = (g \circ h, \beta)$, so we must have $\beta = \alpha$. This shows that F^{fib} is a fibered category and $\mathcal{K}$ is a splitting.

We omit the discussion of how these constructions behave on morphisms and the proof that they are indeed inverses.   □
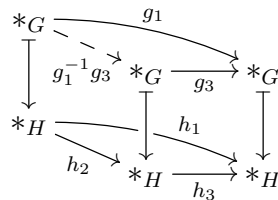
Remark 23.3. If $F : \mathsf{C}^{op} \to \mathsf{Set}$ is a presheaf, we can think of it as a functor to Cat. In this case F^{fib} is exactly the fibered category defined in Definition 22.2, so there is no conflict in the notation.   ◇

Remark 23.4. Similarly, we could show that the category of fibered categories with cleavage (in which morphisms respect cleavage) is equivalent to the category of lax 2-functors $\mathsf{C}^{op} \to \mathsf{Cat}$.   ◇

The following example illustrates that splittings need not exist in general, but Theorem 23.6 tells us that every fibered category is equivalent to one with a splitting.

Example 23.5. A group G can be thought of as a category with one object, whose morphisms are the elements of G (with composition given by multiplication). If G and H are groups, then a functor $p : G \to H$ is the same thing as a homomorphism.
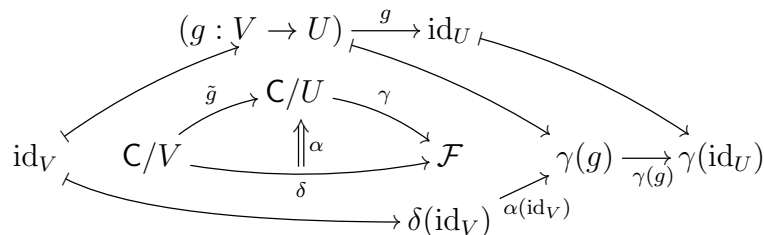
Note that $p : G \to H$ a fibered category if and only if p is surjective: every arrow in G is cartesian (see diagram below), and you need to find an arrow (element) in G lying over every arrow (element) in H.

$$
\begin{array}{ccc}
*_G & \xrightarrow{\;g_1\;} & \\
\downarrow & \searrow^{g_1^{-1}g_3}\;*_G \xrightarrow{\;g_3\;} & *_G \\
*_H & \xrightarrow{\;\;\;} & \downarrow \\
& \searrow_{h_2}\;*_H \xrightarrow{\;h_3\;} & *_H
\end{array}
$$

(2) If p is surjective, then a splitting of $p : G \to H$ is a section $s : H \to G$ of p which is a group homomorphism (the "homomorphism" part follows from the subcategory condition). Such a section may not exist in general.   ◇

Theorem 23.6. *Let $\mathcal{F} \to \mathsf{C}$ be a fibered category. Then there exists a (canonical!) split fibered category $(\tilde{\mathcal{F}}, \mathcal{K})$ and an equivalence $\tilde{\mathcal{F}} \to \mathcal{F}$.*

Proof. Take the $\tilde{\mathcal{F}}$ to be the fibered category associated to the functor $\mathsf{C}^{op} \to \mathsf{Cat}$ given by $U \mapsto \mathrm{HOM}_\mathsf{C}(\mathsf{C}/U, \mathcal{F})$ (as constructed in the proof of Lemma 23.2. An object in $\tilde{\mathcal{F}}$ is of the form (U, γ), with $\gamma \in \mathrm{HOM}_\mathsf{C}(\mathsf{C}/U, \mathcal{F})$, and a morphism $(V, \delta) \to (U, \gamma)$ is a pair (g, α) where $g : V \to U$ and $\alpha : \delta \to \gamma \circ \tilde{g}$ is a base-preserving natural transformation ($\tilde{g} : \mathsf{C}/V \to \mathsf{C}/U$ is the morphism of fibered categories associated to g).

$$
\begin{array}{c}
(g : V \to U) \xmapsto{\;\;g\;\;} \mathrm{id}_U \\[2mm]
\mathrm{id}_V \quad \mathsf{C}/V \xrightarrow[\delta]{\tilde{g}\;\;\mathsf{C}/U\;\;\gamma} \mathcal{F} \quad \gamma(g) \xrightarrow[\gamma(g)]{} \gamma(\mathrm{id}_U) \\[2mm]
\delta(\mathrm{id}_V) \xrightarrow{\alpha(\mathrm{id}_V)}
\end{array}
$$

We define $e : \tilde{\mathcal{F}} \to \mathcal{F}$ by $e(U, \gamma) = \gamma(\mathrm{id}_U : U \to U)$, and $e(g, \alpha) = \gamma(g) \circ \alpha(\mathrm{id}_V)$. This is a morphism of fibered categories. It is easy to see that the fibers $\tilde{\mathcal{F}}(U)$ are $\mathrm{HOM}_{\mathsf{C}}(\mathsf{C}/U, \mathcal{F})$ and that e restricts to the evaluation map on fibers. By the 2-Yoneda lemma, e is an equivalence on each fiber. By Proposition 21.15, e is an equivalence of fibered categories. $\qquad\square$

Warning 23.7. You may be thinking to yourself, "every fibered category is equivalent to a split fibered category, and split fibered categories are equivalent to functors $\mathsf{C}^{op} \to \mathsf{Cat}$, so the category of fibered categories over C is equivalent to the category $\mathsf{Fun}(\mathsf{C}^{op}, \mathsf{Cat})$." This is WRONG! The *category of fibered categories* is not equivalent to the *category of split fibered categories*. It is true that every fibered category is equivalent to split fibered category, but a morphism of fibered categories need not respect the splitting. That is, $\mathsf{Fun}(\mathsf{C}^{op}, \mathsf{Cat})$ injects into the category of fibered categories over C, and the injection is faithful and essentially surjective, but is is not *fully faithful*; there are extra morphisms. $\qquad\lrcorner$

Example 23.8. Consider the fibered category $G = \mathbb{Z}/4 \to \mathbb{Z}/2 = H$. Just for fun, let's calculate $\tilde{G}$. First we need to calculate the fibered category $H/*_H \to H$, which is pretty easy; there are two elements, $x_0 = \mathrm{id} : *_H \xrightarrow{0} *_H$ and $x_1 : *_H \xrightarrow{1} *_H$, and the morphisms are pretty straightforward too. It is pictured on the left. The morphism $x_1 \to x_0$ should be labeled 1, but we call it 1^{-1} for ease of reference.
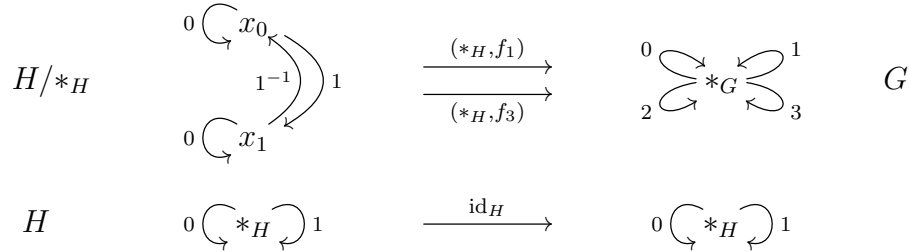

The objects of $\tilde{G}$ are pairs (U, γ), where $U = *_H$ and $\gamma \in \mathrm{HOM}_H(H/*_H, G)$. There are two such objects, $(*_H, f_i)$, where f_i sends the morphism 1 to the morphism i (then it must send 1^{-1} to $4 - i$); since f_i must be base-preserving, i may be 1 or 3. The morphisms in the fiber $\tilde{G}(*_H)$ are base-preserving natural transformations $\eta : (*_H, f_i) \to (*_H, f_j)$. Such a transformation consists of two maps, η_0 and η_1, such that $\eta_0 + j = \eta_1 + i$.

Moreover, the condition "base-preserving" forces η_0 and η_1 to be 0 or 2. If $\eta_0 = k$ and $\eta_1 = l$, then we will write the morphism $(0, \eta)$ in $\tilde{G}$ as kl.

The morphism $1 : *_H \to *_H$ induces the automorphism of $H/*_H$ which switches x_0 and x_1. In particular, pre-composing with this automorphism switches f_1 and f_3. If $\eta : f_i \to f_j \circ \tilde{1}$ has $\eta_0 = k + i$ and $\eta_1 = l - i$, then we write the morphism $(1, \eta)$ in $\tilde{G}$ as kl (this weird notation will make the composition and evaluation more transparent).
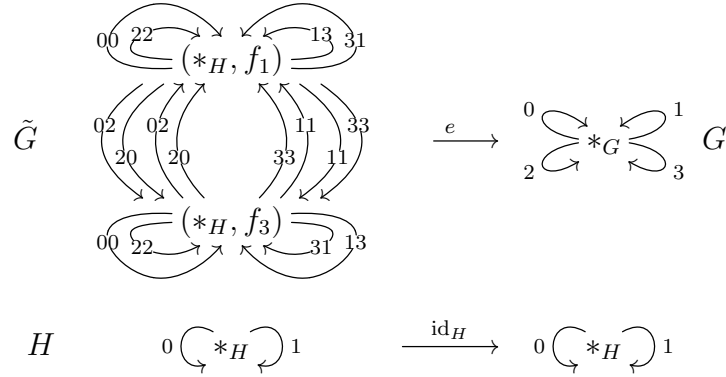
The morphisms in $\tilde{G}$ compose by adding modulo 4 (in the obvious way). The evaluation map e sends ij to i.

The upshot is that if you choose a splitting, you really have no idea what's going on. ◇

24   Stacks

Let C be a site. For simplicity, assume that coproducts are representable in C (so we can always replace coverings by a single map) [[★★★ does that follow from the axioms of a site that a coproduct of maps in a cover is a cover?]] Let $\mathcal{F}$ be a fibered category over C. Choosing some cleavage of $\mathcal{F}$ (one always exists by the axiom of choice), we get a lax 2-functor $\mathcal{F} : \mathsf{C}^{op} \to \mathsf{Cat}$. For a morphism $q : V \to U$ in C, we define $\mathcal{F}(U \to V)$ as in Definition 7.1.

**Definition 24.1.** A fibered category $\mathcal{F}$ is a *stack over* C if for every covering $q : V \to U$ in C, the functor $\mathcal{F}(U) \to \mathcal{F}(V \to U)$, given by $u \mapsto (q^*u, \mathrm{can})$, is an equivalence of categories.   ◇

Remark 24.2. For a morphism $q : V \to U$, the category $\mathcal{F}(V \to U)$ depends on the choice of cleavage of $\mathcal{F}$, but it turns out that they are all equivalent. If you like, you can define a stacks independent of cleavage in the following way. For a fibered category $\mathcal{F} \to \mathsf{C}$, and a morphism $q : V \to U$, define $\mathcal{F}_c(V \to U)$ as the category of commuting diagrams in $\mathcal{F}$ like in the picture below; all arrows are cartesian, lying over the indicated arrows in C, σ is an isomorphism (as are the arrows in the fiber over $V \times_U V \times_U V$ which are pullbacks of σ), and the "equalities" are the canonical isomorphisms (e.g. $p_{23}^* p_1^* \cong p_{12}^* p_1^*$; if your cleavage is a splitting, these are actual equalitites). The dotted arrows are dotted to make the picture look less cluttered.
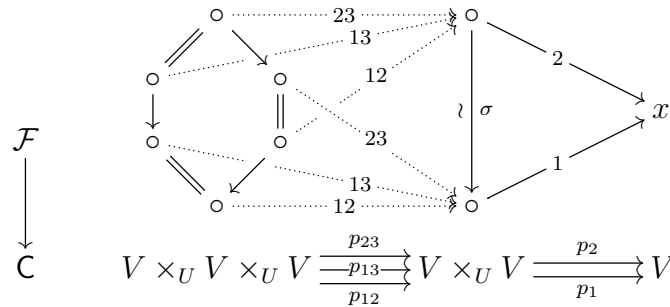
That is, $\mathcal{F}_c(V \to U)$ is the category of all $x \in \mathcal{F}(V)$ with all isomorphisms between the pullbacks along the two projections, and all pullbacks along the other projections which satisfy the cocycle condition. A choice of cleavage picks out some full subcategory (which we called $\mathcal{F}(V \to U)$) of $\mathcal{F}_c(V \to U)$ which contains at least one element from each isomorphism class. This inclusion is fully faithful and essentially surjective (because any two pullbacks of an object are canonically isomorphic), so it is an equivalence.

For an object $U \in \mathsf{C}$, define $\mathcal{F}_c(U)$ to be the category of diagrams
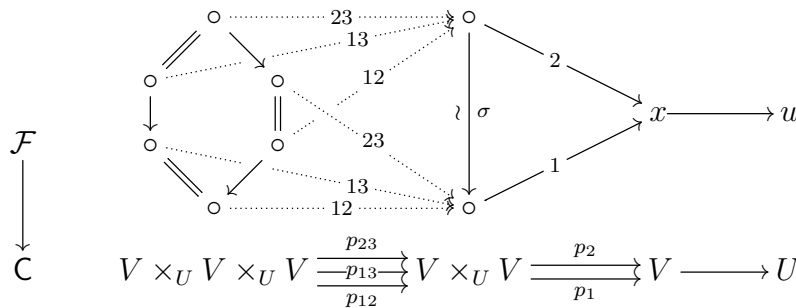
Note that this category depends on the morphism $V \to U$. $\mathcal{F}_c(U)$ is the category of all $u \in \mathcal{F}(U)$, all pullbacks $x \in \mathcal{F}(V)$, all isomorphic pairs of pullbacks along the two projections, and all pullbacks along the other projections which satisfy the cocycle condition. A choice of cleavage picks out some full subcategory (isomorphic to $\mathcal{F}(U)$) with at least one element in each isomorphism class. In particular $\mathcal{F}_c(U)$ is equivalent to $\mathcal{F}(U)$, so it is independent (up to equivalence) of the morphism $V \to U$.

Now $\mathcal{F}$ is a stack if the functor $\mathcal{F}_c(U) \to \mathcal{F}_c(V \to U)$, given by forgetting u, is an equivalence.   ◇

Example 24.3 (Stacks generalize sheaves)**.** If $F : \mathsf{C}^{op} \to \mathsf{Set}$ is a presheaf, then when is F^{fib} is a stack? Recall that objects in F^{fib} are pairs (U, x), where $x \in F(U)$, and morphisms $(U', x') \to (U, x)$ are morphisms $f : U' \to U$ with $Ff(x) = x'$. Thus, we have that $F^{fib}(V \to U) = \{(V, y) | Fp_2(y) = Fp_1(y) \text{ in } F(V \times_U V)\}$, and the pullback functor $F^{fib}(U) \to F^{fib}(f : V \to U)$ is given by $(U, x) \mapsto \big(V, Ff(x)\big)$. This is an equivalence if and only if $F(U) = F^{fib}(U)$ equalizes the two restrictions $F(V) \rightrightarrows F(V \times_U V)$. That is, F^{fib} is a stack if and only if F is a sheaf.   ◇

Example 24.4. When we proved descent for Qcoh and $\mathcal{M}_g$ (with $g \geq 2$ or $g = 0$) in the fppf topology on Sch, we described Qcoh and $\mathcal{M}_g$ as lax 2-functors from Sch^{op} to Cat (i.e. we implicitly chose cleavages). Now we can reinterpret Theorem 7.13 and Proposition 8.2 as saying that Qcoh and $\mathcal{M}_g$ (with $g \geq 2$ or $g = 0$) are stacks in the fppf topology on Sch.   ◇

Remark 24.5. For $g = 1$, we didn't prove a descent theorem for $\mathcal{M}_g$. If we modify the definition of $\mathcal{M}_g$ to have objects $(S, C/S)$, where C is a genus g curve over S, where C is allowed to be an algebraic space (i.e. doesn't have to be a scheme), then descent is almost a tautology. [[★★★ I don't see this tautology yet]] If $S = \operatorname{Spec} k$, then such a C is actually a scheme (Problem set 5, problem 3).   ◇

Definition 24.6. Let $\mathcal{F}$ be a fibered category with some choice of cleavage, and let $x, y \in \mathcal{F}(U)$. The presheaf $\underline{\operatorname{Hom}}(x, y) : (\mathsf{C}/U)^{op} \to \mathsf{Set}$ is defined by $(f : V \to U) \mapsto \operatorname{Hom}_{\mathcal{F}(V)}(f^*x, f^*y)$, and if $g : W \to V$ is a morphism over U, the restriction map is given by

$$\operatorname{Hom}_{\mathcal{F}(V)}(f^*x, f^*y) \xrightarrow{g^*} \operatorname{Hom}_{\mathcal{F}(W)}(g^*f^*x, g^*f^*y) \underset{\mathrm{can}}{\cong} \operatorname{Hom}_{\mathcal{F}(W)}\big((fg)^*x, (fg)^*y\big).$$

Because we define restriction in this way, $\underline{\mathrm{Hom}}(x, y)$ respects composition on the nose, even if the cleavage you choose is not a splitting. Moreover, different cleavages give canonically isomorphic presheaves. ◇

Lemma 24.7. *The following are equivalent.*

1. *For all $U \in \mathsf{C}$ and all $x, y \in \mathcal{F}(U)$, $\underline{\mathrm{Hom}}(x, y)$ is a sheaf.*

2. *For every covering $V \to U$, $\mathcal{F}(U) \to \mathcal{F}(V \to U)$ is fully faithful.*

Proof. Observation: Let $f : V \to U$ be a covering, and let $g = f p_1 = f p_2 : V \times_U V \to U$. Then consider the following sequence.

$$\begin{array}{ccccc}
\underline{\mathrm{Hom}}(x,y)(U) & \longrightarrow & \underline{\mathrm{Hom}}(x,y)(V) & \rightrightarrows & \underline{\mathrm{Hom}}(x,y)(V \times_U V) \\
\| & & \| & & \| \\
\mathrm{Hom}_{\mathcal{F}(U)}(x,y) & \longrightarrow & \mathrm{Hom}_{\mathcal{F}(V)}(f^*x, f^*y) & \rightrightarrows & \mathrm{Hom}_{\mathcal{F}(V \times_U V)}(g^*x, g^*y)
\end{array}$$

An element $\phi \in \underline{\mathrm{Hom}}(x, y)(V)$ whose restrictions along the two projections are equal is exactly a morphism $(f^*x, \mathrm{can}) \to (f^*y, \mathrm{can})$ in $\mathcal{F}(V \to U)$ (after unraveling the funny definition of restriction). Thus, exactness of the above sequence (the sheaf axiom on $\underline{\mathrm{Hom}}(x, y)$ with respect to the cover $V \to U$) is equivalent to full faithfulness of $\mathcal{F}(U) \to \mathcal{F}(V \to U)$.

$(1 \Rightarrow 2)$ This follows immediately from the observation above.

$(2 \Rightarrow 1)$ Let $x, y \in \mathcal{F}(U)$, let $W \to U$ and $f : V \to U$ be objects in C/U, and let $W \to V$ be a covering over U. Then we want to verify the sheaf axiom for $\underline{\mathrm{Hom}}(x, y)$ with respect to the cover $W \to V$. But this is exactly the sheaf axiom for $\underline{\mathrm{Hom}}(f^*x, f^*y)$ with respect to the cover $W \to V$ in C/V. By the observation, this is equivalent to full faithfulness of $\mathcal{F}(V) \to \mathcal{F}(W \to V)$, which is given to us by (2). □

Thus, we may reformulate the stack condition as the following conditions.

1. For every $U \in \mathsf{C}$ and every $x, y \in \mathcal{F}(U)$, $\underline{\mathrm{Hom}}(x, y)$ is a sheaf. That is, given two objects in the fiber over U and a locally defined morphisms which should glue, do glue.

2. (Effectivity of descent) Objects in $\mathcal{F}$ can be defined locally. That is, given a cover $V \to U$ in C, $\mathcal{F}(U) \to \mathcal{F}(V \to U)$ is essentially surjective.

Definition 24.8. A fibered category $\mathcal{F} \to \mathsf{C}$ is a *prestack* if for every $U \in \mathsf{C}$ and every $x, y \in \mathcal{F}(U)$, $\underline{\mathrm{Hom}}(x, y)$ is a sheaf. ◇

Remark 24.9. The terminology is unfortunate because "prestack" is not the generalization of "presheaf"; it is the generalization of "separated presheaf". The generalization of "presheaf" is "fibered category". ◇

25   Groupoids. Stackification.

Definition 25.1. A *groupoid* is a category where all morphisms are isomorphisms.   ◇

We'll give some examples of groupoids in a moment, but first consider the following definition.

Definition 25.2. Let C be a category. We define a *groupoid object in* C to be a 7-tuple $X_\bullet = (X_0, X_1, s, t, i, \varepsilon, m)$, where X_0 (objects) and X_1 (morphisms) are objects in C, with morphisms $s, t : X_1 \to X_0$ (source and target), $\varepsilon : X_0 \to X_1$ (identity map), $i : X_1 \to X_1$ (inverse), and $m : X_1 \times_{s, X_0, t} X_1 \to X_1$ (composition), subject to the following relations.

$$
\begin{array}{ccc}
X_0 \xrightarrow{\ \varepsilon\ } X_1 & X_1 \xrightarrow{\ i\ } X_1 & X_1 \xleftarrow{\ p_1\ } X_1 \times_{s, X_0, t} X_1 \xrightarrow{\ p_2\ } X_1 \\
\varepsilon\downarrow \quad \searrow^{\mathrm{id}} \ \downarrow t & \mathrm{id}\downarrow \quad \swarrow^{i} \ \downarrow s & s\downarrow \qquad\qquad m\downarrow \qquad\qquad \downarrow t \\
X_1 \xrightarrow{\ s\ } X_0 & X_1 \xrightarrow{\ t\ } X_1 & X_0 \xleftarrow{\ s\ } X_1 \xrightarrow{\ t\ } X_0
\end{array}
$$

$$
\text{(inverse)} \qquad\qquad\qquad \text{(identity)}
$$

$$
\begin{array}{ccc}
X_1 \xrightarrow{i \times \mathrm{id}} X_1 \times_{s, X_0, t} X_1 \xleftarrow{\mathrm{id} \times i} X_1 & \quad & X_0 \times_{\mathrm{id}, X_0, t} X_1 = X_1 = X_1 \times_{s, X_0, \mathrm{id}} X_0 \\
s\downarrow \qquad\qquad m\downarrow \qquad\qquad \downarrow t & & \varepsilon\times\mathrm{id}\downarrow \qquad\qquad \| \qquad\qquad \downarrow \mathrm{id}\times\varepsilon \\
X_0 \xrightarrow{\ \varepsilon\ } X_1 \xleftarrow{\ \varepsilon\ } X_0 & & X_1 \times_{s, X_0, t} X_1 \xrightarrow{\ m\ } X_1 \xleftarrow{\ m\ } X_1 \times_{s, X_0, t} X_1
\end{array}
$$

$$
\text{(associativity)}
$$

$$
\begin{array}{ccc}
X_1 \times_{s, X_0, t} X_1 \times_{s, X_0, t} X_1 & \xrightarrow{m \times \mathrm{id}} & X_1 \times_{s, X_0, t} X_1 \\
m\times\mathrm{id}\downarrow & & \downarrow m \\
X_1 \times_{s, X_0, t} X_1 & \xrightarrow{\quad m \quad} & X_1
\end{array}
$$

A *groupoid in spaces* is a groupoid in the category of algebraic spaces (over some scheme S).   ◇

Remark 25.3. A (small) groupoid is the same thing as a groupoid object in Set.   ◇

Example 25.4. A group G, thought of as a category, is a groupoid. An equivalence relation $R \subseteq X \times X$ on a set X can be thought of as a groupoid.

	G	$R \subseteq X \times X$
X_0, X_1	$\{*_G\}, \{g \in G\}$	X, R
s, t	$G \to *$	$p_2, p_1 : R \to X$
ε	identity element $: * \to G$	$\Delta : X \to R$
i	inverse $: G \to G$	"flip" $: R \to R$
m	multiplication $: G \times G \to G$	$X \times X \times X \supseteq R \times_{p_2, X, p_1} R \xrightarrow{p_{13}} R \subseteq X \times X$

In the case of the equivalence relation, the existence of the maps ε, i, and m exactly states that R is reflexive, symmetric, and transitive, respectively. In general, a groupoid $X_\bullet$ is an equivalence relation if and only if $X_1 \xrightarrow{s \times t} X_0 \times X_0$ is an inclusion. $\diamond$

Note that $X_\bullet$ is a groupoid object if and only if for every object $U \in \mathsf{C}$, the 7-tuple $X_\bullet(U) = \big(X_0(U), X_1(U), s(U), t(U), i(U), \varepsilon(U), m(U)\big)$ is naturally a groupoid. That is, a groupoid object $X_\bullet$ is the same as a functor $\mathsf{C}^{op} \to \mathsf{Gpoid}$ whose "object functor" is X_0 and whose "arrow functor" is X_1.

Definition 25.5. Let $X_\bullet$ be a groupoid object in C. Then $[X_\bullet]^{ps}$ is the (split) fibered category over C associated to the functor $X_\bullet : \mathsf{C}^{op} \to \mathsf{Gpoid}$. $\diamond$

Lemma 25.6. *If C is a category with a subcanonical topology (representable functors are sheaves), then $[X_\bullet]^{ps}$ is a prestack.*

Proof. Let $U \in \mathsf{C}$, and let $x, y \in X_0(U)$. We have that U, X_0, and X_1 are sheaves on C, so they are sheaves on C/U (by restriction), so the fibered product P is a sheaf on C/U. For an object $(f : T \to U) \in \mathsf{C}/U$, we can compute $P(T \to U)$ explicitly.

$$
\begin{array}{ccc}
P & \longrightarrow & X_1 \\
\downarrow{\scriptstyle \ulcorner} & & \downarrow{\scriptstyle s \times t} \\
U & \xrightarrow{x \times y} & X_0 \times X_0
\end{array}
\qquad
\begin{aligned}
P(T \xrightarrow{f} U) &= \{\alpha \in X_1(T) | x \circ f = s \circ \alpha, y \circ f = t \circ \alpha\} \\
&= \{\alpha \in X_1(T) | f^*x = s(\alpha), f^*y = t(\alpha)\} \\
&= \{\alpha \in \mathrm{Hom}_{X_\bullet(T)}(f^*x, f^*y)\} \\
&= \underline{\mathrm{Hom}}(x, y)(T \xrightarrow{f} U)
\end{aligned}
$$

Thus, $\underline{\mathrm{Hom}}(x, y) = P$ is a sheaf on C/U, so $[X_\bullet]^{ps}$ is a prestack. $\square$

Example 25.7 ($[X_\bullet]^{ps}$ need not be a stack). If $X_1 = R \rightrightarrows X = X_0$ is an étale equivalence relation, then $X_\bullet(U)$ has objects $x \in X(U)$ and $\mathrm{Hom}(x, y)$ is one point if $x \sim y$ and empty otherwise. It follows that the map $X_\bullet(U) \to X(U)/R(U)$ is an equivalence of categories (the second category is the set of connected components of $X_\bullet(U)$). Thus, $[X_\bullet]^{ps}$ is equivalent to the presheaf X/R. Since X/R is not a sheaf in general, $[X_\bullet]^{ps}$ need not be a stack. $\diamond$

This is not so good, because we want to get the algebraic space quotient X/R, which you only get after sheafifying the presheaf quotient. For this, we need the following proposition, which tells us that we can "stackify" a prestack.

Proposition 25.8 (Stackification). *Let C be a site with coproducts, and let $p : \mathcal{F} \to \mathsf{C}$ be a prestack. Then there exists a morphism of prestacks $\iota : \mathcal{F} \to \tilde{\mathcal{F}}$ with $\tilde{\mathcal{F}}$ a stack, such that for every stack $\mathcal{G}$, the functor $\mathrm{HOM}(\tilde{\mathcal{F}}, \mathcal{G}) \xrightarrow{\iota^*} \mathrm{HOM}(\mathcal{F}, \mathcal{G})$ is an equivalence of categories.*

Remark 25.9. This characterizes $\tilde{\mathcal{F}}$ and $\iota : \mathcal{F} \to \tilde{\mathcal{F}}$ uniquely up to an isomorphism which is unique up to unique isomorphism of morphisms of stacks. $\diamond$

Proof. Choose a cleavage for $\mathcal{F}$ and define $\tilde{\mathcal{F}}$ as follows. The objects of $\tilde{\mathcal{F}}$ are triples $(\pi : V \to U, x, \sigma)$ where $\pi : V \to U$ is a covering and $(x, \sigma) \in \mathcal{F}(V \to U)$. A morphism $(\pi : V \to U, x, \sigma) \to (V' \to U', x', \sigma')$ is a pair $(f, \tilde{f})$, where $f : U \to U'$ is a morphism in C and $\tilde{f} : p^*(\pi, x, \sigma) \to q^*g^*(\pi', x', \sigma')$ is a morphism in $\mathcal{F}(V \times_{U'} V' \to U)$.[1]
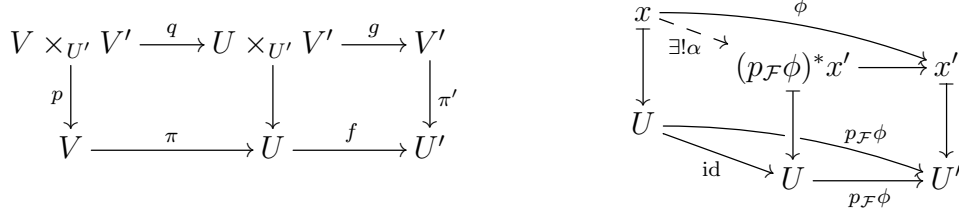
The functor $\iota : \mathcal{F} \to \tilde{\mathcal{F}}$ sends an object $x \in \mathcal{F}(U)$ to $(\mathrm{id} : U \to U, x, \mathrm{can})$ and a morphism $\phi : x \to x'$ to $(p_{\mathcal{F}}\phi, \alpha)$, where α is the dashed arrow in the diagram on the right.

We'll omit the verification that this works. You have effectivity of descent basically for free.[[★★★ I'd like to work it out]]   □

Definition 25.10. $[X_\bullet]$ is the stackification of $[X_\bullet]^{ps}$.   ◇

Remark 25.11. From the proof, we see that ι is fully faithful. This is analogous to the fact that the morphism from a separated presheaf to its sheafification is injective. One can prove a similar theorem which says that you can stackify a fibered category, but then ι will not be fully faithful. [[★★★ does the same construction work for fibered categories, or do you have to pre-stackify before you stackify, like we had to do for sheaves?]]   ◇

Lemma 25.12. *If $\mathcal{F}$ is a prestack in sets (resp. groupoids), then $\tilde{\mathcal{F}}$ is (equivalent to) a stack in sets (resp. groupoids).*

Proof. First let $\mathcal{F}$ be a prestack in sets. Let $U \in \mathsf{C}$, let $\tilde{x} = (\pi : V \to U, x, \sigma), \tilde{x}' = (\pi' : V' \to U, x', \sigma') \in \tilde{\mathcal{F}}(U)$. Then we have that

$$\mathrm{Hom}_{\tilde{\mathcal{F}}(U)}(\tilde{x}, \tilde{x}') = \mathrm{Hom}_{\mathcal{F}(V \times_U V' \to U)}(p^*\tilde{x}, q^*\tilde{x}') \subseteq \mathrm{Hom}_{\mathcal{F}(V \times_U V')}(p^*\tilde{x}, q^*\tilde{x}').$$

Since $\mathcal{F}(V \times_U V)$ is a set, there is at most one element in the right hand side. In general, if an isomorphism in $\mathcal{F}(V \times_U V \to U)$ respects some descent data, so does its inverse. Thus, the one element of $\mathrm{Hom}_{\tilde{\mathcal{F}}(U)}(\tilde{x}, \tilde{x}')$, if it exists, is an isomorphism. Thus, $\tilde{\mathcal{F}}$ is a stack in discrete groupoids, so it is equivalent to a stack in sets.

Similarly, if $\mathcal{F}$ is a prestack in groupoids, then all elements of $\mathrm{Hom}_{\tilde{\mathcal{F}}(U)}(\tilde{x}, \tilde{x}')$ are isomorphisms, so $\tilde{\mathcal{F}}$ is a stack in groupoids.   □

---

[1]Intuitively, $\tilde{f}$ should be a morphism from (π, x, σ) to $g^*(\pi', x', \sigma')$, but to make sense of such a morphism you need to take the common refinement of $V \to U$ and $U \times_{U'} V' \to U$, which is $V \times_U (U \times_{U'} V') = V \times_{U'} V'$.

Corollary 25.13. *If F is a separated presheaf on C and $\tilde{F}$ is the sheafification of F, then $(\tilde{F})^{fib} \cong \widetilde{F^{fib}}$.*

Proof. Since F^{fib} is fibered in sets, $\widetilde{F^{fib}}$ is also fibered in sets, so it is R^{fib} for some sheaf R (by Proposition 22.10). Then for any sheaf G, we have the following natural bijection.

$$\operatorname{Hom}(R, G) = \operatorname{HOM}(R^{fib}, G^{fib}) = \operatorname{HOM}(\widetilde{F^{fib}}, G^{fib}) \xrightarrow{\sim} \operatorname{HOM}(F^{fib}, G^{fib}) = \operatorname{Hom}(F, G)$$

But the unique sheaf R which satisfies such a natural bijection is the sheafification $\tilde{F}$. Thus, we have that $(\tilde{F})^{fib} \cong \widetilde{F^{fib}}$. $\qquad\qquad\square$

Example 25.14 (Example 25.7 continued)**.** Let $R \subseteq X \times X$ be an étale equivalence relation on an algebraic space X. We have shown that $[X/R]^{ps}$ is equivalent to the fibered category associated to the presheaf X/R. By the corollary, the stack $[X/R]$ is equivalent to the fibered category associated to the algebraic space quotient X/R. $\diamond$

26   Quotients by group actions

The construction of the stackification of a prestack is difficult to work with. Corollary 25.13 tells us how to deal with the stackification of a prestack in sets, but we want to understand other stackifications. In practice, you guess what the stackification is, and then use the following lemma.

Lemma 26.1. *Let $\mathcal{F} \to \mathsf{C}$ be a prestack, let $\rho : \mathcal{F} \to \mathcal{G}$ be a morphism to a stack, and assume that*

1. *for every $U \in \mathsf{C}$, the functor $\mathcal{F}(U) \to \mathcal{G}(U)$ is fully faithful, and*

2. *for every $U \in \mathsf{C}$ and every $x \in \mathcal{G}(U)$, there exists a covering $V \to U$ and $y \in \mathcal{F}(V)$ with a cartesian arrow $\rho(y) \to x$ over $V \to U$.*

Then the functor $\tilde{\rho} : \tilde{\mathcal{F}} \to \mathcal{G}$ is an equivalence.

Proof. (Essential surjectivity) Condition (1) implies ρ is fully faithful by Lemma 21.14. Given $x \in \mathcal{G}(U)$, condition (2) produces some cover $\pi : V \to U$ and some $y \in \mathcal{F}(V)$ with $\rho(y) = \pi^* x$. Since $\pi^* x$ comes with descent data with respect to π and since ρ is fully faithful, we get descent data for y. Together with this descent data, y gives an element of $\tilde{\mathcal{F}}$ which maps to x.

(Full faithfulness) By Lemma 21.14, it is enough to check full faithfulness on fibers. Given $x, z \in \tilde{\mathcal{F}}(U)$, we can represent them as elements $\xi, \zeta \in \mathcal{F}(V \to U)$ for some cover $V \to U$ (take a common refinement if necessary). Then we have that $\mathrm{Hom}_{\mathcal{F}(U)}(x, z) = \mathrm{Hom}_{\mathcal{F}(V \to U)}(\xi, \zeta)$,[1]. By condition (1), $\mathrm{Hom}_{\mathcal{F}(V \to U)}(\xi, \zeta) \cong \mathrm{Hom}_{\mathcal{G}(V \to U)}\big(\rho(\xi), \rho(\zeta)\big)$. The sheaf axiom for $\underline{\mathrm{Hom}}$ tells us that $\mathrm{Hom}_{\mathcal{G}(V \to U)}\big(\rho(\xi), \rho(\zeta)\big) \cong \mathrm{Hom}_{\mathcal{G}(U)}\big(\tilde{\rho}(x), \tilde{\rho}(z)\big)$. Thus, the composition $\mathrm{Hom}_{\tilde{\mathcal{F}}(U)}(x, z) \to \mathrm{Hom}_{\mathcal{G}(U)}\big(\tilde{\rho}(x), \tilde{\rho}(z)\big)$ is a bijection. $\square$

Let $\mathsf{C} = \mathsf{Sch}/S$ with the étale topology, let $Y \to S$ be an algebraic space, and let G be a group space over S acting on Y. Then we get the groupoid below, from which we form the pre-stack $[Y_\bullet]^{ps}$, which gives us the stack $[Y_\bullet]$.

$$Y_0 = Y, \qquad Y_1 = Y \times G$$
$$s : Y \times G \to Y, \quad (y, g) \mapsto y \qquad\qquad \varepsilon : Y \to Y \times G, \quad y \mapsto (y, e)$$
$$t : Y \times G \to Y, \quad (y, g) \mapsto gy \qquad\qquad i : Y \times G \to Y \times G, \quad (y, g) \mapsto (gy, g^{-1})$$
$$m : (Y \times G) \times_{s, Y, t} (Y \times G) \to Y \times G, \quad \big((gy, h), (y, g)\big) \mapsto (y, hg)$$

We'd like to get our hands on the stack $[Y_\bullet]$, but we don't want to deal with the stackification construction, so we guess the answer and use the lemma above.

Definition 26.2. Let G be a group object in a site C (in which finite projective limits are representable), let X be an object in C, and let $P \in \mathsf{C}/X$ have a G-action over X. We say that P is a *G-torsor* over X if it is locally trivial (i.e. if there exists a cover $X' \to X$ so that $X' \times_X P \cong X' \times G$). [[★★★ Is this the right definition? everybody keeps saying the words "acts transitively" on something]] ◇

---

[1]Ok, so we should really replace V by $V \times_U V$, but that is confusing.

Definition 26.3. If G is a group space and Y is an algebraic space with a G-action, then $[Y/G]$ is defined to be the fibered category whose objects (over a scheme U) are diagrams $U \leftarrow P \xrightarrow{\rho} Y$, where $P \to U$ is a G-torsor (which may be an algebraic space)) and $\rho : P \to Y$ is a G-equivariant morphism, and whose morphisms are diagrams as below, where $\tilde{f} : P' \to P$ is G-equivariant.

$$
\begin{array}{ccc}
U' & \longleftarrow & P' \\
\scriptstyle f \downarrow & & \scriptstyle \tilde{f} \downarrow \;\; \searrow^{\rho'} \\
U & \longleftarrow & P \;\;\;\; \xrightarrow{\rho} \;\; Y
\end{array}
$$

Note that G-equivariance of $\tilde{f}$ implies that the square is cartesian. In particular, the fibers of $[Y/G]$ are groupoids. $\diamond$

Proposition 26.4. *If the diagonal morphism $G \to G \times G$ belongs to an effective descent class (e.g. if G is a scheme), then $[Y/G]$ is a stack.*

Proof. [[★★★ this proof is quick, does it need to be expanded?]] First note that if $G \to G \times G$ is in some effective descent class, then for any G-torsor $P \to U$, $P \to P \times_U P$ belongs to the same effective descent class. To see this, let $U' \to U$ be an étale cover of U, over which P becomes trival. Then observe that that the diagram below consists of two cartesian squares.

$$
\begin{array}{ccccc}
U' & \longleftarrow & U' \times G & \xrightarrow{\mathrm{id}_{U'} \times \Delta_G} & U' \times G \times G & \cong & (U' \times G) \times_{U'} (U' \times G) \\
\downarrow & & \downarrow & & \downarrow \\
U & \longleftarrow & P & \xrightarrow{\;\;\Delta\;\;} & P \times_U P
\end{array}
$$

Now let $U' \to U$ be an étale cover, and let $U' \leftarrow P' \xrightarrow{\rho'} Y$ be an element of $[Y/G](U')$ together with descent data for the morphism $U' \to U$. By descent for algebraic spaces (Theorem A4.3), P' is the pull-back of an algebraic space P over U. Moreover, we get the descent of all the diagrams that make P' a G-torsor, so P is a G-torsor over U. Finally, since h_Y is a sheaf on AlgSp with the étale topology, we get a G-equivariant morphism $P \to Y$. $\square$

Proposition 26.5. $[Y_\bullet] \cong [Y/G]$.

Proof. We define a morphism of fibered categories $[Y_\bullet]^{ps} \to [Y/G]$ by sending $y \in Y(U)$ to the trivial torsor $U \times G \to U$ with the G-equivariant morphism $U \times G \to Y$ given by $(u, g) \mapsto gy$. We see that $[Y_\bullet]^{ps}(U) \to [Y/G](U)$ is fully faithful with essential image equal to the subcategory of diagrams $U \leftarrow P \to Y$ where P is a trivial torsor [[★★★ exercise]]. Since any torsor is étale locally trivial, Lemma 26.1 says that the induced map $[Y_\bullet] \to [Y/G]$ is an isomorphism. $\square$

Example 26.6. Let Y be a scheme and let G be a finite group which acts freely on Y. Then $[Y/G]$ (the stack quotient) is isomorphic to Y/G (the algebraic space quotient). Define a map $\sigma : [Y/G] \to Y/G$ as follows. If U is a scheme and P is a G-torsor over U with $P \to Y$ G-equivariant, then we get a map $U = P/G \to Y/G$, and this is natural in P and U, so it gives us a map σ. To go in the other direction, if we have $U \to Y/G$, then we get a G-torsor $U \times_{Y/G} Y$ over Y with a G-equivariant projection to Y. That is, $Y/G \leftarrow Y \xrightarrow{\text{id}} Y$ is an object in $[Y/G](Y/G)$.   ◇

Example 26.7. The non-free actions are more interesting. If $Y = X$, with the trivial G-action, then we call $[Y/G]$ the *classifying stack $B_X G$*.   ◇

A while back, we proved that if C is a category, then presheaves on C are equivalent to categories fibered in sets. If $\mathcal{F}$ is any fibered category over C in groupoids, you can define $\bar{\mathcal{F}}$ to be the presheaf given by $U \mapsto \{\text{iso-classes in } \mathcal{F}(U)\}$. There is a morphism of fibered categories $\mathcal{F} \to \bar{\mathcal{F}}$ [[★★★]]. When is this an equivalence? Given two objects, there must exist at most one arrow between them in $\mathcal{F}(U)$ (i.e. $\mathcal{F}$ is discrete). That is, there are no non-trivial automorphisms.

Example 26.8. In the case of $U \leftarrow P \xrightarrow{\rho} Y$, we need to check that there are no non-trivial automorphisms $\alpha : P \to P$ such that $\rho \circ \alpha = \rho$. To check that α must be the identity, we can replace U by a cover and assume P is trivial; choose a section $s : U \to P$. Then $\alpha(s) = gs$ for some $g \in G(U)$. Then we get $\rho(s) = \rho\alpha(s) = \rho(gs) = g\rho(s)$. But G acts freely on $Y(U)$, so we get that $g = \text{id}$.   ◇

27   Algebraic Stacks

Definition 27.1 (Coproducts)**.** If C is a site and $\{\mathcal{X}_i\}_{i \in I}$ are fibered categories over C, then we define $\coprod \mathcal{X}_i$ to be a fibered category with objects $(i \in I, x \in \mathcal{X}_i)$, and
$$\mathrm{Hom}\big((j, y \in \mathcal{X}_j), (i, x \in \mathcal{X}_i)\big) = \begin{cases} \varnothing & i \neq j \\ \mathrm{Hom}_{\mathcal{X}_i}(y, x) & \text{else} \end{cases}. \text{ Note that if all the } \mathcal{X}_i \text{ are}$$
stacks (resp. stacks fibered in groupoids), then so is the coproduct.   ◇

Definition 27.2 (Fiber Products)**.** If $\mathcal{Z} \xrightarrow{F} \mathcal{Y} \xleftarrow{G} \mathcal{X}$ is a diagram of fibered categories over C, then define $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ to be the fibered category over C whose objects are 4-tuples (U, x, z, η), where $U \in \mathsf{C}$, $x \in \mathcal{X}(U)$, $z \in \mathcal{Z}(U)$, and $\eta : Fz \xrightarrow{\sim} Gx$ is an isomorphism in $\mathcal{Y}(U)$. A morphism $(U', x', z', \eta') \to (U, x, z, \eta)$ is a pair of morphisms $\chi : x' \to x$ in $\mathcal{X}$ and $\zeta : z' \to z$ in $\mathcal{Z}$ lying over the same morphism $U' \to U$ in C, so that the following diagram in $\mathcal{Y}$ commutes.

$$\begin{array}{ccc} Fz' & \xrightarrow{\eta'} & Gx' \\ {\scriptstyle F\zeta}\downarrow & & \downarrow{\scriptstyle G\chi} \\ Fz & \xrightarrow{\eta} & Gx \end{array}$$
◇

One can check that if $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ are stacks (resp. stacks in groupoids), then so is $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$.[[★★★ exercise]]

We saw in Lemma A5.11 that fiber products exist in the category of categories. One can see that the fibered product defined above has the property that for any fibered category $\mathcal{F}$ over C, the functor $\mathrm{HOM}_{\mathsf{C}}(\mathcal{F}, \mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}) \to \mathrm{HOM}_{\mathsf{C}}(\mathcal{F}, \mathcal{X}) \times_{\mathrm{HOM}_{\mathsf{C}}(\mathcal{F}, \mathcal{Y})} \mathrm{HOM}_{\mathsf{C}}(\mathcal{F}, \mathcal{Z})$ is an equivalence of categories, and that this functor being an equivalence of categories is exactly the condition that $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is the fibered product in the sense of Definition A5.8. That is, *fiber products are representable in the 2-category of stacks over* C.

Example 27.3. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of fibered categories, then we can form $\mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, and we get the diagonal $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, given by $x \mapsto (x, x, \mathrm{id}_{f(x)})$.   ◇

The basic model for the rest of the course are things of the form $[X_{\bullet}]$, in which all the fibers are groupoids. ***From now on, "stack" will mean "stack in groupoids" unless otherwise stated.***

Remark 27.4. Being a category fibered in groupoids is equivalent to all arrows being cartesian. Suppose $f : V \to U$, with $z \to u$ over f. There is some cartesian arrow $f^*u \to u$, and $z \to u$ factors through it by the cartesian property. Since the fiber over V is a groupoid, we get that $z \cong f^*u$, so $z \to u$ is cartesian.

Conversely, if every arrow is cartesian, then any arrow in the fiber is a pullback along the identity morphism, so it is isomorphic to the identity pullback (i.e. it is an isomorphism; in fact, any pullback along an isomorphism is an isomorphism).   ◇

Remark 27.5. Every base-preserving functor to a category fibered in groupoids takes cartesian arrows to cartesian arrows, so it is a morphism of fibered categories.   ◇

Notation: All morphisms in fibers are now isomorphisms, so instead of $\underline{\mathrm{Hom}}(x, y)$ (see Definition 24.6) we'll write $\underline{\mathrm{Isom}}(x, y)$.

Example 27.6. Let $\mathcal{X}$ be a stack in groupoids over C, and U and V be object in C (which are sheaves), and let $u \in \mathcal{X}(U)$, $v \in \mathcal{X}(V)$. By the 2-Yoneda lemma, we may think of these as morphisms $u : U \to \mathcal{X}$ and $v : V \to \mathcal{X}$ (where we interpret U and V as their associated fibered categories, C/U and C/V). If $p_U : U \times V \to U$ and $p_V : U \times V \to V$ are the projections, then we see that $U \times_{\mathcal{X}} V = \underline{\mathrm{Isom}}(p_U^* u, p_V^* v)$ (we can think of this as a fibered category over $\mathsf{C}/(U \times V)$, which is fibered over C, so we can think of $\underline{\mathrm{Isom}}(p_U^* u, p_V^* v)$ as fibered over C) [[★★★ does this need to be expanded?]]. As usual, this is independent of cleavage up to equivalence.

Note that

1. $U \times_{\mathcal{X}} V$ is fibered in sets (because it is the sheaf $\underline{\mathrm{Isom}}(p_U^* u, p_V^* v)$).

2. $U \times_{\mathcal{X}} V \to U \times V$ need not be a monomorphism. That is, given $g : T \to U \times V$, there may be more than one isomorphism between $g^* p_U^* u$ and $g^* p_V^* v$ in the fiber $\mathcal{X}(T)$.   ◇

Definition 27.7. A stack is *representable* if it is equivalent to an algebraic space. A morphism of stacks $f : \mathcal{X} \to \mathcal{Y}$ is *representable* if for every scheme U over S and every $u : U \to \mathcal{Y}$, the fiber product $U \times_{\mathcal{Y}} \mathcal{X}$ is representable.   ◇

Definition 27.8. Let $\mathcal{P}$ be an (étale) stable property of morphisms of algebraic spaces. Then a representable morphism of stacks $f : \mathcal{X} \to \mathcal{Y}$ *has* $\mathcal{P}$ if for every scheme U over S and $u : U \to \mathcal{Y}$, the map $U \times_{\mathcal{Y}} \mathcal{X} \to U$ has $\mathcal{P}$.   ◇

Example 27.9. surjective, universally open (or closed), separated, quasi-compact, locally of finite type, flat, smooth, etc.   ◇

Definition 27.10. A stack in groupoids $\mathcal{X}$ over $(\mathsf{Sch}/S)_{et}$ is *algebraic* if the following hold.

1. $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable. [[★★★ does this imply that $\mathcal{X}$ is a stack *in groupoids*? probly not; it's just that groupoids are the next level up from sets ...next would be simplicial objects with no homotopy higher than 2?]]

2. There exists a scheme X over S and a smooth surjection $X \to \mathcal{X}$. This makes sense because condition (1) implies that any morphism from a scheme to $\mathcal{X}$ is representable.

◇

Remark 27.11. This definition of representability of a morphism of stacks is unfortunate. It really should say "$\mathcal{X} \to \mathcal{Y}$ is representable if for every *algebraic space U* and every $u : U \to \mathcal{Y}$, the fiber product $U \times_{\mathcal{Y}} \mathcal{X}$ is equivalent to an algebraic space". Consider the following variations of condition (2) in the definition above.

1. $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable,

2. for every scheme U, and every $x, y \in \mathcal{X}(U)$, $\underline{\text{Isom}}(x, y)$ is an algebraic space,

3. for every scheme U, every $u : U \to \mathcal{X}$ is representable,

4. for every algebraic space U, every $u : U \to \mathcal{X}$ is representable.

We saw in Example 27.6 that (2) is equivalent to (3), and (1) implies (3) (the proof of Lemma 10.15 works). To see that (3) implies (1), observe that for a morphism from a scheme $f \times g : T \to \mathcal{X} \times \mathcal{X}$, we have that $T \times_{\mathcal{X} \times \mathcal{X}} \mathcal{X} \cong (T \times_{f, X, g} T) \times_{T \times T} T$. It is clear that (4) implies (3). Under certain circumstances [[★★★ which are?]], we will see that (3) implies (4). ◇

28   More about Algebraic Stacks; Examples

Lemma 28.1. *Let $f : \mathcal{X} \to X$ be a representable morphism from a stack to an algebraic space. Then $\mathcal{X}$ is an algebraic space.*

Proof. First we check that $\mathcal{X}$ is equivalent to a sheaf (i.e. that it is fibered in discrete groupoids). To do this, it is enough to show that there are no non-identity automorphisms in the fibers of $\mathcal{X}$. Let $x \in \mathcal{X}(T)$ for some scheme T, thought of as a morphism $x : T \to \mathcal{X}$.

(1) Let $F : (\mathsf{Sch}/S)^{op}$ be the presheaf given by $(T \to X) \mapsto \{$isomorphism classes in $\mathcal{X}(T)\}$. Then $\mathcal{X} \to F^{fib}$ is an equivalence. That is, $\mathcal{X}(T) \to \pi_0\big(\mathcal{X}(T)\big)$ is an equivalence of categories ([[★★★ exercise]] it is enough to check that objects of $\mathcal{X}(T)$ have no non-trivial automorphisms). To see this, let $x \in \mathcal{X}(T)$, then we have

$$(x, \mathrm{id}, \mathrm{id}) \; \in \; \begin{array}{ccc} \mathcal{X} \times_X T & \longrightarrow & T \\ \downarrow & & \downarrow {\scriptstyle f(x)} \\ \mathcal{X} & \longrightarrow & X \end{array}$$

An automorphism of x gives an automorphism of $(x, \mathrm{id}, \mathrm{id})$, which can't happen because the product is an algebraic space (so fibered in sets).

(2) Check that F is an algebraic space. It is a sheaf for free.

(a) $F \to F \times_X F$ is representable (by schemes). Let $T \to F \times F$ be a morphism from a scheme, then $T \times_{F \times F} F \cong (F \times_X T) \times_{(F \times_X T) \times_T (F \times_X T)} T$, so it is a scheme (because $F \times_X T$ is an algebraic space, so its diagonal is representable). This implies that $F \to F \times F$ is representable [[★★★ as in appendix]]

(b) Let $U \to X$ be an étale cover by a scheme, let $V = F \times_X U$ (which is an algebraic space), and let $V' \to V$ be an étale cover by a scheme. Then $V' \to F$ is an étale cover.   □

Corollary 28.2. *If $\mathcal{X}$ is an algebraic stack, then any morphism $x : X \to \mathcal{X}$ from an algebraic space is representable.*

Proof. Let T be a scheme with a morphism to $\mathcal{X}$.

$$\begin{array}{ccc} P_U & \longrightarrow & U \\ \downarrow & & \downarrow \\ P & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & \mathcal{X} \end{array}$$

We see that $P \to X$ is representable (for any U, P_U is an algebraic space because it is $U \times_{\mathcal{X}} T$), so P is an algebraic space by the lemma.   □

Example 28.3. This is a main source of algebraic stacks. Let Y/S be an algebraic space, and let G/S be a smooth group scheme. Then $\mathcal{X} = [Y/G]$ is an algebraic stack. Recall that $[Y/G](T \to S)$ is the groupoid of diagrams $T \xleftarrow{G_T} P \xrightarrow{\rho} Y$, where $P \to T$ is a G-torsor and $P \to Y$ is G-equivariant.

Proof. Representability of the diagonal: consider

$$
\begin{array}{ccc}
\underline{\mathrm{Isom}}\big((P_1, \rho_1), (P_2, \rho_2)\big) & \longrightarrow & T \\
\downarrow & & \downarrow {\scriptstyle (P_1,\rho_1)\times(P_2,\rho_2)} \\
\mathcal{X} & \xrightarrow{\quad\Delta\quad} & \mathcal{X} \times \mathcal{X}
\end{array}
\qquad
\begin{array}{c}
P_1 \\
\\
P_2 \xrightarrow{\rho_2} Y \\
\\
T
\end{array}
$$

(with $\rho_1 \colon P_1 \to Y$ and projections to T)

$I : (\mathsf{Sch}/T)^{op} \to \mathsf{Set}$ is given by $(T' \to T) \mapsto \{\sigma : P_{1,T'} \xrightarrow{\sim} P_{2,T'}$ such that $\rho_1 = \rho_2 \circ \sigma\}$. To show that I is an algebraic space, it is enough to consider the case where there exist sections $s_1 : T \to P_i$ (because we can work étale locally, and torsors are locally trivial).

We see that $I \subseteq G_T$. G_T is in bijection with isomorphisms $\sigma : P_1 \to P_2$, given by $\sigma \mapsto$ the unique g such that $\sigma(s_1) = gs_2$. We have

$$
\begin{array}{ccc}
I & \longrightarrow & Y \\
\downarrow & \ulcorner & \downarrow \\
G_T & \longrightarrow & Y \times Y \\
\downarrow & & \\
T & &
\end{array}
\qquad
g \mapsto \big(\rho_1(s_1), g\rho_2(s_2)\big)
$$

(Aside: If Y is separated, then I is a closed subscheme of G_T. If G was affine, we'd get descent for schemes, so I would be a scheme.)

Smooth cover: $Y \to [Y/G]$. What does Y represent? If $T \to [Y/G]$, what does it mean for the map to factor through Y? It is the same as choosing a section $T \to T \times_{[Y/G]} Y$. Thus, Y represents the functor of triples (P, ρ, s), where s is a trivialization of the torsor P. $\diamond$

$\diamond$

Example 28.4. If $Y = \operatorname{Spec} k$ and G is a smooth group scheme, then the universal torsor is $\operatorname{Spec} k \to [\operatorname{Spec} k/G]$, which looks kind of boring, but thats what it is. $\diamond$

Example 28.5 (Weighted projective stack)**.** Let $\alpha_0, \ldots, \alpha_n \in \mathbb{Z}$. Then we define the *weighted projective stack* $\tilde{\mathbb{P}}(\alpha_0, \ldots, \alpha_n) = [(\mathbb{A}^{n+1} \smallsetminus \{0\})/\mathbb{G}_m]$, where the action is given by $u \cdot (x_0, \ldots, x_n) = (u^{\alpha_0} x_0, \ldots, u^{\alpha_n} x_n)$. $\diamond$

Application to $\mathcal{M}_{1,1}$. Recall that objects of $\mathcal{M}_{1,1}$ are pairs $(S,\ E \overset{\leftharpoonup}{\to} S\,)$ and morphisms are cartesian diagrams.

$$
\begin{array}{ccc}
E' & \longrightarrow & E \\
\updownarrow\!\!\uparrow & & \updownarrow\!\!\uparrow \\
S' & \longrightarrow & S
\end{array}
$$

Goal: see this in a weighted projective stack (in fact, $\overline{\mathcal{M}}_{1,1} \cong \tilde{\mathbb{P}}(4,6)$ over $\mathbb{Z}[1/6]$).

Let's start with $Y : (\mathsf{Sch}/\mathbb{Z}[1/6])^{op} \to \mathsf{Set}$, given by $S \mapsto \{(E/S, e, b\ :\ \mathcal{O}_S \xrightarrow{\sim} \omega_{E/S})\}/\cong$, where $\omega_{E/S} = f_*\Omega^1_{E/S}$ (this is locally free of rank 1 and commutes with arbitrary base change), and where an isomorphism of elliptic curves $\sigma\ :\ E \xrightarrow{\sim} E'$ induces an isomorphism $f_*(\sigma) : \omega_{E/S} \xrightarrow{\sim} \omega_{E'/S}$, and we only allow the isomorphisms which work well with b and b'. In fact, we could make a category, but it turns out that that category is equivalent to this set.

As a stack, $\mathcal{M}_{1,1}$ should be $Y/\mathbb{G}_m$ because b is unique up to the action of $\mathbb{G}_m$.

Proposition 28.6. *Y is represented by $Y' = \operatorname{Spec}\mathbb{Z}[1/6][g_2, g_3][1/\Delta] \subseteq \mathbb{A}^2_{\mathbb{Z}[1/6]}$, where $\Delta = g_2^3 - 27g_3^2$.*

The action of $\mathbb{G}_m$ is given by $g_2 \mapsto u^4 g_2$, $g_3 \mapsto u^6 g_3$. We'll see that $\mathcal{M}_{1,1} = [Y/\mathbb{G}_m]$.

Proof. (This is in [Har77]) Define $E' \overset{e}{\underset{\leftharpoonup}{\to}} Y'$ by $(2y)^2 = 4x^3 - g_2 x - g_3$, with $b' = -dx/2y$ giving us an isomorphism $\mathcal{O}_{Y'} \xrightarrow{\sim} f_*\Omega^1_{E'/Y'}$. So this gives us a morphism of functors $Y' \to Y$. To check that it is an isomorphism, we need to show that for every scheme $S = \operatorname{Spec} A$ (we can assume affine because we could define it as a stack and check sheafy, so we can work locally) and $(E, e, b) \in Y(S)$, there exists a unique g_2, g_3 such that (E, e, b) is given by $(E', e', b')_{(g_2, g_3)}$.

We have $E \overset{e}{\underset{\leftharpoonup}{\to}} \operatorname{Spec} A$. Let $\hat{\mathcal{O}}_{E,e} = \varprojlim \mathcal{O}_E/\mathcal{I}_e^n \cong A[[T]]$. Choose T such that $b = (1+\text{higher terms}) \cdot dT$; this choice is unique up to $T \mapsto T+\text{higher terms}$. We have that $f_*\mathcal{I}_e^n$ is locally free of rank n for $n \geq 2$. Shrink a little to make it free for $n = 2$. Choose a basis $1, x$ for $f_*\mathcal{I}_e^2$ so that $x = \frac{1}{T^2}(1+\text{higher})$, and choose a basis $1, x, y$ for $f_*\mathcal{I}_e^3$ so that $y = \frac{1}{T^3}(1+\text{higher})$. Then you get that you can write $y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$. Exercise: There is a unique choice of y such that $a_1 = a_3 = 0$ and a unique choice of x so that $a_2 = 0$. Thus, we get $y^2 = x^3 + a_4 x + a_6$. Let $g_2 = -4a_4$ and $g_3 = -4a_6$.

It remains to see what the action is. Let $u \in \mathbb{G}_m(A) = A^\times$. Then T is replaced by uT, x gets replaced by $u^{-2}x$, and y gets replaced by $u^{-3}y$. So g_2 gets replaced by $u^4 g_2$ and g_3 gets replaced by $u^6 g_3$. $\qquad\square$

Upshot: $\mathcal{M}_{1,1} \cong [Y/\mathbb{G}_m]$.

29   Hilb and Quot

Recall that last time we saw that $\mathcal{M}_{1,1}$ was algebraic over $\mathbb{Z}[1/6]$. We did this by finding some open subset $U \subseteq \mathbb{A}^2$ which was invariant under the $(4,6)$ $\mathbb{G}_m$ action on $\mathbb{A}^2$. And we saw that $\mathcal{M}_{1,1} \cong [U/\mathbb{G}_m]$. In general, you don't have such a delicate analysis, but this is the way to show algebraicity. You use the Hiblert scheme or the Quot functor.

Aside on the Quot functor. (Grothendieck, Seminars Boubaki, 1960/61, no. 221) Let $f : X \to S$ be a separated morphism of finite presentation. Let $\mathcal{L}$ be relatively ample on X, and let $P \in \mathbb{Q}[z]$ (the Hilbert polynomial). Also, fix a quasi-coherent sheaf $\mathcal{F}$ on X. Then we can define $\mathrm{Quot}^P(\mathcal{F}/X/S) : (\mathsf{Sch}/S)^{op} \to \mathsf{Set}$ by $(S' \to S) \mapsto \Big\{ \mathcal{F}_{S'} \twoheadrightarrow \mathcal{G} | \mathcal{F}_{S'}$ the pullback of $\mathcal{F}$ to $X_{S'} = X \times_S S'$, $\mathcal{G}$ quasi-coherent and locally finitely presented (basically coherent), $\mathcal{G}$ has support proper over S', and for every $s' \in S'$, the Hilbert polynomial (makes sense because may as well replace X by the support [[★★★ I missed some stuff]]) of $\mathcal{G}_{S'} = \mathcal{G}|_{X_{S'}}$ is equal to $P \Big\}/ \cong$, where isomorphism is

$$
\begin{array}{ccc}
\mathcal{F}_{S'} & \!\!\!\!\longrightarrow\!\!\!\!\twoheadrightarrow & \mathcal{G}_1 \\
& \searrow & \wr \downarrow f \\
& & \mathcal{G}_2
\end{array}
$$

Such an iso is unique if it exists because of surjectivity.

Theorem 29.1 (Grothendieck). *If $\mathcal{F}$ is locally finitely presented, $\mathrm{Quot}^P(\mathcal{F}/X/S)$ is a quasi-projective S-scheme (projective if $X \to S$ is proper).*

In fact, you get a nice projective embedding into a Grassmanian.

Remark 29.2. If $\mathcal{F} = \mathcal{O}_X$, then $\mathrm{Quot}^P(\mathcal{F}/X/S)$ is usually written $\mathrm{Hilb}^P_{X/S}$, the *Hilbert scheme.*   ◇

Example 29.3 (Hartshorne's comment). Suppose $\mathcal{F} = \bigoplus \mathcal{O}_X$, then we can take $\mathcal{G}$ to be a quotient of any one of them, so it looks like we'll get an infinite disjoint union. Maybe the theorem only holds if $\mathcal{F}$ is locally finitely presented.   ◇

Remark 29.4. If $f : X \to S$ is locally finitely presented and separated morphism of algebraic spaces, and $\mathcal{F}$ is quasi-coherent on X, then we can still define $\mathrm{Quot}(\mathcal{F}/X/S)$. The ample part is only used for the Hilbert polynomial part. This is an algebraic space and is quasi-proper (satisfies the valuative criterion, but may not be finite something [[★★★]]) if $X \to S$ is proper.   ◇

Example 29.5. If $g \geq 2$, then $\mathcal{M}_g$ is algebraic. The idea is the same as for $\mathcal{M}_{1,1}$. Put some extra structure to get something representable by a scheme and then quotient out the extra structure. If $(\pi : C \to S) \in \mathcal{M}_g(S)$, then one can show (by Riemann-Roch)

– $(\Omega^1_{C/S})^{\otimes 3}$ is relatively very ample.

– $V_C = \pi_*(\Omega^1_{C/S})^{\otimes 3}$ has rank $5(g-1)$.

– $\pi_*(\Omega^1_{C/S})^{\otimes n}$ has rank $(2n-1)(g-1)$ for $n \geq 3$.

This stuff implies that for all $s \in S$, the Hilbert polynomial of C_s is $P = (6z-1)(g-1)$ (here $\mathcal{L} = (\Omega^1_{C/S})^{\otimes 3}$) because $n = 3z$. You have to compute the dimension of $H^0(\mathcal{L}^{\otimes m}) = (6m-1)(g-1)$.

Let $\tilde{\mathcal{M}}_g$ be the functor $\mathsf{Sch}^{op} \to \mathsf{Set}$ given by $S \mapsto \{(C/S, \sigma : \mathbb{P}^{5g-6)} \xrightarrow{\sim} \mathbb{P}V_C)\}/\cong$, where $(C_1, \sigma_1) \cong (C_2, \sigma_2)$ if there is an isomorphism $i : C_1 \xrightarrow{\sim} C_2$ over S such that

$$
\begin{array}{ccc}
\mathbb{P}^{5g-6} \xrightarrow{\sigma_1} \mathbb{P}V_{C_1} & \longleftarrow & C_1 \\
\llap{σ_2}\searrow \quad \downarrow i & & \downarrow i \\
\mathbb{P}V_{C_2} & \longleftarrow & C_2
\end{array}
$$

There is at most one such isomorphism because some stuff has to respect embeddings. Concretely, they are equivalent if the images of the curves in $\mathbb{P}^{5g-6}$ are equal.

Thus, we see that we get a subfunctor $j : \tilde{\mathcal{M}}_g \hookrightarrow \mathrm{Hilb}^{(6z-1)(g-1)}_{\mathbb{P}^{5g-6}}$. Let $P = (6z-1)(g-1)$.

Lemma 29.6. *Let $X \hookrightarrow \mathbb{P}^{5g-6}$ (a morphism over S) be an S-valued point of $\mathrm{Hilb}^P_{\mathbb{P}^{5g-6}}$. Then the set of points $s \in S$ for which X_s is a smooth genus g curve is an open set. This lemma is true without anything about $5g - 6$; genus is constant in a flat family.*

Proof. (1) the condition that $X \to S$ is smooth is open in S. After some reduction, you can assume S and X are noetherian, so the set of points where stuff is not smooth is closed, and $X \to S$ is proper, so the image in S is closed.

(2) the semi-continuity theorem implies that the set of points where the fiber is geometrically connected is open.

Now after shrinking on S, we can assume $X \to S$ is smooth proper with fibers geometrically connected.

(3) X_s smooth geometrically connected and the Hilbert polynomial is $(6z-1)(g-1)$ implies that X_s is a genus g curve.   ◇

[[★★★ Hartshorne: you need that some embedding is given by the tri-canonical system, so it looks like $\tilde{\mathcal{M}}_g$ is a closed subset of this open thing]]

Corollary 29.7. *$\tilde{\mathcal{M}}_g$ is represented by an open subscheme of $\mathrm{Hilb}^P_{\mathbb{P}^{5g-6}}$.*

To be corrected

Then we want to say that $\mathcal{M}_g \cong [\tilde{\mathcal{M}}_g/PGL_{5g-6}]$ (quotient out by the choice of the isomorphism σ). We have a map $\tilde{\mathcal{M}}_g \to \mathcal{M}_g$, given by $(C, \sigma) \mapsto C$. By uniqueness of isomorphisms, this gives a morphism of fibered categories. This induces a morphism of fibered categories

$$[\tilde{\mathcal{M}}_g/PGL_{5g-6}]^{ps} \to \mathcal{M}_g. \tag{*}$$

Lemma 29.8. *This map induces an isomorphism $[\tilde{\mathcal{M}}_g/PGL_{5g-6}] \xrightarrow{\sim} \mathcal{M}_g$.*

Proof. It is enough to show that $(*)$ is fully faithful, because it is clear that every point is locally in the image (we can always locally choose a basis for the locally free sheaf). Equivalently, given (C_1, σ_1) and (C_2, σ_2) in $\tilde{\mathcal{M}}_g(S)$ and an isomorphism $\iota : C_1 \to C_2$, there is a unique $h \in PGL_{5g-6}(S)$ such that

$$
\begin{array}{ccc}
\mathbb{P}^{5g-6} & \xrightarrow{\ \sigma_1\ } & \mathbb{P}V_{C_1} \\
\downarrow{\scriptstyle h} & & \wr\downarrow{\scriptstyle \iota} \\
\mathbb{P}^{5g-6} & \xrightarrow{\ \sigma_2\ } & \mathbb{P}V_{C_2}
\end{array}
$$

and we see that there is no choice for h, it has to be the composition.   ◇

◇

Definition 29.9. An algebraic stack $\mathcal{X}$ over S is *Deligne-Mumford* if there exists an étale surjection $U \to \mathcal{X}$ with U a scheme. (sometimes people require the diagonal to be finite).   ◇

Remark 29.10. $\mathcal{M}_{1,1}$ and $\mathcal{M}_g$ are Deligne-Mumford. The way we've done things, it looks like it might be hard to show this. We'll see that an algebraic stack $\mathcal{X}$ is Deligne-Mumford if and only if "objects have no infinitesimal automorphisms". It is a purely deformation-theoretic thing. For the $\mathcal{M}_g$ and $\mathcal{M}_{1,1}$ cases, this will follow from the fact that $H^0(C, T) = 0$.   ◇

Corrected proof that $\mathcal{M}_g$ is algebraic.

It is still correct that $\mathcal{M}_g$ is represented by a locally closed subscheme of the Hilbert scheme, but this is hard (it requires knowledge of the relative Picard functor)[[★★★ there are some diagrams about this in Ed's notes, but I don't understand them]]. The following argument will bypass this.

Let S be a fixed base scheme. Let X be quasi-projective, flat, and finitely presented over S, and let Y be quasi-projective and finitely presented over S. We define the functor $\underline{\operatorname{Hom}}(X, Y) : (\mathsf{Sch}/S)^{op} \to \mathsf{Set}$ by $(S' \to S) \mapsto \operatorname{Hom}_{S'}(X_{S'}, Y_{S'})$.

Proposition 29.11. $\underline{\operatorname{Hom}}(X, Y)$ *is a scheme.*

Proof. To give an element of $\operatorname{Hom}_{S'}(X_{S'}, Y_{S'})$, it is enough to give its graph (a subscheme of $X_{S'} \times_{S'} Y_{S'} = (X \times_S Y)_{S'}$), so we get a morphism $\underline{\operatorname{Hom}}(X, Y) \to \operatorname{Hilb}_{X \times_S Y}$, given by $(f : X_{S'} \to Y_{S'}) \mapsto \Gamma_f \subseteq (X \times_S Y)_{S'}$. Note that Γ_f is proper and flat over the base since X is projective and flat [[★★★]], so Γ_f is a point of the Hilbert scheme. Also, Y is [[★★★ something]]

Claim: This identifies $\underline{\operatorname{Hom}}(X, Y)$ with an open subscheme of $\operatorname{Hilb}_{X \times_S Y}$. To see this, stare that the following diagram.

$$
\begin{array}{ccccc}
\Gamma & \hookrightarrow & (X \times_S Y) \times_S \operatorname{Hilb}_{X \times_S Y} & \longrightarrow & X \times_S \operatorname{Hilb}_{X \times_S Y} \\
 & \searrow & \downarrow & \swarrow & \\
 & & \operatorname{Hilb}_{X \times_S Y} & &
\end{array}
$$

[[★★★ I haven't thought about this yet.]]

Lemma 29.12. *Let $f : \Gamma \to X$ be a morphism of proper, flat, finitely presented S-schemes. Then there is an open set $U \subseteq S$ so that a morphism $S' \to S$ factors through U if and only if $\Gamma_{S'} \to X_{S'}$ is an isomorphism.*

Proof. There is a fiber-wise criterion for being étale [EGA, IV.17.8.2], which implies that if $\gamma \in \Gamma$ is a point with image $s \in S$, then $\Gamma_S \to X_S$ is étale at γ if and only $\Gamma \to X$ is étale at γ.

Let $Z \subseteq \Gamma$ be the closed subset where f is not étale, and replace S by the complement of the (closed) image of Z in S (Γ is proper over the base). This reduces to the case where f is proper and étale (and therefore finite étale). Such a morphism has a rank, which we want to be 1. Let $W \subseteq X$ be the open and closed [[★★★]] subset where $rk(f_*\mathcal{O}_\Gamma) > 1$, and take U to be the complement of the image of W in S.
$\square_{\text{Lemma}}$

$\square$

Define $\mathcal{M}'_g$ to be the category of pairs (C, τ), where $\pi : C \to S$ is a genus g curve over S and $\tau : \mathcal{O}_S^{5(g-1)} \xrightarrow{\sim} V_C = \pi_*(\Omega^1_{C/S})^{\otimes 3}$ is an isomorphism; the morphisms are what you expect. Define a morphism $\mathcal{M}'_g \to \mathrm{Hilb}_{\mathbb{P}^{5g-6}}$ by taking (C, τ) to the closed subscheme of $\mathbb{P}_S^{5g-6}$ corresponding to the closed immersion $j_\tau : C \xhookrightarrow{\text{can}} \mathbb{P}V_C \xrightarrow[\sim]{\mathbb{P}\tau} \mathbb{P}_S^{5g-6}$, or $(C, (\Omega^1_{C/S})^{\otimes 3}) \hookrightarrow (\mathbb{P}V_C, \mathcal{O}_{\mathbb{P}V_C}(1)) \xrightarrow{\tau} (\mathbb{P}_S^{5g-6}, \mathcal{O}(1))$ [[★★★ I don't think this is clearer]]

Given $\pi : C \to S$, what is $\{\tau : \mathcal{O}_S^{5g-5} \xrightarrow{\sim} V_C\}$? It is close to $\{(j, \iota) | j : C \hookrightarrow \mathbb{P}^{5g-6}, \iota : j^*\mathcal{O}(1) \xrightarrow{\sim} (\Omega^1_{C/S})^{\otimes 3}\}$. In fact, there is a map $\tau \mapsto (j_\tau, \text{can})$. You can go in the other direction: given (j, ι), you get a map by the adjunction

$$\begin{array}{ccc} \mathcal{O}_X^{5g-5} & \longrightarrow & \pi_* j^* \mathcal{O}_{\mathbb{P}^{5g-6}}(1) \\ \vdots & & \downarrow{\iota} \\ V_C & = & \pi_*(\Omega^1_{C/S})^{\otimes 3} \end{array}$$

Let T be the total space of the $\mathbb{G}_m$-torsor of isomorphisms, and let Φ be the fiber product shown (Φ classifies $(C/S, j, \iota)$)

$$\begin{array}{ccc} \Phi & \longrightarrow & \underline{\mathrm{Hom}}(C_U, T) \\ \downarrow & \ulcorner & \downarrow \\ U & \xrightarrow{\text{id}} & \underline{\mathrm{Hom}}(C_U, C_U) \end{array} \qquad \begin{array}{ccc} T & \longrightarrow & C_U \xhookrightarrow{j_U} \mathbb{P}_U^{5g-6} \\ & & \downarrow \swarrow \\ \Phi & \longrightarrow & U \quad \subseteq \quad \mathrm{Hilb}_{\mathbb{P}^{5g-6}} \end{array}$$

More precisely, Φ is the functor $S \mapsto \{(C/S, j : C \hookrightarrow \mathbb{P}^{5g-6}, \iota : j^*\mathcal{O}(1) \xrightarrow{\sim} (\Omega^1_{C/S})^{\otimes 3})\}$ and U is the funct $S \mapsto (C/S, j)$.

To get $\mathcal{M}_g$: $\mathcal{M}'_g$ has an action of GL_{5g-5}. Say $g \in GL_{5g-5}$, then we define $g \cdot (C/S, \tau : \mathcal{O}_S^{5g-5} \xrightarrow{\sim} V_C) = (C/S, \tau \cdot g : \mathcal{O}_S^{5g-5} \xrightarrow{\sim} V_C)$

30   Sheaf cohomology and Torsors

[[★★★ the following stuff needs to go somewhere]]

If C is a site and F is a fibered category over C, then a *global section* of F is, for each object $X \in \mathsf{C}$, an element $\alpha_X \in F(X)$ such that for every morphism $f : X \to Y$ in C, we have $f^*\alpha_Y \cong \alpha_X$. Two such choices α and α' are considered the same if α_X and α'_X are isomorphic in the fiber $F(X)$ for every X. In the case where $\mathsf{C} = \mathsf{Sch}/X$ (or any other site with a final object) for some scheme X and F is a sheaf on X, this really corresponds to a section of F over X since all the others can be obtained by pullback. The set of global sections of F on C will be denoted $\Gamma(\mathsf{C}, F)$ or $\Gamma(\mathsf{T}, F)$, where T is the topos of C. To see that this is independent of the site (for F a sheaf), just note that $\Gamma(\mathsf{C}, F) = \mathrm{Hom}_{\mathsf{T}}(*, F)$, where $*$ is the punctual sheaf (terminal object in T). [[★★★ how to do this if F a fibered category? I guess define T as the 2-category of fibered categories over C, then $\Gamma(\mathsf{T}, F)$ is parametrized by isoclasses of $\mathrm{HOM}(*, F)$]]

Define cohomology. First prove that the category $\mathsf{Ab}(\mathsf{C})$ of sheaves of abelian groups on C has enough injectives [[★★★ The topos has to have "enough points" and not all Topoi do have enough points. See Lemmas 19.4 and 19.5. Deligne has some criteria for when a topos has enough points in [SGA, 4.VI.9]. See [SGA, 4.IV.7] for a topos with no points]]. Given a sheaf of abelian groups μ, take an injective resolution $0 \to \mu \to I^{\bullet}$. Then $H^i(\mathsf{C}, \mu)$ is defined to be the i-th homology of the complex $\Gamma(\mathsf{C}, I^{\bullet})$.

[[★★★ /stuff]]

Definition 30.1. Let C be a site, let μ be a sheaf of groups on C, let $P \to X$ be a morphism from a sheaf P on C to an object $X \in \mathsf{C}$, and assume μ acts on P over X. We say P is a *μ-torsor* over X if there is a cover $Z \to X$ in C and a μ-equivariant isomorphism of sheaves $Z \times_X P \cong Z \times \mu$. We call $X \times \mu$ *the trivial μ-torsor*. [[★★★ It's okay that we insisted torsors be algebraic spaces before by Corollary 30.9. We used descent for algebraic spaces in the proof of Prop 26.4, but we could have just as easily used descent for torsors (which follows almost immediately from decent for sheaves) together with that corollary.]]

A morphism of μ-torsors $(P' \to X') \xrightarrow{(f, \bar{f})} (P \to X)$ is a morphism $\bar{f} : X' \to X$ and a μ-equivariant morphism $f : P' \to P$ such that the following square commutes.

$$
\begin{array}{ccc}
P' & \xrightarrow{f} & P \\
\downarrow & & \downarrow \\
X' & \xrightarrow{\bar{f}} & X
\end{array}
$$

The category of μ-torsors is denoted $\mathsf{Tors}(\mu)$.   ◇

Remark 30.2 (An alternative definition)**.** Let μ is a sheaf of abelian groups on X (i.e. on C/X), and let P be a sheaf on X with an action of μ. Then P is a μ-torsor if and only if there is some cover $Z \to X$ so that $P|_Z \cong \mu|_Z$ as sheaves on Z.

To see that these definitions are equivalent, use Exercise 4.2: the category of sheaves on C with a morphism to X is equivalent the category of sheaves on C/X.

Incidently, this allows us to define a tosor on a site. If μ is a sheaf of abelian groups on C and P is a sheaf on C with a μ action, P is a μ-torsor if for every object X, $P|_X$ is a $\mu|_X$-torsor. This is equivalent to an element of $\Gamma(\mathsf{C}, \mathsf{Tors}(\mu))$.   ◇

Note that $\mathsf{Tors}(\mu)$ is fibered over C. A torsor $P \to X$ lies over the object X and the pullback of a torsor is a torsor.

Lemma 30.3. *The category* $\mathsf{Tors}(\mu)$ *is a stack over* C.

Proof. This follows almost immediately from descent for sheaves (Theorem 7.5). If $P' \to Z$ is a μ-torsor with descent data with respect to the cover $Z \to X$, then the theorem tells us that there is a unique sheaf P over X whose pullback is P'. The theorem also descends the action of μ on P' to an action of μ on P (note that the pullback of μ to Z is just the restriction of μ to the site C/Z). [[★★★ does this need to be clearer?]]   □

The following lemma shows that it is a stack in groupoids.

Lemma 30.4. *Any morphism of torsors over an object* X *is an isomorphism.*

Proof. First note that any morphism of trivial torsors is an isomorphism. If $f : X \times \mu \to X \times \mu$ is a morphism over X, then for a test object $T \in \mathsf{C}$, we have a $\mu(T)$-equivariant map $X(T) \times \mu(T) \to X(T) \times \mu(T)$ over $X(T)$, which is automatically a bijection. Thus, f is an isomorphism.

Let $f : P \to P'$ be a morphism of μ-torsors over X (i.e. $\bar{f} = \mathrm{id}_X$). Then there is some cover $Z \to X$ so that both P and P' are trivial over Z. Now we have that $f_Z : P_Z \to P'_Z$ is a morphism of trivial torsors, so it is an isomorphism. It follows that f is an isomorphism (the descent data on f_Z induces descent data for the inverse of f_Z, which descends to an inverse of f).   □

Corollary 30.5. *Let* $f : P \to X$ *be a* μ-torsor. *A section* $s : X \to P$ *of* f *induces an isomorphism of torsors* $P \cong X \times \mu$ *(a* trivialization *of* P*).*

Proof. We get a morphism $X \times \mu \to P$ over X given by $(h, g) \mapsto s(h) \cdot g$. By the previous lemma, this is an isomorphism.   □

Proposition 30.6 (Another alternative definition of torsors)**.** *Let* μ *be a sheaf of abelian groups on a site* C, *and let* $P \to X$ *be a morphism from a sheaf* P *to an object* X. *Then* P *is a* μ-torsor *if and only if*

1. *there exists a cover* $g : Z \to X$ *which factors through* P, *and*

2. $\mu(T)$ *acts simply transitively on the fibers of* $P(T) \to X(T)$ *whenever* $P(T)$ *is non-empty.*

Proof. ($\Rightarrow$) Assume $P \to X$ is a μ-torsor. Then condition (1) follows immediately. (2) Let $f : T \to X$ be an element of $X(T)$. A lifting $\tilde{f} : T \to P$ is equivalent to a section of the torsor $T \times_X P \to T$.

$$
\begin{array}{ccc}
T \times_X P & \longrightarrow & P \\
\uparrow\downarrow & \overset{\tilde{f}}{\nearrow} & \downarrow \\
T & \underset{f}{\longrightarrow} & X
\end{array}
$$

If $P(T)$ is non-empty, then we have some section $s : T \to T \times_X P$. By Corollary 30.5, we get that $T \times_X P \to T$ is the trivial torsor, isomorphic to $T \times \mu \to T$. A section of $T \times \mu \to T$ is equivalent to an element of $\mu(T)$. Since $\mu(T)$ acts simply transitively on itself, we get that it acts simply transitively on the inverse image of f in $P(T)$.

($\Leftarrow$) Let $\tilde{g} : Z \to P$ be the factorization of the cover $g : Z \to X$. There is a bijection between such factorizations and sections of $Z \times_X P \to Z$. Let s be the section corresponding to $\tilde{g}$. Define $h(T) : Z(T) \times \mu(T) \to Z(T) \times_{X(T)} P(T)$ by $(z, m) \mapsto \big(z, s(z) \cdot m\big)$. It is clear that this defines a μ-equivariant morphism h of sheaves over Z. Since $\mu(T)$ acts transitively on $P(T)$ over $X(T)$, $h(T)$ is always a bijection (if $P(T)$ is empty, then $Z(T)$ is empty, so $h(T)$ is still a bijection). Thus, h is an isomorphism, so P is a μ-torsor. $\qquad\square$

Theorem 30.7. *If μ is a sheaf of abelian groups, there is an natural bijection between isomorphism classes of μ-torsors over X and $H^1(\mathsf{C}/X, \mu)$. [[★★★ more generally, I think this proof should show that $H^1(\mathsf{C}, \mu) \cong \Gamma(\mathsf{C}, \mathsf{Tors}(\mu))$]]*

Proof. $\big(H^1(\mathsf{C}/X, \mu) \to \{\mu\text{-torsors}\}\big)$ Let $\mu \to I$ be an injection into an injective sheaf of abelian groups and let K be the cokernel sheaf. Then we have an exact sequence

$$0 \to \mu \to I \xrightarrow{d} K \to 0$$

and an element of $H^1(\mathsf{C}/X, \mu)$ is represented by some element $\alpha \in (I/\mu)(X)$. Given such an element, define $P_\alpha : \mathsf{C}^{op} \to \mathsf{Set}$ by $T \mapsto \big\{ \big(x \in X(T), s \in I(T)\big) \mid ds = x^*\alpha \big\}$. This presheaf comes with a projection map to X. Thinking of it as a presheaf on C/X, we see that it is a sheaf because it is the following fibered product.

$$
\begin{array}{ccc}
P_\alpha & \longrightarrow & * \\
\downarrow & & \downarrow{\scriptstyle \alpha} \\
I & \xrightarrow{d} & I/\mu
\end{array}
$$

There is an obvious action of μ on P_α over X (given by addition in the second coordinate). Finally, we need to show that P_α is locally the trivial torsor. Since d is surjective as a morphism of sheaves, there is some cover $f : Z \to X$ so that $f^*\alpha \in K(Z)$ is di for some $i \in I(Z)$. The calculation on the right verifies that the diagram on the left is

cartesian.

$$\begin{array}{ccc} Z \times \mu & \longrightarrow & P \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

$$\begin{aligned} (Z \times_X P)(T) &= \{(z,x,s)|ds = x^*\alpha, \ fz = x\} \\ &= \{(z,s)|ds = z^*f^*\alpha = z^*(di)\} \\ &= \{(z,s)|s - z^*i \in \mu(T)\} \\ &\cong (Z \times \mu)(T) \end{aligned}$$

Note that for the third equality we're using the fact that the kernel presheaf is the same as the kernel sheaf.

P is independent of the choice of representative in $H^1(\mathsf{C}/X, \mu)$. Assume $\alpha, \alpha' \in K(X)$ represent the same cohomology class, with $\gamma \in I(X)$ and $d\gamma = \alpha' - \alpha$. Then we get a morphism of torsors $P_\alpha \to P_{\alpha'}$, given by $(x,s) \mapsto (x, s+\gamma)$. By Lemma 30.4, this is an isomorphism.

Moreover, P_α is independent of the choice of injective resolution of μ. Let $J^\bullet$ be another injective sheaf with an injection $\mu \to J$ with cokernel R, and let $\alpha \in K$ and $\beta \in R$ represent the same cohomology class. By the usual arguments with injective resolutions, there is is a morphism of complexes $I^\bullet \to J^\bullet$ inducing the identity on homology. Changing β to another representative in the same cohomology class, we may assume this map of complexes sends α to β. This induces a morphism $P_\alpha \to P_\beta$, which must be an isomorphism by Lemma 30.4.

$\left(\{\mu\text{-torsors}\} \to H^1(\mathsf{C}/X, \mu)\right)$ Given a μ-torsor $P \to X$, let $f : Z \to X$ be a cover over which P is trivial. That is, the sheaves $Z \times \mu$ and $Z \times_X P$ are isomorphic. That is, the sheaves $\mu|_Z$ and $P|_Z$ are isomorphic. Note that we have a canonical map $P \to f_*(P|_Z)$; explicitly, for an object T over X, the map $P(T \to X) \to f_*(P|_Z)(T \to X) = P(T \times_X Z \to Z)$ is induced by the projection map $T \times_X Z \to T$. Since $T \times_X Z \to T$ is a cover, the sheaf axiom on P tells us that this map is an injection. Similarly, we have an injection $\mu \to f_*(\mu|_Z)$. Let I be an injective sheaf of abelian groups such that there is an injection $f_*(\mu|_Z) \hookrightarrow I$. Then we have the following diagram (the rows are not exact).

$$\begin{array}{ccc} 0 \longrightarrow P & \hooklongrightarrow & f_*(P|_Z) \\ & & \downarrow{\wr} \\ 0 \longrightarrow \mu \hooklongrightarrow f_*(\mu|_Z) & \hooklongrightarrow I \longtwoheadrightarrow I/\mu \longrightarrow 0 \end{array}$$

Since $Z \times_X P \to Z$ is a trivial μ-torsor, it has a section. That is, $P(Z)$ has an element. Let $\beta \in (I/\mu)(Z)$ be the image of this point. The claim is that β is the pullback of a point $\alpha \in (I/\mu)(X)$ along f. [[★★★ I want to try to prove this with the sheaf axiom on I/μ (really on $f_*\mu/\mu$), but I can't get my hands around the point β to check that $p_2^*\beta = p_1^*\beta$.]]

[[★★★ show that these two procedures are inverse]]   □

[[★★★ Before, we always took torsors to be algebraic spaces. The following lemma says that that was ok.]]

Lemma 30.8. *If X is a scheme and G is a group scheme over X (i.e. a group object in Sch/X),[1] then any G-torsor $P \to X$ is an algebraic space. Furthermore, any stable property (in the same topology which makes P a torsor) of the diagonal map $G \to G \times_X G$ is inherited by the diagonal $P \to P \times_X P$.*

Proof. Let $P \to X$ be a G-torsor. Then P is already a sheaf. To check representability of the diagonal $P \to P \times P$, it is enough to check representability of $P \to P \times_X P$ by Lemma A4.1, so let $T \to P \times_X P$ be a morphism from a scheme given by $p_1 \times p_2$ and let Z be the fiber product as shown in the diagram on the left.

$$
\begin{array}{ccc}
Z & \longrightarrow & T \\
\downarrow & & \downarrow{\scriptstyle p_1 \times p_2} \\
P & \xrightarrow{\ \Delta\ } & P \times_X P
\end{array}
\qquad\qquad
\begin{array}{ccc}
Z & \longrightarrow & T \\
\downarrow & & \downarrow{\scriptstyle p_1 \times \mathrm{id}_T\ \times p_2 \times \mathrm{id}_T} \\
P \times_X T & \xrightarrow{\ \Delta\ } & (P \times_X T) \times_X (P \times_X T)
\end{array}
$$

Note that Z is then also the fiber product shown in the diagram the right. Since $P \times_X T \to T$ has a section ($p_1 \times \mathrm{id}_T$, for example), it is the trivial G-torsor $G \times_X T$, so it is a scheme. Thus, Z is a fiber product of schemes over a scheme, so it is a scheme.

Let $f : U \to X$ be a cover of X so that $P \times_X U \cong G \times_X U$. Then the projection $G \times_X U \cong P \times_X U \to P$ is an étale cover of P by a scheme.

Finally, if $\Delta : G \to G \times_X G$ has some stable property, then we have the following diagram. It is easy to verify that the squares are cartesian and that the double headded arrows are covers.

Any stable property of the diagonal of G pulls up, over, and down to a property of the diagonal of P. $\square$

Corollary 30.9. *If X is a scheme over S and G is a group scheme over S, then any G-torsor $P \to X$ is an algebraic space. Furthermore, any stable property of $G \to G \times_S G$ is inherited by $P \to P \times_X P$.*

Proof. A G-torsor $P \to X$ is the same thing as a $(G \times_S X)$-torsor over X, where $G \times_S X$ is thought of a a group scheme over X. [[★★★ put insightful remark here to make this really clear]] Any stable property of $\Delta_{G,S} : G \to G \times_S G$ is inherited by $\Delta_{G \times_S X, X} = \Delta_{G,S} \times \mathrm{id}_X : G \times_S X \to (G \times_S X) \times_X (G \times_S X) = (G \times_S G) \times_S X$. $\square$

---

[1] A group object in Sch/X is not the same as a group scheme with a morphism to X. One of them has group structure morphism $G \times G \to G$, and the other has $G \times_X G \to G$. If you like, the *fibers* of a group object $G \to X$ in Sch/X are groups.

Remark 30.10. [[★★★ I don't know where we'll want this, but it will be somewhere]] If $G \to S$ is locally of finite type, then $G \to G \times_S G$ is of finite type. "This takes an argument, but we won't give it here". Thus, if G is, say, a smooth group scheme over S, then all G-torsors $P \to X$ are algebraic spaces with finite type diagonal.   ◇

31   Gerbes

It may look like everything is a quotient of a variety by some group action, but this is not the case. Gerbes are the next examples of stacks.

Definition 31.1. Let μ be a sheaf of abelian groups on a site C. A μ-*gerbe* is a stack in groupoids $\mathcal{F}$ over C and for each $x \in \mathcal{F}$ an isomorphism $\iota_x : \mu \xrightarrow{\sim} \underline{\mathrm{Aut}}_x$ such that

1. locally there is an object in $\mathcal{F}$ (i.e. for any object $X \in \mathsf{C}$, there is a cover $X' \to X$ such that $\mathcal{F}(X')$ is non-empty),

2. any two objects of $\mathcal{F}$ are locally isomorphic, and

3. for any morphism $y \to x$ in $\mathcal{F}(U)$ (automatically an isomorphism since $\mathcal{F}$ is fibered in groupoids), the diagram on the left commutes.

$$
\begin{array}{ccc}
 & \mu & \\
{}_{\iota_y}\nearrow & & \searrow{}^{\iota_x} \\
\underline{\mathrm{Aut}}_x & \xrightarrow{\ \sim\ } & \underline{\mathrm{Aut}}_y
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & \mu & \\
{}_{\iota_x}\nearrow & & \searrow{}^{\iota'_{f(x)}} \\
\underline{\mathrm{Aut}}_x & \xrightarrow{\ \sim\ } & \underline{\mathrm{Aut}}_{f(x)}
\end{array}
$$

A *morphism of μ-gerbes* is a morphism of stacks $f : \mathcal{F} \to \mathcal{F}'$ such that the diagram on the right commutes.   ◇

Remark 31.2. We could make the above definition for a sheaf of (not necessarily abelian) groups μ. However, if $f : x \to y$ is a morphism in the gerbe (over some S, say) and $g \in \mu(S)$, then condition (3) implies that the following diagram commutes.

$$
\begin{array}{ccc}
x & \xrightarrow{\ g\ } & x \\
{}_{f}\big\downarrow & & \big\downarrow{}^{f} \\
y & \xrightarrow{\ g\ } & y
\end{array}
$$

On the other hand, we could take $y = x$, so f is some other element of $\mu(S)$. Then commutativity of the diagram gives us that $fg = gf$, so μ is abelian.   ◇

Lemma 31.3. *Any morphism of μ-gerbes $f : \mathcal{G}_1 \to \mathcal{G}_2$ is an equivalence.*

Proof. (Full faithfulness) By Lemma 21.14 it is enough to check full faithfulness on fibers. We want the natural map $\underline{\mathrm{Isom}}_{\mathcal{G}_1}(x, y) \to \underline{\mathrm{Isom}}_{\mathcal{G}_2}\big(f(x), f(y)\big)$ to be an isomorphism for every pair $x, y \in \mathcal{G}_1(U)$. Since $\mathcal{G}_1$ and $\mathcal{G}_2$ are μ-gerbes, both of these $\underline{\mathrm{Isom}}$ sheaves are μ-torsors ($\underline{\mathrm{Aut}}_y \cong \mu \cong \underline{\mathrm{Aut}}_{f(y)}$ acts on the right), and the morphism is μ-equivariant (because f is a morphism of gerbes), so it must be an isomorphism by Lemma 31.3.

(Essential surjectivity) First recall that taking pullbacks along morphisms commutes with f (this is just because a morphism of fibered categories sends cartesian arrows to cartesian arrows). Given any object $x \in \mathcal{G}_2(U)$, there is some cover $h : V \to U$

such that $\mathcal{G}_1(V)$ is non-empty; let $y \in \mathcal{G}_1(V)$. Since any two objects in $\mathcal{G}_2$ are locally isomorphic, we may assume that $f(y) \cong h^*x$ (possibly replacing V by a further cover). Then we have that $f(p_i^*y) \cong p_i^*h^*x$, where $p_i : V \times_U V \to V$ are the projections. Since $hp_1 = hp_2$, we have that $p_2^*h^*x \cong p_1^*h^*x$. By full faithfulness, we have an isomorphism $\sigma : p_2^*y \xrightarrow{\sim} p_1^*y$. Similarly, we see that σ satisfies the usual cocycle condition. Since $\mathcal{G}_1$ is a stack, there is an object $z \in \mathcal{G}_1(S)$ such that $y \cong h^*z$. Note that $f(z)$ is specified by the same descent data as x, so since $\mathcal{G}_2$ is a stack, we have that $x \cong f(z)$. $\square$

Lemma 31.4 (The fundamental example)**.** *If μ is a sheaf of abelian groups on a site* C*, then the automorphism sheaf of a μ-torsor $P \to X$ is naturally isomorphic to μ. In particular,* $\mathsf{Tors}(\mu)$ *is a μ-gerbe over* C*.*

Proof. If $f : X \times \mu \to X \times \mu$ is an automorphism of the trivial torsor over X, then for a test object $T \in$ C, we have a $\mu(T)$-equivariant map $X(T) \times \mu(T) \to X(T) \times \mu(T)$ over $X(T)$. Such a map is given by addition of an element of $\mu(T)$. Thus, the automorphism sheaf of the trivial torsor is naturally isomorphic to μ.

It follows that the automorphism sheaf $\underline{\mathrm{Aut}}_P$ is locally isomorphic to μ, and the naturality of this isomorphism implies that it comes with descent data. By descent for sheaves, we see that $\underline{\mathrm{Aut}}_P$ is naturally isomorphic to μ. $\square$

Remark 31.5. Let $\mathcal{F}$ is a μ-gerbe with a global section α. For any object $y \in \mathcal{F}(U)$, consider the sheaf $\underline{\mathrm{Isom}}(\alpha_U, y)$ on U. Note that there is a right action of $\underline{\mathrm{Aut}}_y \cong \mu$. Since $\mathcal{F}$ is a gerbe, there is some cover $h : V \to U$ so that α_V and h^*y are isomorphic. Such an isomorphism induces an isomorphism of sheaves $\underline{\mathrm{Isom}}(\alpha_U, y) \times_U V = \underline{\mathrm{Isom}}(\alpha_V, h^*y) \cong V \times \mu$. Thus, $\underline{\mathrm{Isom}}(\alpha_U, y)$ is a μ-torsor over U. We get a morphism of gerbes $\mathcal{F} \to \mathsf{Tors}(\mu)$ sending $y \in \mathcal{F}(U)$ to the torsor $\underline{\mathrm{Isom}}(\alpha_U, y) \to U$. By Lemma 31.3, this is an equivalence of gerbes.

Since every gerbe locally has a section, we may think of a μ-gerbe as a fibered category which is locally equivalent to the category of μ-torsors. Therefore, we refer to $\mathsf{Tors}(\mu)$ as the *trivial μ-torsor*. $\diamond$

Next we'll show that μ-gerbes on a site C are parameterized by $H^2(\mathsf{C}, \mu)$. But first, we need the following lemma.

Lemma 31.6. *If μ is an injective sheaf of abelian groups over a site* C$/X$*, then any μ-gerbe $\mathcal{G}$ is trivial (i.e. $\mathcal{G}(X) \neq \varnothing$). [[★★★ I don't see how to make this lemma work on a site without a terminal object]]*

Proof. We may assume $\mathcal{G}$ has a splitting. Choose a cover $f : Z \to X$ so that $\mathcal{G}(Z)$ is non-empty. Let $z \in \mathcal{G}(Z)$. Consider the functor $F : (\mathsf{C}/X)^{op} \to \mathsf{Set}$ given by $(h : V \to X) \mapsto \{(v, \iota) | v \in \mathcal{G}(V), \iota : p_Z^*z \xrightarrow{\sim} p_V^*v \text{ in } \mathcal{G}(Z \times_X V)\}/\sim$, where $(v_1, \iota_1) \sim (v_2, \iota_1)$ if there exists $\delta : s_2 \to s_1$ in $\mathcal{G}(V)$ such that $\iota_1 = p_V^*\delta \circ \iota_2$. Note that if such a δ exists, then it is unique: if $\delta, \delta' : v_2 \to v_1$ with $p_V^*\delta = \iota_1\iota_2^{-1} = p_V^*\delta'$, then we have that $\delta'\delta^{-1} \in \mathrm{Hom}_{\mathcal{G}(V)}(v_1, v_1) \cong \mu(V)$ restricts to the trivial element $p_V^*\delta'p_V^*\delta^{-1} \in$

$\mathrm{Hom}_{\mathcal{G}(Z\times_X V)}(p_V^* v_1, p_V^* v_1) \cong \mu(Z \times_X V)$, but $Z \times_X V \to V$ is a cover, so by the sheaf axiom on μ, we get $\delta = \delta'$.

$$
\begin{array}{ccc}
Z \times_X V & \xrightarrow{\ p_V\ } & V \\
\downarrow{\scriptstyle p_Z} & & \downarrow{\scriptstyle h} \\
Z & \xrightarrow{\ f\ } & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
 & p_V^* v_2 \longrightarrow v_2 & \\
p_Z^* z \nearrow{\scriptstyle \iota_2} & \downarrow{\scriptstyle p_V^*\delta} \quad \vdots\,{\scriptstyle \delta} & \\
\searrow{\scriptstyle \iota_1} & p_V^* v_1 \longrightarrow v_1 & \\
\downarrow & & \downarrow \\
z & &
\end{array}
$$

It suffices to show that $F(\mathrm{id}_X) \neq \varnothing$. We will do this by showing that F is a torsor under an acyclic sheaf.

First we must show that F is a sheaf. Let $V' \to V$ be a cover, and let $V'' = V' \times_V V'$. (Injectivity) Let $(v_1, \iota_1), (v_2, \iota_2) \in F(V)$ have the same image in $F(V')$. Then there is some $\delta' : v_2' \to v_1'$ in $\mathcal{G}(V')$ such that $\iota_1' = p_{V'}^* \delta' \circ \iota_1'$. By the uniqueness of δ, we have that $p_2^* \delta' = p_1^* \delta'$. Since $\mathcal{G}$ is a (pre)stack, we get that δ' is the pullback of some $\delta : v_2 \to v_1$ in $\mathcal{G}(V)$ such that $\iota_1 = p_V^* \delta \circ \iota_2$. (Exactness in the middle) Similarly, if $(v', \iota') \in F(V')$ such that there is some $\delta'' : p_2^* v' \to p_1^* v'$ in $\mathcal{G}(V'')$ with $p_1^* \iota' = p_{V''}^* \delta'' \circ p_2^* \iota'$. The uniqueness of δ implies the usual cocycle condition. Since $\mathcal{G}$ is a stack, we get that (v', ι') is the pullback of some $(v, \iota) \in F(V)$.

Next observe that we have an action of $f_* \mu$ on F. Given an element $(v, \iota) \in F(V)$, an element $g \in f_*\mu(V) = \mu(Z \times_X V) \cong \mathrm{Hom}_{\mathcal{G}(Z\times_X V)}(p_V^* v, p_V^* v)$ acts by $g \cdot (v, \iota) = (v, g \circ \iota)$. Note that the stabilizer of (v, ι) is exactly the image of $\mu(V)$ in $f_*\mu(V)$. That is, $(v, g \circ \iota) \sim (v, \iota)$ exactly when $g \in f_*\mu(V) = \mu(Z \times_X V)$ is the restriction of an element of $\mu(V)$. Also note that the action of $f_*\mu(V)$ on $F(V)$ is transitive if: given any pair $(v_1, \iota_1), (v_2, \iota_2) \in F(V)$, we have the isomorphism $\iota_1 \iota_2^{-1} : p_V^* v_2 \xrightarrow{\sim} p_V^* v_1$, so we may always represent an element of $F(V)$ as (v_0, ι), where $v_0 \in \mathcal{G}(V)$ is fixed; now transitivity follows immediately from the isomorphism $f_*\mu(V) \cong \mathrm{Hom}_{\mathcal{G}(Z\times_X V)}(p_V^* v_0, p_V^* v_0)$. Finally, there is some cover $Y \to Z$ so that $F(Y)$ is non-empty ($\mathcal{G}$ is a gerbe, so $p_1^* z$ and $p_2^* z$ must be isomorphic over some cover), so F is a $f_*\mu/\mu$-torsor (Proposition 30.6).

Since μ is injective, $\mathrm{Hom}(-, \mu)$ is exact. Also, f^* is exact because it commutes with finite projective limits [[★★★ here we're using that $f^* : \mathcal{O}\text{-mod} \to f^*\mathcal{O}\text{-mod}$ is left adjoint to f_* and commutes with finite projective limits, and that $f^*\mathbb{Z} = \mathbb{Z}$ (constant sheaf)]], so $\mathrm{Hom}(-, f_*\mu) \cong \mathrm{Hom}(f^*-, \mu)$ is exact, so $f_*\mu$ is injective. It follows that $f_*\mu/\mu$ is acyclic. By Theorem 30.7, any torsor under an acyclic sheaf is trivial, so F has a global section. $\qquad\square$

Theorem 31.7. *Let μ be a sheaf of abelian groups on a site* C. *There is a bijection between isomorphism (equivalence) classes of μ-gerbes over* C *and* $H^2(\mathsf{C}, \mu)$.

Proof. $\big(H^2(\mathsf{C}, \mu) \to \{\mu\text{-gerbes}\}\big)$ Choose an injective resolution $\mu \to I^\bullet$, where the I^i are sheaves of abelian groups on C. Let $K = \ker(I^2 \to I^3)$, so we have the exact sequence of sheaves

$$0 \to \mu \xrightarrow{j} I^0 \xrightarrow{d_0} I^1 \xrightarrow{d_1} K \to 0.$$

Let $\alpha \in \Gamma(\mathsf{C}, K)$ represent a class in $H^2(\mathsf{C}, \mu)$. Define $\mathcal{G}_\alpha$ as the category with objects pairs (V, γ), where $V \in \mathsf{C}$ and $\gamma \in I^1(V)$ with $d_1\gamma = \alpha_V$, and a morphism $(V', \gamma') \to (V, \gamma)$ is an X-morphism $g : V' \to V$ and an element $\sigma \in I^0(V)$ such that $d_0\sigma = g^*\gamma - \gamma'$.

Claim. $\mathcal{G}_\alpha$ *is a stack.*

Proof. First we check that $\mathcal{G}_\alpha$ is a prestack. Given $V \in \mathsf{C}$ and $\gamma_1, \gamma_2 \in I^1(V)$, we have that $\underline{\mathrm{Isom}}\big((V, \gamma_1), (V, \gamma_2)\big) = \{\sigma \in I^0 | d_0(\sigma) = \gamma_2 - \gamma_1\}$ is a sheaf on V because it is the following fibered product of sheaves on V.

$$
\begin{array}{ccc}
\underline{\mathrm{Isom}}(\gamma_1, \gamma_2) & \longrightarrow & * \\
\downarrow {\scriptstyle \Gamma} & & \downarrow {\scriptstyle \gamma_1 - \gamma_2} \\
I^0|_V & \xrightarrow{\ d_0\ } & I^1|_V
\end{array}
$$

The hard part is to check effectivity of descent. Let $h : V \to U$ be a cover in C. An object in $\mathcal{G}_\alpha(V \to U)$ is of the form (γ, σ) where $\gamma \in I^1(V)$ (with $d_1\gamma = \alpha_V$) and $\sigma \in I^0(V \times_U V)$ such that $d\sigma = p_2^*\gamma - p_1^*\gamma$ and $p_{13}^*\sigma = p_{12}^*\sigma + p_{23}^*\sigma$. Given such an object, we want to find (ε, δ) with $\varepsilon \in I^1(U)$ and $\delta \in I^0(V)$ with $d\varepsilon = \alpha_U$, $d_0\delta = h^*\varepsilon - \gamma$ and $d\sigma = p_2^*d_0\delta - p_1^*d_0\delta$ (compatibility of descent data).

$$
\begin{array}{ccc}
p_2^*\gamma & \xrightarrow{\ d\sigma\ } & p_1^*\gamma \\
{\scriptstyle p_2^*d_0\delta} \downarrow & & \downarrow {\scriptstyle p_1^*d_0\delta} \\
p_2^*h^*\varepsilon & = & p_1^*h^*\varepsilon
\end{array}
$$

Note that for any morphism $f : T \to U$, we can pull the descent data (γ, σ) back to $V \times_U T$ to get an object (γ_T, σ_T) in $\mathcal{G}_\alpha(h_T : V \times_U T \to T)$. Define $F : (\mathsf{C}/U)^{op} \to \mathsf{Set}$ by

$$
T \mapsto \big\{(f, \varepsilon_T, \delta_T) \mid f \in U(T),\ \varepsilon_T \in I^1(T),\ \delta_T \in I^0(V \times_{U,f} T),
$$
$$
d\varepsilon_T = \alpha_T,\ d_0\delta_T = h_T^*\varepsilon_T - \gamma_T,\ \text{and}\ d\sigma_T = p_2^*d_0\delta_T - p_1^*d_0\delta_T \big\}.
$$

[[★★★ It is enough to specify δ_T satisfying the condition $d\sigma_T = p_1^*d^0\delta_T - p_2^*d^0\delta_T$, because then it follows from the definition of σ and the sheaf condition on I^0 that $d^0\delta_T + \gamma_T$ is the restriction of some ε_T satisfying $d\varepsilon_T = \alpha_T$. I think F is a sheaf by the same sort of fiber product trick]]

For an element $\lambda \in I^0(T)$, we define $\lambda \cdot (f, \varepsilon_T, \delta_T) = (f, \varepsilon_T + d\lambda, \delta + h_T^*\lambda)$, so F has an action of I^0 over U.

[[★★★ This paragraph is the one I'm confused about]] In fact, F is an I^0-torsor over U. To see this, it is enough to show that $I^0(T)$ acts simply transitively on $F(T)$ when $F(T)$ is non-empty. Given (ε, δ) and (ε', δ'), we'd like to find a λ so that $\lambda \cdot (\varepsilon, \delta) = (\varepsilon', \delta')$. If such a λ exists, we have $h_T^*\lambda = \delta' - \delta$, so λ is unique (h_T^* is injective because h_T is a cover and I^0 is a sheaf). We know that $(p_2^* - p_1^*)\big(d(\delta - \delta')\big) = d\sigma_T - d\sigma_T = 0$, but this doesn't show that $(p_2^* - p_1^*)(\delta - \delta') = 0$. [[★★★ now what?]]

But I^0-torsors are parameterized by $H^1(\mathsf{C}/U, I^0)$, which is trivial because I^0 is injective. Thus, F is the trivial torsor; in particular, it has a global section. This proves effectivity of descent. $\square_{\text{Claim}}$

In fact, $\mathcal{G}_\alpha$ is a μ-gerbe:

1. it is étale locally non-empty (because d_1 is a surjective map of étale sheaves),

2. any two objects are locally isomorphic (use the resolution to see this [[★★★]]),

3. we will see compatibility of the ι_α later. $\underline{\mathrm{Aut}}_{(\varepsilon, \delta)} = \{\delta' | d\delta' = 0\} \cong \mu$.

$\mathcal{G}_\alpha$ is independent of the choice of representative within the cohomology class. If $\alpha, \alpha' \in K$ represent the same cohomology class, with $d\gamma_0 = \alpha' - \alpha$, then we get a map $\mathcal{G}_\alpha \to \mathcal{G}_{\alpha'}$, given by $(V, \gamma) \mapsto (V, \gamma + \gamma_0)$ (note that $d\gamma = \alpha$, so $d(\gamma + \gamma_0) = \alpha + \alpha' - \alpha = \alpha'$). By the lemma, the two gerbes are isomorphic.

$\mathcal{G}_\alpha$ is independent of the choice of resolution. To see this, let $J^\bullet$ be another resolution of μ. Let $\alpha_I \in \ker(I^2 \to I^3)$ and $\alpha_J \in \ker(J^2 \to J^3)$ represent the same cohomology class. Then we get a morphism of complexes $I^\bullet \to J^\bullet$ inducing the identity on cohomology (by the usual arguments with injective resolutions). Changing α_J to something within the same class (so not changing $\mathcal{G}_{\alpha_J}$), we may assume that the map $I^2 \to J^2$ maps α_I to α_J. From the definition of $\mathcal{G}_\alpha$, this induces a morphism $\mathcal{G}_{\alpha_I} \to \mathcal{G}_{\alpha_J}$, which must be an isomorphism by the lemma.

$\bigl(\{\mu\text{-gerbes}\} \to H^2(\mathsf{C}, \mu)\bigr)$ Let $\mathcal{G}$ be a μ-gerbe. Include μ into an injective abelian sheaf $\mu \hookrightarrow I$. Define $\mathcal{G} \times^\mu I$ to be the stack associated to the prestack whose objects are objects in $\mathcal{G}$ and morphisms are elements of $\underline{\mathrm{Hom}}_\mathcal{G}(x, y) \times^\mu I = \underline{\mathrm{Hom}}_\mathcal{G}(x, y) \times I/\sim$, where $(g \circ \zeta, \iota) \sim (\zeta, \iota - g)$ for $g \in \mu$. Define composition as $(\zeta, i) \circ (\varepsilon, j) = (\zeta \circ \varepsilon, i + j)$. To check that this is well defined, let $g, g' \in \mu$. Then we have

$$(g\zeta, i) \circ (g'\varepsilon, j) = (g\zeta g'\varepsilon, i + j) \qquad\qquad \text{(definition)}$$
$$= (gg'\zeta\varepsilon, i + j) \qquad\qquad \text{(gerbe axiom 3)}$$
$$= \bigl((g + g')\zeta\varepsilon, i + j\bigr) \qquad\qquad \text{(that's how group actions work)}$$
$$\sim (\zeta\varepsilon, i + j - g - g') = (\zeta, i - g) \circ (\varepsilon, j - g')$$

Anyway, we get that $\mathcal{G} \times^\mu I$ is an I-gerbe. Since I is injective, $\mathcal{G} \times^\mu I$ is trivial, so there is some section $s_0 \in (\mathcal{G} \times^\mu I)(X)$. Moreover, we have a map $j : \mathcal{G} \to \mathcal{G} \times^\mu I$. Define $P_{s_0} : (\mathsf{Sch}/X)^{op} \to \mathsf{Set}$ by $(V \to X) \mapsto \{(s \in \mathcal{G}(V), \iota : j(x) \xrightarrow{\sim} s_0 \text{ in } \mathcal{G} \times^\mu I(V))\}$. Then P is an I/μ-torsor. From the short exact sequence

$$0 \to \mu \to I \to I/\mu \to 0$$

we get a map $\partial : H^1(X, I/\mu) \to H^2(X, \mu)$. You have the map $\mathcal{G} \mapsto \partial(P_{s_0})$ (recall that I/μ-torsors are parameterized by $H^1(X_{ET}, I/\mu)$).

Exercise: Show that this is inverse to the map $H^2(X_{ET}, \mu) \to \{\mu\text{-gerbes}\}$ from last time. [[★★★]] $\square$

Remark 31.8 (Aside on non-abelian cohomology). We have only defined cohomology for sheaves of abelian groups, but suppose that for a non-abelian sheaf of groups μ, we *define* $H^1(\mathsf{C}/X, \mu)$ to be the isomorphism classes of μ-torsors. Note that $H^1(\mathsf{C}/X, \mu)$ no longer has a group structure, but it is a pointed set (the distinguished element is the trivial torsor). Suppose you have an exact sequence of sheaves of groups

$$1 \to G_1 \to G_2 \to G_3 \to 1$$

in which G_1 is abelian, but G_2 and G_3 need not be abelian. Then we expect an exact sequence

$$H^1(\mathsf{C}/X, G_1) \to H^1(\mathsf{C}/X, G_2) \to H^1(\mathsf{C}/X, G_3) \xrightarrow{\partial} H^2(\mathsf{C}/X, G_1).$$

A G_1-torsor $P \to X$ is taken to the G_2-torsor $P \times^{G_1} G_2 := P \times G_2/\sim$, where $(pg_1, g_2) \sim (p, g_1 g_2)$ for $g_1 \in G_1$. Similarly, we see how to take a G_2-torsor to a G_3-torsor.[[★★★ is it easy to see exactness?]]

The boundary map ∂ is more interesting. Let $P \to X$ be a G_3-torsor. From this we want to obtain a G_1-gerbe. Define ∂P to be the category (fibered over C/X) whose objects are triples $(V, \tilde{P}, \iota)$, where $V \to X$ is an object over X, $\tilde{P} \to V$ is a G_2-torsor, and $\iota : \tilde{P} \times^{G_2} G_3 \to P$ is a G_3-equivariant morphism over $V \to X$. A morphism $(V', \tilde{P}', \iota') \to (V, \tilde{P}, \iota)$ is a G_2-equivariant morphism $f : \tilde{P}' \to \tilde{P}$ such that the following diagram commutes.
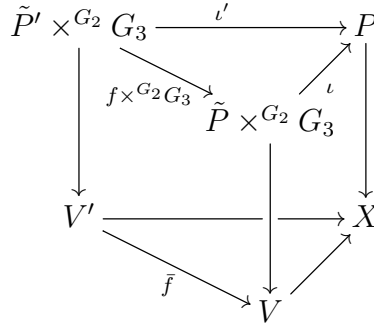
[[★★★ check that this is a stack]] It is clear that if P is trivial over $V \to X$, then $\partial P(V)$ is non-empty. Similarly, it is clear that any two objects are locally isomorphic.

Given an automorphism f of such a $(V, \tilde{P}, \iota)$, we have that f is an automorphism of $\tilde{P}$, so it is given by an element of G_2, and $f \times^{G_2} G_3 = \mathrm{id}$. It follows that the image of f (as an element of G_2) is trivial in G_3, so f is given by an element of G_1. It is clear that this isomorphism $\underline{\mathrm{Aut}}_{(V, \tilde{P}, \iota)} \cong G_1$ is natural.

[[★★★ how to check exactness]]

Upshot: Consider the sequence

$$1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$$

Then we have an exact sequence

$$H^1(\mathsf{C}/X, GL_n) \to H^1(\mathsf{C}/X, PGL_n) \to H^2(\mathsf{C}/X, \mathbb{G}_m) =: Br'(X).$$

An element of $H^1(\mathsf{C}/X, GL_n)$ is the same as a vector bundle on X. One direction: if $\mathcal{E}$ is a vector bundle, then the corresponding GL_n-torsor is $\underline{\mathrm{Isom}}\big(\mathcal{E}nd\,(\mathcal{O}_X^n), \mathcal{E}nd\,(\mathcal{E})\big)$. Anyway, we get a map $\bigcup_n \{PGL_n\text{-torsors}\}/(\mathcal{E}nd\,\mathcal{E} \sim 0) \hookrightarrow H^2(X, \mathbb{G}_m)$. $\diamond$

Proposition 31.9. *Let G be a smooth (commutative) group scheme over a scheme X. Then any G-gerbe $\mathcal{G}$ [[★★★ in any topology?]] is algebraic.*

Proof. (Representability of the diagonal) Let T be a scheme

$$
\begin{array}{ccc}
\underline{\mathrm{Isom}}(s_1, s_2) & \longrightarrow & T \\
\downarrow & & \downarrow{\scriptstyle (s_1, s_2)} \\
\mathcal{G} & \xrightarrow{\ \Delta\ } & \mathcal{G} \times \mathcal{G}
\end{array}
$$

$\underline{\mathrm{Isom}}(s_1, s_2)$ is a G-torsor, so by Lemma 30.9 it is an algebraic space.

(Smooth cover) Find $X' \to X$ étale so that $\mathcal{G}(X')$ is non-empty, and let s' be an object.

$$
\begin{array}{ccccccc}
X' & & & & & & \\
\downarrow & & & & & & \\
[X'/G] & = & B_{X'}G & = & \mathcal{G} \times_X X' & \underset{s'}{\rightrightarrows} & X' \\
& & & & \downarrow & & \downarrow{\scriptstyle et} \\
& & & & \mathcal{G} & \longrightarrow & X
\end{array}
$$

To check that the map $X' \to [X/G]$ is smooth, check that for any T and let P be a G-torsor.

$$
\begin{array}{ccc}
 & Z \longrightarrow X' & \\
Z' \longrightarrow G_{X'} & & \\
\downarrow \quad T \xrightarrow{\ P\ } [X'/G] & & \\
T' \longrightarrow X \quad {\scriptstyle P_0} & &
\end{array}
$$

$\square$

32   Properties of algebraic stacks

Properties of algebraic stacks over S.

By *a presentation* of an algebraic stack S, we mean a smooth surjective morphism $X \to \mathcal{X}$, with X an algebraic space.

Let $\mathcal{P}$ be a property of algebraic spaces over S which is local in the smooth topology (if $X' \to X$ is a smooth cover, then X has $\mathcal{P}$ if and only if X' has $\mathcal{P}$), then an algebraic stack $\mathcal{X}$ over S *has* $\mathcal{P}$ if there exists a presentation $X \to \mathcal{X}$ where X has $\mathcal{P}$.

Example 32.1. Locally noetherian, reduced, normal, regular, characteristic p, etc. ⋄

This is coarser than étale locally (e.g. dimension is no good)

Remark 32.2. If an algebraic stack has $\mathcal{P}$, then for every presentation $X' \to \mathcal{X}$, X' has $\mathcal{P}$.

$$
\begin{array}{ccc}
X \times_{\mathcal{X}} X' & \longrightarrow & X \quad \mathcal{P} \\
\downarrow & & \downarrow \\
X' & \longrightarrow & \mathcal{X}
\end{array}
$$

All morphisms are smooth surjections, so we can push $\mathcal{P}$ around like we did before. ⋄

Definition 32.3. $\mathcal{X}$ over S is *quasi-compact* if there exists a presentation $X \to \mathcal{X}$ with X quasi-compact. ⋄

Definition 32.4. Say $\mathcal{X}$ is an algebraic stack over S. An open (resp. closed, resp. locally closed) substack of $\mathcal{X}$ is a fully faithful substack $\mathcal{Y} \subseteq \mathcal{X}$ such that $\mathcal{Y} \to \mathcal{X}$ is representable by an open (resp. closed, resp. locally closed) immersion. ⋄

Lemma 32.5. *Let $\mathcal{Y} \to \mathcal{X}$ be a representable morphism of stacks with $\mathcal{X}$ algebraic. Then $\mathcal{Y}$ is algebraic.*

Aside: $\mathcal{M}_{g,n} \to \mathcal{M}_g$ is representable because for any T,

$$
\begin{array}{ccccc}
C \times_T \cdots \times_T C & \supseteq & U & \longrightarrow & T \\
& & \downarrow & & \downarrow \\
& & \mathcal{M}_{g,n} & \longrightarrow & \mathcal{M}_g
\end{array}
$$

Then you get that $\mathcal{M}_{g,n}$ is algebraic.

Proof. Claim: $\Delta : \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ is representable.

$$
\begin{array}{ccc}
P & \longrightarrow & T \\
\downarrow & & \downarrow \\
\mathcal{Y} & \longrightarrow & \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}
\end{array}
\qquad\qquad
\begin{array}{ccc}
P & \longrightarrow & T \\
\downarrow & & \downarrow \\
\mathcal{Y} \times_{\mathcal{X}} T & \xrightarrow{\ \Delta\ } & (\mathcal{Y} \times_{\mathcal{X}} T) \times_T (\mathcal{Y} \times_{\mathcal{X}} T)
\end{array}
$$

so P is a scheme.

For the smooth surjection, choose a presentation $X \to \mathcal{X}$, then let $Y = \mathcal{Y} \times_{\mathcal{X}} X$, which is an algebraic space which smoothly surjects onto $\mathcal{Y}$. □

Lemma 32.6. *Let $\mathcal{Z} \xrightarrow{g} \mathcal{Y} \xleftarrow{f} \mathcal{X}$ be a diagram of algebraic stacks. Then the stack $\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}$ is algebraic.*

Proof. Representability of the diagonal:

$$
\begin{array}{ccc}
I = \underline{\mathrm{Isom}}\big((x_1, z_1, \iota_1), (x_2, z_2, \iota_2)\big) & \longrightarrow & T \\
\downarrow & & \downarrow {\scriptstyle (x_1,z_1,\iota_1)\times(x_2,z_2,\iota_2)} \\
\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z} & \longrightarrow & (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z}) \times (\mathcal{X} \times_{\mathcal{Y}} \mathcal{Z})
\end{array}
$$

Where $x_i \in \mathcal{X}(T)$, $z_i \in \mathcal{Z}(T)$ and $\iota_i : f(x_i) \xrightarrow{\sim} g(z_i)$. An isomorphism between two such collections of data is a pair $(\rho_x, \rho_z) : (x_1, z_1, \iota_1) \to (x_2, z_2, \iota_2)$, where $\rho_x : x_1 \xrightarrow{\sim} x_2$ in $\mathcal{X}(T)$ and $\rho_z : z_1 \xrightarrow{\sim} z_2$ in $\mathcal{Z}(T)$ such that

$$
\begin{array}{ccc}
f(x_1) & \xrightarrow{\ \rho_x\ } & f(x_2) \\
{\scriptstyle \iota_1}\downarrow & & \downarrow{\scriptstyle \iota_2} \\
g(z_1) & \xrightarrow{\ \rho_z\ } & g(z_2)
\end{array}
$$

So we see that I is the equalizer of $\underline{\mathrm{Isom}}_{\mathcal{X}}(x_1, x_2) \times \underline{\mathrm{Isom}}_{\mathcal{Z}}(z_1, z_2) \rightrightarrows \underline{\mathrm{Isom}}(f(x_1), g(z_2))$

Smooth surjection. Choose a presentation $Y \to \mathcal{Y}$ and choose $Z \to Y_{\mathcal{Z}}$ be a smooth surjection[[★★★ I didn't quite catch how you get this]].

Then $Z \times_Y X \to \mathcal{Z} \times_{\mathcal{Y}} \mathcal{X}$ is a smooth surjection. □

Let $\mathcal{P}$ be a property of morphisms of algebraic spaces which is local on source and target in the smooth topology.

$$
\begin{array}{ccc}
X'' \xrightarrow{\ f'\ } X' & \longrightarrow & X \\
\ \searrow{\scriptstyle f''}\ \ \downarrow {\scriptscriptstyle \ulcorner} & & \downarrow {\scriptstyle f} \\
\qquad Y' & \xrightarrow{\ q\ } & Y
\end{array}
$$

If q and p ar smooth surjections then f has $\mathcal{P}$ if and only if f'' has $\mathcal{P}$.

Definition 32.7. A morphisms of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ has $\mathcal{P}$ if there exists a diagram

$$
\begin{array}{ccc}
X' \xrightarrow{\;P'\;} & \mathcal{X}' \longrightarrow & \mathcal{X} \\
& \downarrow & \downarrow f \\
& Y \xrightarrow{\;Q\;} & \mathcal{Y}
\end{array}
$$

P', Q presentations, then f' has $\mathcal{P}$.   ◇

So we can talk about surjective, universally open, locally of finite presentation (resp. type), flat, smooth.

33　$\mathcal{X}$ Deligne-Mumford $\Leftrightarrow$ $\Delta_{\mathcal{X}}$ formally unramified

Recall that if $\mathcal{P}$ is a property of morphisms of algebraic spaces which is local on source and target in the smooth topology, then we say that a morphism $f : \mathcal{X} \to \mathcal{Y}$ of algebraic stacks has $\mathcal{P}$ if there is a commutative diagram

$$
\begin{array}{ccc}
X' \xrightarrow{\;H\;} & \mathcal{X}' \longrightarrow & \mathcal{X} \\
& \downarrow^{\ulcorner} & \downarrow f \\
& Y \xrightarrow{\;Q\;} & \mathcal{Y}
\end{array}
$$

where H and Q are presentations an f' has $\mathcal{P}$.

Remark 33.1. If f has $\mathcal{P}$, then for every diagram of the form above, f' has $\mathcal{P}$.　　$\diamond$

Definition 33.2. A morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ is *quasi-compact* if for every quasi-compact scheme Y and morphism $Y \to \mathcal{Y}$ (need not be smooth), the fiber product $Y \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact.　　$\diamond$

Then finite presentation (resp. type) means locally of finite presentation (resp. type) and quasi-compact.

Definition 33.3. An algebraic stack $\mathcal{X}$ is *Deligne-Mumford* if there exists an étale surjection $X \to \mathcal{X}$ with X an algebraic space.　　$\diamond$

Theorem 33.4. *Let $\mathcal{X}$ be an algebraic stack over S. Then $\mathcal{X}$ is Deligne-Mumford if and only if $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is formally unramified.*

Remark 33.5. We have to say what unramified means for a morphism of algebraic spaces (since Δ is representable). We say that a morphism of algebraic spaces $g : Z \to W$ is *formally unramified* if for every closed immersion $T_0 \hookrightarrow T$ of affine schemes[[★★★ its weird that you have to use affine schemes ... what is the weird example with a non-affine scheme that goes wrong?]] defined by a nilpotent ideal, the map $Z(T) \to Z(T_0) \times_{W(T_0)} W(T)$ is injective.

$$
\begin{array}{ccc}
T_0 & \longrightarrow & T \\
{\scriptstyle \exists \leq 1} \downarrow & \nearrow & \downarrow \\
Z & \longrightarrow & W
\end{array}
$$

you also see that it is enough to check on ideals which square to zero.

Unramified means locally of finite presentation [[★★★ type?]] and formally unramified. I think we can do the theorem with only formally unramified.

The key point is that g formally unramified is equivalent to saying that $g^* \Omega^1_{W/S} \to \Omega^1_{Z/S}$ [[★★★ i.e. $\Omega^1_{Z/W} = 0$]] is surjective, which is what we'll use in the proof.　　$\diamond$

Note that this makes it easy to check that $\mathcal{M}_g$, for example, is Deligne-Mumford. Consider

$$
\begin{array}{ccc}
\underline{\mathrm{Isom}}(C_1, C_2) & \longrightarrow & S \\
\downarrow & & \downarrow {\scriptstyle (C_1, C_2)} \\
\mathcal{M}_g & \xrightarrow{\ \Delta\ } & \mathcal{M}_g \times \mathcal{M}_g
\end{array}
\qquad
\begin{array}{ccc}
T_0 & \xrightarrow{\ \iota_0\ } & \underline{\mathrm{Isom}}(C_1, C_2) \\
\downarrow & {\scriptstyle \exists_{\leq 1}} & \downarrow \\
T & \longrightarrow & S
\end{array}
$$

[[★★★ other picture]] We have to check that $\underline{\mathrm{Isom}}(C_1, C_2) \to S$ is formally unramified, so draw the picture on the right. We need to show that if $\sigma : C_{1,T} \to C_{1,T}$ is an automorphism reducing to the identity over T_0 then $\sigma \cong \mathrm{id}$. First of all, it is enough to consider $T_0 \hookrightarrow T$ defined by a square zero ideal $J \subseteq \mathcal{O}_T$.

It follows from stuff about gerbes that $H^0(C_{1,T}, T_{C_{1,T_0}/T_0} \otimes_{\mathcal{O}_{T_0}} J)$ is naturally in bijection with the group of automorphisms reducing to the identity. [[★★★ magic?]]

Using $g \geq 2$, we see that $H^0(C_{1,T}, T_{C_{1,T_0}/T_0} \otimes_{\mathcal{O}_{T_0}} J) = 0$. [[★★★ Ω^1 is ample, which implies that T_{C_1/T_0} is negative, so it has no global sections.]]

Proof of Theorem. $DM \Rightarrow \Delta$ formally unramified. Let $X \to \mathcal{X}$ be an étale surjection. Then we have

$$
\begin{array}{ccc}
X \times_{\mathcal{X}} X & \longrightarrow & X \times_S X \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{\ \Delta\ } & \mathcal{X} \times_S \mathcal{X}
\end{array}
$$

(1) Δ is formally unramified if and only if $X \times_{\mathcal{X}} X \to X \times_S X$ is formally unramified. This is because of the usual diagram; checking unramified, we can replace things by étale covers. Check that $X \times_{\mathcal{X}} X \to X \times_S X \xrightarrow{p_1} X$ is étale (it is a base change of $X \to \mathcal{X}$, which is étale).

$$
\begin{array}{ccccc}
X \times_{\mathcal{X}} X & \longrightarrow & X \times_S X & \xrightarrow{\ p_1\ } & X \\
\uparrow & \nwarrow & \uparrow & \nearrow & \\
T_0 & \longrightarrow & T & &
\end{array}
$$

we need to check at most 1 dashed arrow, but then just look at the outer square, where you get the result you want because $X \times_{\mathcal{X}} X \to X$ is étale.

Δ formally unramified $\Rightarrow DM$. Let k be a field and let $y : \mathrm{Spec}\, k \to \mathcal{X}$. Then we need to find an étale morphism $U \to \mathcal{X}$ with U an algebraic space such that $U \times_{\mathcal{X}, y} \mathrm{Spec}\, k$ is non-empty. Then take the disjoint union over all points and that will be an étale cover.

Idea: start with some $X \to \mathcal{X}$ which is smooth. We'd like to take a slice of X which étale covers $\mathcal{X}$. (1) Construct an étale morphism $f : X \to \mathcal{X} \times_S \mathbb{A}^n_S$ which factors $X \to \mathcal{X}$. This is where formally unramified is used. (2) find appropriate $E \subseteq \mathbb{A}^n_S$ étale over S. Take $U = f^{-1}(\mathcal{X} \times_S E)$, then $U \to \mathcal{X} \times_S E$ and $\mathcal{X} \times_S E \to \mathcal{X}$ are étale.

(1) Let $X \to \mathcal{X}$ be a smooth surjection. Need to define $\Omega^1_{X/\mathcal{X}}$.

$$
\begin{array}{ccccc}
Z' = Z \times_X Z & \rightrightarrows & Z & \xrightarrow{p_2} & X \\
\downarrow & & \downarrow{\scriptstyle p_1} & & \downarrow \\
Z = X \times_{\mathcal{X}} X & \rightrightarrows & X & \longrightarrow & \mathcal{X}
\end{array}
$$

We see that $\Omega^1_{Z/X}$ comes with descent data relative to $p_2 : Z \to X$ (because both pullbacks are the sheaf of differentials of Z' over $X \times_{\mathcal{X}} X$). We define $\Omega^1_{X/\mathcal{X}}$ to be the descended sheaf.

Remark: Note that you can also define $\Omega^1_{X/\mathcal{X}}$ as the conormal sheaf of $X \to X \times_{\mathcal{X}} X$, also by descent theory.

There is a map $\Omega^1_{X/S} \to \Omega^1_{X/\mathcal{X}}$ because of the remark above and the diagram below.

$$
\begin{array}{ccc}
X & \xrightarrow{\Delta} X \times_{\mathcal{X}} X & \qquad \Delta^* I \\
 & {\scriptstyle \Delta'}\searrow \quad \downarrow & \\
 & X \times_S X & \qquad \Delta^* \pi^* J
\end{array}
$$

Lemma 33.6. *This map $\Omega^1_{X/S} \to \Omega^1_{X/\mathcal{X}}$ is surjective. (in general, $\Omega^1_{X/\mathcal{X}}$ need not be locally generated by differentials of functions.)*

Proof. We check that the pullback to Z is surjective.

$$
\begin{array}{ccc}
Z & \longrightarrow & X \times X \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X}
\end{array}
$$

and $Z \to X \times X$ is formally unramified (because Δ is). So we have that $p_1^* \Omega^1_{X/S} \oplus p_2^* \Omega^1_{X/S} \to \Omega^1_{Z/S}$ is surjective. Set $\Omega^1_{p_1} = \mathrm{coker}(p_1^* \Omega^1_{X/S} \to \Omega^1_{Z/S}) = \Omega^1_{Z/X} = p_2^* \Omega^1_{X/\mathcal{X}}$.

$$
\begin{array}{ccc}
Z & \xrightarrow{p_2} & X \\
{\scriptstyle p_1}\downarrow & & \downarrow \\
X & \longrightarrow & \mathcal{X}
\end{array}
$$

So $p_2^* \Omega^1_{X/S} \to p_2^* \Omega^1_{X/\mathcal{X}}$ is surjective, and this is exactly the map we were considering [[★★★ exercise]]. $\qquad \Box_{\mathrm{Lemma}}$

So we have the following diagram. We may assume X is a scheme.

$$
x' \longmapsto x
$$

$$
\begin{array}{ccc}
X_y & \longrightarrow & X \\
\downarrow & & \downarrow \\
\mathrm{Spec}\, k & \xrightarrow{y} & \mathcal{X}
\end{array}
$$

In a neighborhood of x, there exist $f_1, \ldots, f_r \in \mathcal{O}_X$ such that images of $df_1, \ldots, df_r$ form a basis for the sheaf $\Omega^1_{X/\mathcal{X}}$ (it is a locally free sheaf; choose f_i such that df_i map to a basis for $\Omega^1_{X/\mathcal{X}}(x)$, then they form a basis because locally free). This gives a map to affine space.

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & \mathcal{X} \times_S \mathbb{A}^r_S \\
\downarrow & \swarrow & \\
\mathcal{X} &
\end{array}
$$

Lemma 33.7. *f is étale.*

Proof. If $\mathcal{X}$ were a scheme, this would be [EGA, IV.17.11.1]. To reduce to this case, make a base change

$$
\begin{array}{ccccccc}
X & \xleftarrow{\ f\ } & \mathbb{A}^r_{\mathcal{X}} & \longleftarrow & X_W & \xrightarrow{\ f_W\ } & \mathbb{A}^r_W \\
\downarrow & \swarrow & & & & & \\
\mathcal{X} & & & & W & &
\end{array}
$$

f_W is étale by EGA. $\square_{\text{Lemma}}$

$\square$

Recall that we are trying to prove that if $\mathcal{X}/S$ is an algebraic stack, then $\mathcal{X}$ is Deligne-Mumford if and only if $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is formally unramified. Last time we did $\Rightarrow$. For the other direction, let k be a field and let $y : \operatorname{Spec} k \to \mathcal{X}$ be a point. Then we want to produce an étale $U \to \mathcal{X}$ so that $U_y \neq \varnothing$. Last time we took a smooth presentation $X \to \mathcal{X}$ and we factored this map through an étale map $\pi : X \to \mathcal{X} \times_S \mathbb{A}^r_S$ (this is where we used formally unramified).

Claim: there exists a subscheme $E \subseteq \mathbb{A}^r_S$ which is étale over S such that if $U = \pi^{-1}(X \times_S E)$, $U_y \neq \varnothing$. Then we have a composition of étale morphisms $U \to \mathcal{X} \times_S E \to \mathcal{X}$, so it is étale.

Let k_0 be the residue field of the image of $\operatorname{Spec} k$ in S and let k_0^s be a separable closure of k_0. Consider the morphism $\pi_y : X \times_{\mathcal{X}, y} \operatorname{Spec} k = X_y \to \mathbb{A}^r_k = (\mathcal{X} \times_S \mathbb{A}^r) \times_{\mathcal{X}, y} \operatorname{Spec} k$. This is étale, so it has open image, so it contains $D(F)$, where $F \in k[x_1, \ldots, x_r]$. Since k_0^s is infinite, there exist $z_1, \ldots, z_r \in k_0^s$ such that $F(z) \neq 0$. This implies that there exists a closed point $Q \in \mathbb{A}^r_{k_0}$ with $\kappa(Q)$ finite separable over k_0 such that Q is in the

image of π_y composed with $\mathbb{A}_k^r \to \mathbb{A}_{k_0}^r$. [[★★★]]

$$
\begin{array}{ccccc}
\varnothing \neq Z_y \hookrightarrow & X_y \hookrightarrow & & X \\
& & \searrow & & \downarrow \\
& \mathcal{X}_0 \times_{k_0} Q \hookrightarrow & \mathcal{X} \times_S \mathbb{A}_S^r \\
& \downarrow & & \downarrow \\
& \mathcal{X}_0 \hookrightarrow & & \mathcal{X} \\
& \downarrow & & \downarrow \\
& \operatorname{Spec} k_0 \hookrightarrow & & S
\end{array}
$$
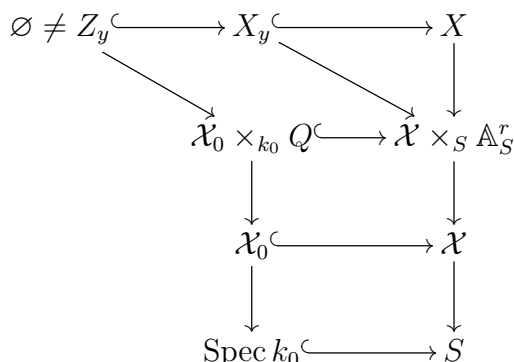
So we need to lift (extend, spread) $Q \subseteq \mathbb{A}_{k_0}^r$ to a subscheme $E \subseteq \mathbb{A}_S^r$ which is étale over S.

We can assume Q is a closed point because ... oh, it's not important.

Lemma 33.8. *Let A be a ring, let $x \in \operatorname{Spec} A$ be a closed point, and let $k(x) \to k'$ be a finite étale map of algebras. Then there exists a morphism $\operatorname{Spec} A' \to \operatorname{Spec} A$ which is finite étale over its image such that $A' \otimes_A k(x) \cong k'$.*

Proof. Can assume k' is a field (it is a finite product of separable field extensions). So $k' = k(x)[t]/p(t)$, where $p(t)$ is monic. Let $\tilde{p}(t) \in A[t]$ be a monic lifting of $p(t)$, and consider $B = A[t]/\tilde{p}(t)$. Then B is a finite flat A-algebra (because monic). To check étaleness at points, it is enough to check the fibers.

$$
\begin{array}{c}
\operatorname{Spec} B \\
\downarrow g \\
\operatorname{Spec} A
\end{array}
$$
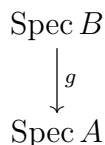
The above is étale at points lying over x. Let $Z \subseteq \operatorname{Spec} B$ be the closed set where this morphism is not étale. Then $\operatorname{Spec} B \times_{\operatorname{Spec} A} g(Z)^c \to \operatorname{Spec} A$ is finite étale over its image. $\qquad \square$

For a non-closed point, apply the lemma to $\mathcal{O}_{S, \operatorname{Spec} k_0}$. This gives you a finite étale $\operatorname{Spec} A' \to \operatorname{Spec} \mathcal{O}_{S, \operatorname{Spec} k_0}$ (local ring, so it's image is everything) and then "spread out".

This completes the proof of the whole theorem.

Remark 33.9 (Characterization of Ω^1 [[★★★ Random thing that should go somewhere]]). Let $T_0 \subseteq T$ be a subscheme of an affine scheme defined by some nilpotent [[★★★ square-zero?]] ideal I, let $f : Z \to W$ be a morphism of schemes, and consider the following diagram.

$$
\begin{array}{ccc}
T_0 & \xrightarrow{x_0} & Z \\
{\scriptstyle I}\downarrow & \nearrow & \downarrow {\scriptstyle f} \\
T & \longrightarrow & W
\end{array}
$$
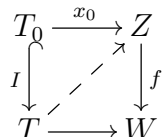
The set of dashed arrows is a $\mathrm{Hom}(x_0^*\mathcal{W}_{Z/W}^1, I)$-torsor. This characterizes $\Omega_{Z/W}^1$.

$\diamond$

Artin's Theorem

Example 33.10. Say X/S is an algebraic space and G/S is a smooth group scheme acting freely on X. Then the quotient $[X/G]$ is an algebraic space [[★★★ explain]]. This is not clear because it only comes with a smooth covering. $\diamond$

Corollary 33.11. *Let $\mathcal{X}$ be an algebraic stack over S. Then the following are equivalent.*

1. *$\mathcal{X}$ is an algebraic space,*

2. *for every $x \in \mathcal{X}$, $\underline{\mathrm{Aut}}_x = \{\mathrm{id}\}$.*

Proof. $(1 \Rightarrow 2)$ is clear. $(2 \Rightarrow 1)$ Let's check that Δ is formally unramified. This is clear because doing something, we get a non-trivial automorphism of x_1.

$$
\begin{array}{ccc}
T_0 & \hookrightarrow & T \\
\downarrow & \searrow & \downarrow \\
\underline{\mathrm{Isom}}(x_1, x_2) & \longrightarrow & Z \\
\downarrow & & \downarrow {\scriptstyle (x_1, x_2)} \\
\mathcal{X} & \longrightarrow & \mathcal{X} \times_S \mathcal{X}
\end{array}
$$

So there exists an étale cover $X \to \mathcal{X}$. Also, 2 implies that $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is a monomorphism, which implies that $\mathcal{X} \cong |\mathcal{X}|$ (the sheaf of isomorphism classes of things in the fibers). Thus, we can assume $\mathcal{X} = F$ is a sheaf, and it has an étale cover $X \to F$. The remaining point is that $F \to F \times F$ is representable (by algebraic spaces); we have to check representability by schemes. Let T be a scheme.

$$
\begin{array}{ccc}
\Delta^{-1}(T) & \hookrightarrow & T \\
\downarrow & & \downarrow \\
F & \longrightarrow & F \times F
\end{array}
$$

where $\Delta^{-1}(T) \hookrightarrow T$ is separated and quasi-finite (the fibers are empty or one point), so scheme. $\square$

This should make you happy. Usually when we have moduli, they are presented as something with a group action.

Example 33.12. $B\mu_p$ over the field $\mathbb{F}_p$. Recall that $\mu_p = \ker(\mathbb{G}_m \to \mathbb{G}_m; u \mapsto u^p)$. We see that $B\mu_p = [\mathbb{G}_m/\mathbb{G}_m]$, where the action is $u * v = u^p v$. So $B\mu_p$ is an Artin (algebraic) stack, but it is not Deligne-Mumford in general.

$$
\begin{array}{ccc}
\mu_p & \xrightarrow[\text{unram.}]{\text{not}} & \operatorname{Spec} \mathbb{F}_p \\
\downarrow & & \downarrow {\scriptstyle (p_1,p_2)} \\
B\mu_p & \longrightarrow & B\mu_p \times B\mu_p
\end{array}
$$

you can't make an étale cover.   ◇

35   The Lisse-étale site on an algebraic stack

<u>Quasi-coherent sheaves.</u>   We have to define a good topology on a stack where the sheaves will live. Let $\mathcal{X}$ be an algebraic stack over S, then we want a good topos of sheaves on $\mathcal{X}$.

- étale. This is OK for Deligne-Mumford stacks (and is clearly the good topology). In general, if $\mathcal{X}$ is an algebraic stack, you might try to take objects to be $\mathcal{Y} \to \mathcal{X}$ representable étale morphisms of algebraic stacks. Then any morphism between two such things something. Morphisms are $\mathcal{X}$-morphisms. This gives a site, but it doesn't have enough sheaves.

  Let $k = \bar{k}$ and let G/k be a connected smooth group scheme (like $G = \mathbb{G}_m$). Then BG has no non-trivial (non-identity) étale covers. Reason: assume $\mathcal{Y} \to BG$ is étale, then

$$
\begin{array}{ccc}
Y & \longrightarrow & \mathcal{Y} \\
{\scriptstyle et}\downarrow & & \downarrow{\scriptstyle et} \\
\operatorname{Spec} k & \longrightarrow & BG
\end{array}
$$

  Then $Y = \coprod \operatorname{Spec} k$ with an action of G. Since G is connected, this is the trivial action. This implies that $\mathcal{Y}$ is a disjoint union of copies of BG.

- flat site (fppf). This is OK, but not great. It is bad for differential geometry.

- lisse-étale, which has its own problems, but it is the best we can do.

Definition 35.1. Let $\mathcal{X}$ be an algebraic stack over S. The *lisse-étale site Lis-Et($\mathcal{X}$)* of $\mathcal{X}$ has objects smooth morphisms $u : U \to \mathcal{X}$ with U an algebraic space (notation: we will denote such an object by (U, u)) and morphisms $(U, u) \to (V, v)$ are pairs $(f, f^\flat)$ where $f : U \to V$ is an S-morphism and $f^\flat : u \to v$ is an (automatically cartesian) arrow in $\mathcal{X}$ over f. A collection $\{(U_i, u_i) \xrightarrow{(f_i, f_i^\flat)} (V, v)\}$ is a covering if $\coprod U_i \to V$ is étale surjective. Let $\mathcal{X}_{lis\text{-}et}$ be the associated topos. We define the structure sheaf $\mathcal{O}_{\mathcal{X}_{lis\text{-}et}}$ to be $(U, u) \mapsto \Gamma(U_{et}, \mathcal{O}_{U_{et}})$. ◇

Remark 35.2. Concretely, to give a sheaf F in this site is equivalent to a sheaf $F_{(U,u)}$ on the étale site U_{et} for every object $(U, u) \in Lis\text{-}Et(\mathcal{X})$ and transition maps (don't have to be isomorphisms) $\phi_f : f^{-1}F_{(V,v)} \to F_{(U,u)}$ for every morphism $(f, f^\flat) : (U, u) \to (V, v)$ compatible with composition ($\phi_{gf} = \phi_f \circ f^{-1}\phi_g$) [[★★★ I think we have to require that when f is étale, ϕ_f is an isomorphism. This is to verify the sheaf condition on the lisse-étale sheaf you build from such data]].

   [[★★★ write out the verification better]] Given a sheaf $F \in \mathcal{X}_{lis\text{-}et}$, we define $F_{(U,u)}$ to be the restriction of F to the small étale site of U. It is easy to see that the sheaf condition on F implies the sheaf condition on $F_{(U,u)}$. For a morphism $f^\flat : u \to v$ (which projects to $f : U \to V$), we get a morphism $f^{-1}F_{(V,v)} \to F_{(U,u)}$ by the universal property of limits (you have to remember the definition of f^{-1}); since this is defined

by a universal property, it probably automatically plays well with composition. Note that if f is étale, then this map is an isomorphism (because the directed system over which you take a limit has a terminal object).

Conversely, if you have the data, define $F(U,u) := F_{(U,u)}(\mathrm{id}_U)$ and observe that the ϕ_f give you restriction maps. The sheaf axiom follows from the fact that ϕ_f is an isomorphism when f is étale. [[★★★ complete this]] ◇

Remark 35.3. If $\mathcal{X}$ is an algebraic space, then this is as close to the étale topology as you can get: the inclusion $Et(\mathcal{X}) \hookrightarrow Lis\text{-}Et(\mathcal{X})$ is continuous and $Et(\mathcal{X})$ has finite projective limits, so we get a morphism of topoi $\varepsilon : \mathcal{X}_{lis\text{-}et} \to \mathcal{X}_{et}$. Note that ε_* is exact: $\varepsilon_* F = F_{(\mathcal{X},\mathrm{id}_\mathcal{X})}$ is exact because do something on each étale something [[★★★]]. ◇

It might be true that the Lisse-Lisse site gives you the same topos (even though it looks like it should give a smaller topos) because every smooth morphism étale locally has a section. I dunno.

Example 35.4. $Lis\text{-}Et(\mathcal{M}_{1,1})$ has objects pairs (U, E_U) where U is an algebraic space and E_U is an elliptic curve over U such that $E_U : U \to \mathcal{M}_{1,1}$ is smooth [[★★★ how should I think about this condition?]]. Morphisms incorporate the automorphisms: a morphism $(U, E_U) \to (V, E_V)$ is a cartesian diagram

$$
\begin{array}{ccc}
E_U & \xrightarrow{f^\flat} & E_V \\
e_U \big\uparrow\big\downarrow & & \big\uparrow\big\downarrow e_V \\
U & \xrightarrow{f} & V
\end{array}
$$

You have choices for $f^\flat$.

What is an example of a sheaf? If (U, E_U) is an object, define $\omega_U = e^* \Omega^1_{E_U/U}$ (e is the section $U \to E_U$), which is a locally free sheaf of rank 1 on U_{et}. We have a cartesian diagram as above. Then we get $f^* e_V^* \Omega^1_{E_V/V} \cong e_U^* \Omega^1_{E_U/U}$ [[★★★ here upper f^* means f^{-1}; note that $\Omega^1_{E_U/U} = f^{\flat *} \Omega^1_{E_V/V}$]] so we get a sheaf $\omega \in \mathcal{M}_{1,1,lis\text{-}et}$. This would work even in the big fppf site.

Note that this is an $\mathcal{O}_{\mathcal{M}_{1,1}}$-module. ◇

Definition 35.5. A sheaf F of $\mathcal{O}_\mathcal{X}$-modules is *quasi-coherent* if each $F_{(U,u)}$ is quasi-coherent and for every morphism $(f, f^\flat) : (U,u) \to (V,v)$, the map $f^\star F_{(V,v)} \to F_{(U,u)}$ is an isomorphism. ◇

If $F_{(V,v)}$ is a sheaf of ideals and f is not flat, then $f^* F_{(V,v)}$ need not be a sheaf of ideals . . . we'll take care of this later.

Remark 35.6. We can also talk about locally free sheaves of finite rank (each $F_{(U,u)}$ should be locally free of finite rank). In the locally noetherian case, we can talk about coherent sheaves. If $\mathcal{X}$ is an algebraic space, a sheaf F of $\mathcal{O}_{\mathcal{X}_{lis\text{-}et}}$-modules is quasi-coherent if and only if $F_X = \varepsilon_* F$ is quasi-coherent and the adjunction map $\varepsilon^* F_\mathcal{X} = \varepsilon^* \varepsilon_* F \to F$ is an isomorphism. [[★★★ this gives an equivalence of categories of quasi-coherent sheaves?]] ◇

Example 35.7. Let k be a field, and let G be a smooth group scheme over k. Then $\mathsf{Qcoh}(BG) \cong Rep_k(G)$.

$$F \operatorname{Spec} k \xrightarrow{\ f_g\ } \operatorname{Spec} k, F\, k\text{-vector space}$$

$$\downarrow$$

$$BG, \mathcal{F}\,\text{quasi-coherent sheaf}$$

for every $g \in G(k)$, you get a map f_g (something about the $\flat$s). Thus, we get an action of $G(k)$ on F. Exercise to check that this actually defines an equivalence of categories.

What is $H^0(BG, \mathcal{F})$? $\operatorname{Spec} k \to BG$ is a smooth surjection, so $H^0(BG, \mathcal{F}) \hookrightarrow F$.

$$
\begin{array}{ccc}
 & \operatorname{Spec} k & \\
\overset{et\ locally}{\nearrow} & \downarrow & \\
T \longrightarrow & BG &
\end{array}
$$

It turns out that $H^0(BG, \mathcal{F}) \cong F^G$. This can be generalized to $\mathcal{X} = [Y/G]$ over any base S where quasi-coherent sheaves become G-linearized and ... $\diamond$

Functoriality: If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks, we have no hope of getting a morphism of topoi [[★★★ we saw this for schemes?]]. We want adjoint functors $f_* : \mathcal{X}_{lis\text{-}et} \to \mathcal{Y}_{lis\text{-}et}$ and $f^{-1} : \mathcal{Y}_{lis\text{-}et} \to \mathcal{X}_{lis\text{-}et}$. f^{-1} is not exact, so this is not a morphism of topoi.

If f is representable, then we have a continuous map $Lis\text{-}Et(\mathcal{Y}) \to Lis\text{-}Et(\mathcal{X})$ given by $(U \xrightarrow{u} \mathcal{Y}) \mapsto (U \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X})$. To get the functors f_* and f^{-1}, there are two options:

(1) note that in the definition of the lisse-étale site, we could have taken morphisms from stacks instead of from algebraic spaces. Define $\widetilde{Lis\text{-}Et}(\mathcal{X})$ to have objects representable smooth morphisms $\mathcal{U} \to \mathcal{X}$ and morphisms are

$$
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\ f\ } & \mathcal{U}' \\
 & \overset{f^\flat}{\Longrightarrow} & \\
 & \searrow\ \ \swarrow & \\
 & \mathcal{X} &
\end{array}
$$

Then check that this gives you the same topos.

(2) First define f^{-1} on representable sheaves. If $y : Y \to \mathcal{Y}$ is smooth and $u : U \to \mathcal{Y}$, then $h_{(Y,y)}(U,u) = \operatorname{Hom}_{lis-et(\mathcal{Y})}\big((U,u),(Y,y)\big)$. Then $f^{-1}(h_{(Y,y)})(U \to \mathcal{X}) = \left\{ \begin{array}{c} U \longrightarrow Y \\ u\downarrow\ \ \nearrow\ \downarrow y \\ \mathcal{X} \longrightarrow \mathcal{Y} \end{array} \right\}$. Then define $(f_*F)(Y \xrightarrow{y} \mathcal{Y}) = \operatorname{Hom}_{\mathcal{Y}_{lis\text{-}et}}(h_{(Y,y)}, f_*F) = \operatorname{Hom}_{\mathcal{X}_{lis\text{-}et}}(f^{-1}h_{(Y,y)}, F) = \Gamma(\mathcal{X}_{lis\text{-}et}|_{f^{-1}(Y,y)}, F)$

Recall that if T is a topos and $G \in T$, then we can define T/G, which turns out to be a topos. For $G \in \mathcal{Y}_{lis\text{-}et}$, define $(\hat{f}^{-1}G)(U \xrightarrow{u} \mathcal{X})$ to be the limit over $I_{(U,u)}$, the

category of diagrams

$$
\begin{array}{ccc}
U & \longrightarrow & V \\
{\scriptstyle u}\downarrow & \nearrow\!\!\!\!\Rightarrow & \downarrow{\scriptstyle v} \\
\mathcal{X} & \longrightarrow & \mathcal{Y}
\end{array}
$$

of $G(V, v)$. Then $f^{-1}G$ is the sheaf associated to the presheaf $\hat{f}^{-1}G$.

Remark 35.8. $I_{(U,u)}$ is not filtering, which is what ruins the exactness of f^{-1}. ◇

Remark 35.9. There is a natural map $f^{-1}\mathcal{O}_{\mathcal{Y}_{lis\text{-}et}} \to \mathcal{O}_{\mathcal{X}_{lis\text{-}et}}$. So we can define $f^*G = f^{-1}G \otimes_{f^{-1}\mathcal{O}_{\mathcal{Y}_{lis\text{-}et}}} \mathcal{O}_{\mathcal{X}_{lis\text{-}et}}$ for an $\mathcal{O}_{\mathcal{Y}}$-module G. ◇

Proposition 35.10. *Let G be a quasi-coherent sheaf on $\mathcal{Y}$. Then f^*G is also quasi-coherent.*

Proof. Note that $f^*G = (\hat{f}^{-1}G \otimes_{f^{-1}\mathcal{O}_{\mathcal{Y}}} \mathcal{O}_{\mathcal{X}})^a$ because they have the same universal property ... use the adjunction. We need to compute $\lim_{I_{(U,u)}} (\tilde{f}^*G_V)(U)$. Define $\bar{I}_{(U,u)}$ to be the poset whose elements are $(g : U \to V) \in \mathcal{I}_{(U,u)}$ where we say that $(g : U \to V) \geq (g' : U \to V')$ if there exists a $V \to V'$ in $I_{(U,u)}$.

The category $I_{(U,u)}$ is almost filtering; we just squash multiple arrows together.

We have a surjection $I_{(U,u)} \to \bar{I}_{(U,u)}$. The claim is that

$$
\begin{array}{ccc}
I_{(U,u)} & \xrightarrow{\ \ \tilde{f}^*G_V\ \ } & \Gamma(U, \mathcal{O}_U)\text{-mod} \\
& \searrow \qquad \nearrow & \\
& \bar{I}_{(U,u)} &
\end{array}
$$

$\square$

36 Quasi-coherent sheaves on algebraic stacks

Recall that last time we defined the lisse-étale topos $Lis\text{-}Et(\mathcal{X})$ for an algebraic stack $\mathcal{X}$. We started proving the following.

Proposition 36.1. *If $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of algebraic stacks and G is a quasi-coherent sheaf on $\mathcal{Y}$, then $f^\star G$ is quasi-coherent on $\mathcal{X}$.*

Proof. $f^\star G$ is the sheaf associated to the presheaf $(U \to \mathcal{X}) \mapsto \lim_{I_{(U,u)}} g^\star G_V(U)$. The objects of $I_{(U,u)}$ are diagrams

$$
\begin{array}{ccc}
U & \xrightarrow{\ g\ } & V \\
{\scriptstyle u}\downarrow & {\scriptstyle g^\flat}\nearrow & \downarrow{\scriptstyle v} \\
\mathcal{X} & \underset{f}{\longrightarrow} & \mathcal{Y}
\end{array}
$$

and morphisms $V \to V'$ are diagrams

$$
\begin{array}{c}
V' \\
\end{array}
$$

(1) all morphisms in the limit are isomorphisms: $g^\star G_V = g'^\star \pi^\star G_V \xrightarrow{\sim} g'^\star G_{V'}$. If $I_{(U,u)}$ were filtering, this would imply that for any object D_{V_0} the map $g^\star G_{V_0} \to \lim_{I_{(U,u)}} g^\star G_V$ is an isomorphism.

Example 36.2. Say V is a vector space and let α be an automorphism. Then the limit of the diagram $V \underset{\mathrm{id}}{\overset{\alpha}{\rightrightarrows}} V$ is $V/(\alpha(v) - v)$ ◇

The claim is that the functor $F : I_{(U,u)} \to \mathsf{Ab}$ given by $D_V \mapsto g^\star G_V(U)$ factors though $\bar{I}_{(U,u)}$ whose objects are the same as the objects of $I_{(U,u)}$ and where all the morphisms have been crushed together (so there is either one morphism or no morphisms between two objects).

If the claim holds, then $\lim_{I_{(U,u)}} F = \lim_{\bar{I}_{(U,u)}} \overline{F}$ where $\overline{F} : \bar{I}_{(U,u)} \to \mathsf{Ab}$ is the factorization.

$\mathrm{Hom}_{\mathsf{Ab}}(\lim_{I_{(U,u)}} F, M) = \mathrm{Hom}_{\mathsf{Ab}^{I_{(U,u)}}}(F, M) = \mathrm{Hom}_{\mathsf{Ab}^{\bar{I}}}(\overline{F}, M)$.

Proof of claim:

We want the following arrows to be equal

$$g_2^\star G_{V_2} \xrightarrow{\sim} g_1^* p_1^* G_{V_2} \longrightarrow g_1^* G_{V_1}$$
$$g_2^\star G_{V_2} \xrightarrow{\sim} g_1^* p_2^* G_{V_2} \longrightarrow g_1^* G_{V_1}$$

with the diagram: $g_1 : Z \to V_1$, $U \xrightarrow{2} V_2$, $\ulcorner$, $Z \to V_1$, $V_2 \xrightarrow{\Delta} V_2 \times_{\mathcal{Y}} V_2$.

$g_2^* G_{V_2} \Longrightarrow g_2^* \Delta^* G_{V_2 \times_{\mathcal{Y}} V_2} = g_1^* (p_1, p_2)^* G_{V_2 \times_{\mathcal{Y}} V_2} \longrightarrow g_1^* G_{V_1}$. So we need to show that the two maps $g_2^* G_{V_2} \Longrightarrow g_2^* \Delta^* G_{V_2 \times_{\mathcal{Y}} V_2}$ are equal and we have $\phi_\Delta : g_2^* \Delta^* G_{V_2 \times_{\mathcal{Y}} V_2} \xrightarrow{\sim} g_2^* G_{V_2}$ (because G quasi-coherent), and the compositions are the identity, so the two maps must be equal. □

Remark 36.3. In the locally noetherian case, we see that $f^\star$ takes coherent sheaves to coherent sheaves.   ◇

Proposition 36.4. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated ($\Delta : \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact) morphism, and let F be quasi-coherent on $\mathcal{X}$. Then $f_* F$ is quasi-coherent on $\mathcal{Y}$.*

Proof. Reduction to the case when $\mathcal{Y} = Y$ is a quasi-compact algebraic space.
    Let $Y \to \mathcal{Y}$ be a smooth morphism. Then we defined $F(Y) = \Gamma(\mathcal{X}|_{f^{-1}Y}, F)$.

Lemma 36.5. *Let $\pi : \mathcal{Z} \to \mathcal{X}$ be a smooth representable morphism and let F a sheaf on $\mathcal{X}_{lis\text{-}et}$. Then $\Gamma(\mathcal{X}|_{\mathcal{Z}}, F) = \Gamma(\mathcal{Z}_{lis\text{-}et}, \pi^\star F)$. Here $\mathcal{X}|_{\mathcal{Z}}$ is what? $\mathcal{Z}$ is a sheaf ($h_{\mathcal{Z}}$) on $\mathcal{X}_{lis\text{-}et}$ [[★★★ object are triangles $U \to Z \to X$ with 2-morphism]]. On the right side, you have an inclusion $Lis\text{-}Et(\mathcal{Z}) \to Lis\text{-}Et(\mathcal{X})$*

The site $Lis\text{-}Et(\mathcal{X})|_{\mathcal{Z}}$ has objects $(U, z : U \to \mathcal{Z})$ such that U is an algebraic space and $\pi \circ z : U \to \mathcal{X}$ is smooth (this is bigger than $Lis\text{-}Et(\mathcal{Z})$. The morphisms $(U', z') \to (U, z)$ are triangles

$$U' \xrightarrow{f} U'$$

with z' and z mapping to $\mathcal{Z}$, with a 2-morphism $\Uparrow$.

Then $\mathcal{X}|_{\mathcal{Z}}$ is the topos associated to $Lis\text{-}Et(\mathcal{X})|_{\mathcal{Z}}$.
    So there is a natural map $\Gamma(\mathcal{X}|_{\mathcal{Z}}, F) \to \Gamma(\mathcal{Z}, F)$, which we want to check is a bijection.

Proof. This is a bijection because if $Z \to \mathcal{Z}$ is a smooth surjection, then étale locally any morphism $U \to \mathcal{Z}$ factors through Z.

$$U \xrightarrow[u]{} \mathcal{Z}$$

with $\tilde{u} : U \to Z$ and α_Z.

If $\tilde{u}$ exists, then you have to choose $\tilde{u}^*\alpha_Z$. This guy doesn't depend on the choice of lifting. If $\tilde{u}_1, \tilde{u}_2 : U \to Z$ are two liftings, then we have

$$
\begin{array}{ccc}
 & & Z \times_{\mathcal{Z}} Z \\
 & {}^{\gamma}\nearrow & \Downarrow \\
U & & Z \\
 & \searrow & \downarrow \\
 & & \mathcal{Z}
\end{array}
$$

Then we have that $\tilde{u}_1^*\alpha_Z = \gamma^*p_1^*\alpha_Z = \gamma^*\alpha_{Z\times_{\mathcal{Z}}Z}$. That shows that the map is surjective and it also shows that it is injective because of something [[★★★]]   □

Let's finish the reduction to the case of a quasi-compact Y. We know that if $Y \to \mathcal{Y}$, then $(f_*F)_Y = (f_{Y*}F|_{\mathcal{X}\times_{\mathcal{Y}}\mathcal{X}})_Y$ with $f_Y : \mathcal{X} \times_{\mathcal{Y}} Y \to Y$. Then we still have to check something cartesian.

$$
\begin{array}{ccc}
Y & \xleftarrow{p_1} Y \times_{\mathcal{Y}} Y' \xrightarrow{\Gamma_g} & Y' \\
\downarrow & \downarrow{p_2} \swarrow {\scriptstyle\text{id}} & \\
\mathcal{Y} & \xleftarrow{\hspace{2em}} Y' &
\end{array}
$$

$g^*(f_*F)_Y = \Gamma_g^*p_1^*(f_*F)_Y \xrightarrow{\sim} \Gamma_g^*(f_*F)_{Y\times_{\mathcal{Y}}Y'} \to (f_*F)_{Y'}$ want to be an isomorphism. $\Gamma_g^*p_2^*(f_*F)_{Y'} = (f_*F)_{Y'}$ and the first guy maps to the third guy isomorphically and the second guy is equal to the last guy.

Upshot: Enough to consider $\mathcal{Y}$ a quasi-compact scheme

□

37   Push-forward of Quasi-coherent sheaves

Recall that we're trying to prove

Proposition 37.1. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a quasi-compact and quasi-separated morphism of algebraic stacks, and let F be quasi-coherent on $\mathcal{X}$. Then f_*F is quasi-coherent on $\mathcal{Y}$.*

Last time we reduced to the case where $\mathcal{Y} = Y$ is a quasi-compact scheme.

Choose a presentation $X \xrightarrow{p} \mathcal{X}$ such that $X \to Y$ is quasi-compact and quasi-separated. Let $X' = X \times_{\mathcal{X}} X$. Then

$$
\begin{array}{ccc}
X \times_{\mathcal{X}} X \xrightarrow{\text{qcmpt}} X \times_Y X \xrightarrow{\text{qcmpt}} Y \\
\downarrow \qquad\qquad \downarrow \\
\mathcal{X} \xrightarrow{\quad\Delta\quad} \mathcal{X} \times_Y \mathcal{X}
\end{array}
$$

$(g : X \to Y)$ so $\eta : X' \to Y$ is also quasi-compact. Choose an étale surjection $W \to X'$ from a scheme so that $h : W \to Y$ is quasi-compact and quasi-separated (we could take it to be a disjoint union of affines). Then $(f_*F)|_Y = \ker(\, g_*F_X \rightrightarrows h_*F_W \,) = \ker(\, g_*F_X \rightrightarrows \eta_*F_{X'} \,)$ $(\eta_*F_{X'} \hookrightarrow h_*F_W)$. This is because $(f_*F)(Y) = \Gamma(\mathcal{X}_{lis\text{-}et}, F) = Eq(\, \Gamma(X, F_X) \rightrightarrows \Gamma(X', F_{X'}) \,)$ by some stuff we did last time (nontrivial).

So $(f_*F)_Y$ is quasi-coherent.

$$
\begin{array}{ccccc}
\tilde{W} \rightrightarrows \tilde{X} \longrightarrow \mathcal{X}' \xrightarrow{f'} Y' \\
\downarrow \qquad \downarrow \qquad \downarrow{\scriptstyle t} \qquad \downarrow{\scriptstyle s}\, {\scriptstyle sm} \\
W \rightrightarrows X \longrightarrow \mathcal{X} \xrightarrow{f} Y
\end{array}
$$

Then want the map $s^*(f_*F)_Y \to (f'_*t^*F)_{Y'}$ to be an isomorphism. We are using the lemma from last time that something in the restricted topos is the same as something else [[★★★]]. This follows because the formation of f_*F for f a quasi-compact quasi-separated morphism of schemes commutes with flat base change.

This concludes the proof of the proposition.

Example 37.2. Let $\mathcal{X} = BG$ over a field k. Then $\mathsf{Qcoh}(\mathcal{X}) \cong Rep(G)$ and the functor $\Gamma(\mathcal{X}, F) = F^G$. More generally, suppose $H \subseteq G$ is a normal closed subgroup scheme with quotient Q. Then we get a map $f : BG \to BQ$. If you have a G-torsor P over some scheme, then you take $P \times^G Q$ (product with Q and quotient by the diagonal action of G). Then we have

$$
\begin{array}{ccccc}
\mathsf{Qcoh}(BG) \xrightarrow{\ \sim\ } Rep(G) & & M \\
{\scriptstyle f_*}\downarrow \qquad\qquad \downarrow & & \downarrow \\
\mathsf{Qcoh}(BQ) \xrightarrow{\ \sim\ } Rep(Q) & & M^H
\end{array}
$$

$\diamond$

Warning 37.3 (Advertisement for derived categories and cohomological descent). We can define $R^i f_* F$ for some quasi-compact quasi-separated morphism $f : \mathcal{X} \to \mathcal{Y}$, but you have to be very careful. You'd like to say that f_* of an injective is injective (or acyclic, something in abelian groups [[★★★]]), which uses that f^* is exact.

Given $\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z}$, we want a spectral sequence $E_2^{p,q} = R^p g_* R^q f_* F \Rightarrow R^{p+q}(gf)_* F$. This is the Leray spectral sequence, and you really need that f_* of an injective is acyclic, for which you want an exact left adjoint. Things will still work out for quasi-coherent stuff, but not for sheaves of abelian groups in general. Because of these difficulties, we'll postpone this discussion.

We have something $E_2^{p,q} = H^p(Q, H^q(H, M)) \Rightarrow H^{p+q}(G, M)$.   ⌟

38   Keel-Mori

Coarse moduli spaces

Since the first day, we've talked about the j-invariant $\mathcal{M}_{1,1} \to \mathbb{A}^1_j$. (1) This is universal for maps to schemes in the sense that for any $\mathcal{M}_{1,1} \to Z$, there is a unique factorization through $\mathbb{A}^1_j$. (2) if k is an algebraically closed field, then $|\mathcal{M}_{1,1}(k)| \xrightarrow{\sim} \mathbb{A}^1_j(k)$.

   In general, the analogue of $\mathbb{A}^1_j$ will be called a *coarse moduli space*.

   Say S is a scheme and $\mathcal{X}$ is an algebraic stack over S [[★★★ why not any stack? something about relative coarse moduli spaces]] with finite diagonal. Recall that $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is finite if for any $(x_1, x_2) : T \to \mathcal{X} \times_S \mathcal{X}$, $\underline{\mathrm{Isom}}(x_1, x_2) \to T$ is finite (in particular, $\underline{\mathrm{Isom}}(x_1, x_2)$ is a scheme). Quasi-finite is not enough, but maybe you can relax this slightly to say that the $\underline{\mathrm{Aut}}_x$ are finite.

   All the hypothesis in the paragraph above should go below the definition.

Definition 38.1. A *coarse moduli space* for $\mathcal{X}$ over S is a morphism $\pi : \mathcal{X} \to X$ where X is an algebraic space over S such that

   1. π is universal for morphisms to algebraic spaces:

   2. if k is algbraically closed, then the map $|\mathcal{X}(k)| \to X(k)$ is bijective.   ◇

Remark 38.2. $\pi : \mathcal{X} \to X$ is unique up to unique isomorphism (you only need property 1 for this).   ◇

Theorem 38.3 (Keil-Mori (1997 in annals), Conrad (in his web page))**.** *With the above assumptions, there exists a coarse moduli space* $\pi : \mathcal{X} \to X$ *($\mathcal{X}$ need not be Deligne-Mumford). Additionally,*

   1. *X is separated over S and if S is locally noetherian, then X is locally of finite type over S.*

   2. *π is proper (we haven't defined this yet)*

   3. *if $X' \to X$ is a flat morphism of algebraic spaces, then $\mathcal{X}' = \mathcal{X} \times_X X' \to X'$ is a coarse moduli space for $\mathcal{X}'$.*

   This theorem was folklore for a long time, then it was written without proof in places. Keil and Mori proved it in the noetherian case. Usually, once you prove something in the noetherian case, you can get it in general, but there is no reason for that to be true here because of the following warning.

Warning 38.4.   The formation of coarse moduli space *does not* commute with arbitrary base change, only with flat base change.   ⌐

What if you drop the finiteness assumption on the diagonal. You have to throw out condition (2) [[★★★ for some reason]]. For example, it seems like the coarse moduli space for $[\mathbb{A}^1_k/\mathbb{G}_m]$ should be $\operatorname{Spec} k$. This satisfies condition (1), but the proof breaks horribly.

We'll give the proof in the locally noetherian case. The proof in general is along the same lines, but more technical.

Idea: start with your stack $\mathcal{X}$. First (we'll do it last) go to the case where there exists a quasi-finite flat surjection $U \to \mathcal{X}$. This isn't too bad (in [SGA, III]). The really subtle step is to then go to the case where you have a *finite* flat surjection $U \to \mathcal{X}$. Back when we talked about Stein factorization, we had some sort of argument like if we have $U \to X$ quasi-finite, you pick a point $x \in X$ and you can find an étale neighborhood V su that $P \coprod U \to V$ is quasi-finite and finitely presented over V. Then we reduce to the case where there exists a finite flat surjection $\operatorname{Spec} A_1 \to \mathcal{X}$ (in [SGA, III]). This case, we've essentially done. Many of the technical lemmas are also in [SGA, III].

What does it mean to be proper?

Definition 38.5. If $f : \mathcal{X} \to Y$ is a morphism from an algebraic stack to a scheme, then f is *closed* if for every closed substack $\mathcal{Z} \subseteq \mathcal{X}$, the image of $\mathcal{Z}$ in Y is closed. By the image, we mean take all the field-valued points in $\mathcal{Z}$ and they give you field-valued points in Y, and that set is the image; equivalently, cover $\mathcal{Z}$ by a scheme and look at the image in Y. A morphism of algebraic stacks $f : \mathcal{X} \to \mathcal{Y}$ is *universally closed* if for every morphism $Y \to \mathcal{Y}$ with Y a scheme, the morphism $\mathcal{X} \times_{\mathcal{Y}} Y \to Y$ is closed. $f : \mathcal{X} \to \mathcal{Y}$ is *proper* if it is separated (diagonal is proper [[★★★ does this actually make sense?]]), finite type, and universally closed.   ◇

If $\pi : \mathcal{X} \to X$ is a coarse moduli space and π is proper, then $\pi^{-1} : OpenSet(X) \to OpenSubstack(\mathcal{X})$ is bijective. Reason: say $\mathcal{U} \subseteq \mathcal{X}$ is an open substack, then the claim is that $\pi(\mathcal{U})$ is open in X and $\pi^{-1}(\pi(\mathcal{U})) = \mathcal{U}$. Complement of $\mathcal{U}$: take a presentation $p : \tilde{X} \to \mathcal{X}$, then

$$
\begin{array}{ccccc}
\tilde{U}' & \hookrightarrow & \tilde{X}' & \hookleftarrow & \tilde{Z}' \\
\downdownarrows & & \downdownarrows & & \downdownarrows \\
\tilde{U} & \hookrightarrow & \tilde{X} & \hookleftarrow & \tilde{Z} \\
\downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\
\mathcal{U} & \hookrightarrow & \mathcal{X} & \hookleftarrow & \mathcal{Z} \\
\downarrow & & \downarrow & & \downarrow \\
U & \longrightarrow & X & \hookleftarrow & Z
\end{array}
$$

where Z is the complement of U with the reduced structure. Formation of maximal reduced subscheme commutes with smooth base change, so Z descends. π proper so $\pi(\mathcal{Z}) \subseteq X$ is closed. The claim is that $\pi^{-1}(\pi(\mathcal{Z}))_{red} = \mathcal{Z}$. This is because $|\mathcal{X}(k)| \to X(k)$ is bijective. $\tilde{U}$ is an algebraic space because $\mathcal{U} \to \mathcal{X}$ is representable.

In some sense, the coarse moduli space is a universal homeomorphism.

Proof: I

S is a scheme and $\mathcal{X}$ is an algebraic stack over S which is locally of finite presentation (forgot this last time, but it should be there) with finite diagonal.

Theorem 38.6. *There exists a coarse moduli space over S $\pi : \mathcal{X} \to X$ so that all that stuff from last time [[★★★]]. Additionally,*

1. *X is separated over S and locally of finite type if S is locally noetherian.*

2. *π is proper. (i.e. that bijection of opens)*

3. *if $X' \to X$ is flat, then $\mathcal{X}' = \mathcal{X} \times_X X' \to X'$ is a coarse moduli space.*

The typed up note for this are online. We only give the proof in the locally noetherian case.

In the example $[\mathbb{A}^1/\mathbb{G}_m]$, something doesn't work. For BG, we have that $BG \to \operatorname{Spec} k$ is a coarse moduli space (because you only do the testing on algebraically closed fields), but without these special properties. Is there an example where the coarse moduli space doesn't exist.

In general, if you have a quotient $[X/G]$, the ring of invariants should be the coarse moduli space, but it might be infinitely generated or something else bad.

Remark 38.7 (Theorem is Zariski local on $\mathcal{X}$). If $\mathcal{X} = \bigcup \mathcal{X}_i$ where each $\mathcal{X}_i$ is open in $\mathcal{X}$ and each $\mathcal{X}_i$ has a coarse moduli space $\mathcal{X}_i \to X_i$ as in theorem (with the properties), then $\mathcal{X}$ also has a coarse moduli space as in theorem. To see that, take the $\mathcal{X}_i$ and
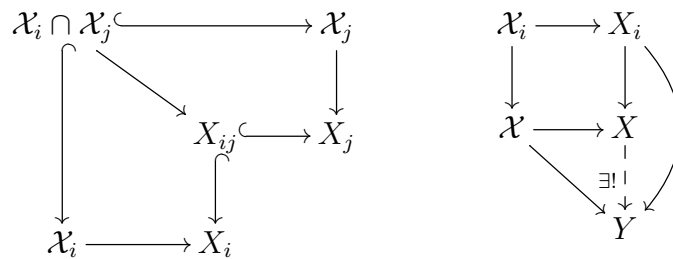
Then you glue to get $\mathcal{X} \to X$. Then we check that property (on the right). Now check the other properties [[★★★]].

Note that this argument doesn't work in the étale topology; there is no reason to expect $X_i \to X$ to be étale in that case. $\diamond$

Thus, we can assume S is a noetherian affine scheme.

Special case 1: assume there exists a faithfully flat surjection $\operatorname{Spec} A_1 \to \mathcal{X}$. Since the diagonal is finite, $\operatorname{Spec} A_1 \times_{\mathcal{X}} \operatorname{Spec} A_1 = \operatorname{Spec} A_2$ is affine (since finite over affine). If you look at the proof from lecture 13 and change étale to flat in some places, then something. You're really using that this is really a groupoid. Set $A_0 = Eq(A_1 \rightrightarrows A_2)$, then

(a) $A_0 \to A_1$ is finite and integral. This implies that A_0 is of finite type over S [AM69, 7.8].

(b) The topological space $|\operatorname{Spec} A_0|$ is the topological quotient of $|\operatorname{Spec} A_1|$ by the equivalence relation defined by $|\operatorname{Spec} A_2| \to |\operatorname{Spec} A_1 \times \operatorname{Spec} A_1|$.

(c) $\pi : \mathcal{X} \to \operatorname{Spec} A_0$ is universal for maps to schemes [[★★★ $\mathcal{X}$ is presented by $\operatorname{Spec} A_2 \rightrightarrows \operatorname{Spec} A_1$, so the map π is induced by the fact that $\operatorname{Spec} A_1 \to \operatorname{Spec} A_0$ coequalizes those arrows]]. We showed that $\operatorname{Spec} A_0$ is the quotient in the category of ringed spaces of $\operatorname{Spec} A_1$ by $\operatorname{Spec} A_2$. This implies that $\pi_* \mathcal{O}_{\mathcal{X}}|_{(\operatorname{Spec} A_0)_{et}} = \mathcal{O}_{\operatorname{Spec} A_0}$. Just look at maps to $\mathbb{A}^1$. This is a general thing if you look at coarse moduli space and push forward the sturcture sheaf you get the structure sheaf.

We need to check that π is universal for maps to algebraic spaces. So take an algebraic space Y. Then the sequence

$$Y(A_0) \to Y(A_1) \rightrightarrows Y(A_2)$$

should be exact.

Fix an étale surjection $U \to Y$ with U a scheme and set $R = U \times_Y U$.

I guess we need to prove something first. The claim is that $\operatorname{Hom}(\mathcal{X}, Y) = Eq\big(Y(A_1) \rightrightarrows Y(A_2)\big)$

$$\operatorname{Spec} A_2$$
$$\Downarrow$$
$$\operatorname{Spec} A_1$$
$$\downarrow$$
$$\mathcal{X} \longrightarrow F$$

This would be clear if something [[★★★]] is étale cover. We need to prove that Y is a sheaf with respect to the fppf topology. [[★★★ what?]] F is the presheaf of isomorphism classes in the fibers of $\mathcal{X}$, so $\operatorname{Hom}(\mathcal{X}, Y) = \operatorname{Hom}(F, Y)$.

We should also correct the theorem. In the definition of coarse moduli space, the universal property should be for morphisms to quasi-separated algebraic spaces. The proof doesn't work for general algebraic spaces; you'll see the reason.

Theorem 38.8. *Let X be a quasi-separated algebraic space over S. Then X is a sheaf with respect to the fppf topology on* Sch/S.

Proof. Let $\overline{X}$ be the sheaf (in the fppf topology) associated to X and let $q : X \to \overline{X}$ be the natural map of presheaves. First note that X is a separated presheaf, which is equivalent to injectivity of q: If $s_1, s_2 \in X(U)$, then $Z = U \times_{(s_1,s_2),X \times X,\Delta} X$ is a scheme (because diagonal is representable).

Surjectivity: say $s \in \overline{X}(U)$. By injectivity, to show that s is in the image of X, we may replace U by an open cover. Thus, we can assume U is quasi-compact. Now we can write $X = \bigcup X_i$ where each X_i is quasi-compact. Now let $\overline{X}_i$ be the fppf sheaf associated to X_i. It is an exercise to check that $X(U) = \varprojlim X_i(U)$ and $\overline{X}(U) = \varprojlim \overline{X}_i(U)$ because U is quasi-compact.

So we may assume X is quasi-compact. Let $X_0 \to X$ be an étale sujection with X_0 quasi-compact scheme. The claim is that $\overline{X}_0 \times_{\overline{X},s} U$ ($\overline{X}_0 = X_0$ and $\overline{U} = U$) is a scheme which is étale and quasi-compact over U. This would do it because lifting to X_0 is even better than lifting to X.

The quasi-compact is there to be able to do descent theory for quasi-affine maps: an étale and quasi-compact map is quasi-affine. We may then work fppf locally on U. Thus, we can assume there exists an $\tilde{s} \in X(U)$ with $q(\tilde{s}) = s$. Then by injectivity, $\overline{X}_0 \times_{\overline{X},s} \overline{U} = X_0 \times_{X,\tilde{s}} U$, which is étale over U (because X_0 is étale over U) and it is quasi-compact because

$$
\begin{array}{ccccc}
X_0 \times_X U & \longrightarrow & X_0 \times_S U & \longrightarrow & U \\
\downarrow & & \downarrow & & \\
X & \xrightarrow{\quad \Delta \quad} & X \times X & &
\end{array}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Now let's go back to the sequence

$$ Y(A_0) \xrightarrow{j} Y(A_1) \rightrightarrows Y(A_2). $$

Let's do injectivity of the first map. As before, $U \to Y$ is an étale surjection from a scheme. Let $\eta_1, \eta_2 \in Y(A_0)$ such taht $j(\eta_1) = j(\eta_2)$. First point: we can replace $\operatorname{Spec} A_0$ by an étale cover $A_0 \to R_0$.

$$
\begin{array}{ccccc}
Y(A_0) & \longrightarrow & Y(A_1) & \rightrightarrows & Y(A_2) \\
\downarrow & & \downarrow & & \downarrow \\
Y(R_0) & \longrightarrow & Y(R_0 \otimes_{A_0} A_1) & \Longrightarrow & Y(T_0 \otimes_{A_0} A_2) \\
\| & & \| & & \| \\
Y(R_0 \underset{A_0}{\otimes} R_0) \longrightarrow & Y(R_0 \underset{A_0}{\otimes} A_1) \underset{A_1}{\otimes} Y(R_0 \underset{A_0}{\otimes} A_1) & \Longrightarrow & Y(R_0 \underset{A_0}{\otimes} A_2) \underset{A_2}{\otimes} Y(R_0 \underset{A_0}{\otimes} A_2)
\end{array}
$$

[[★★★ what do those tensor products over A_1 and A_2 mean?]] Let $R = U \times_Y U \subseteq U \times U$. We want to show that $(\tilde{\eta}_1, \tilde{\eta}_2) \in R(A_0)$. Because of the universal property for

schemes, we have exactness of the right column.

$$
\begin{array}{ccc}
(\tilde\eta_1, \tilde\eta_2) & \in & R(A_0) \\
\big\uparrow & & \big\downarrow \\
U(A_1) \times U(A_1) \longleftarrow\!\!\!\!\supset R(A_1) \\
\| & & \| \\
U(A_2) \times U(A_2) \longleftarrow\!\!\!\!\supset R(A_2)
\end{array}
$$

Now exactness in the middle:

$$
\operatorname{Spec} A_2 \rightrightarrows \operatorname{Spec} A_1 \xrightarrow{\ \overset{\eta}{\frown}\ } \operatorname{Spec} A_0 \dashrightarrow Y
$$

η the map $\operatorname{Spec} A_1 \to Y$. We can work fppf locally on $\operatorname{Spec} A_0$. We can even assume A_0 is strictly henselian local ring (maybe works??). Choose a point $\bar x : \operatorname{Spec} k \to \operatorname{Spec} A_0$ where k is separably closed. Then let $I_{\bar x}$ be the category of all étale maps $U \to \operatorname{Spec} A_0$ which $\bar x$ factors through. Then $\mathcal{O}_{\operatorname{Spec} A_0, \bar x} = \lim_{I_{\bar x}} \Gamma(U, \mathcal{O}_U)$. Properties:

1. $\mathcal{O}_{\operatorname{Spec} A_0, \bar x}$ is local

2. If $R \to k$ where R is strictly henselian and local and $\operatorname{Spec} \mathcal{O}_{\operatorname{Spec} A_0, \bar x} \to k$, then there is a unique morphism $\mathcal{O}_{\operatorname{Spec} A_0, \bar x} \to R$.

$$
\begin{array}{ccccccc}
\operatorname{Spec} A_2 & \rightrightarrows & \operatorname{Spec} A_1 & \longrightarrow & \operatorname{Spec} A_0 & \longrightarrow & Y \\
& & \big\uparrow & & \big\uparrow & & \\
& & \prod \operatorname{Spec} R_i & \longrightarrow & \operatorname{Spec} \mathcal{O}_{\operatorname{Spec} A_0, \bar x} & &
\end{array}
$$

Then $\operatorname{Spec} R_i \to \mathcal{X}$ faithfully flat surjective and $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_{\operatorname{Spec} A_0, \bar x}$. Something about if coarse moduli space then doesn't matter which flat surjection. Reduce to the case when A_0 strictly henselian local rings.

Proof: II

$$
\operatorname{Spec} A_2 \rightrightarrows \operatorname{Spec} A_1 \longrightarrow \mathcal{X} \xrightarrow{\ \overset{\eta}{\frown}\ } \operatorname{Spec} A_0 \dashrightarrow Y
$$

Recall that we have a stack $\mathcal{X}$ and a finite flat $\operatorname{Spec} A_1 \to \mathcal{X}$ and $\operatorname{Spec} A_2 = \operatorname{Spec} A_1 \times_{\mathcal{X}} \operatorname{Spec} A_1$. We also have $\operatorname{Spec} A_0$, which is the invariants. We fix a morphism $\eta : \mathcal{X} \to Y$ and we'd like to show there is a unique $\operatorname{Spec} A_0 \to Y$. Here, Y is a quasi-separated algebraic space. Last time we proved uniqueness.

Theorem 38.9. *Let $\mathcal{X}$ be of finite type over S with finite diagonal. Assume there is a finite flat surjection $U \to \mathcal{X}$ with U a scheme and such that for all $x \in U$ the finite set $s(t^{-1}(x))$ is contained in an affine. Then there is an open covering $\mathcal{X} = \bigcup \mathcal{X}_i$ such that each $\mathcal{X}_i$ admits a finite flat covering by an affine scheme. Here $s, t : R = U \times_{\mathcal{X}} U \rightrightarrows U$.*

We'll prove this in a bit, but first let's use it to see existance of the map $\operatorname{Spec} A_0 \to Y$.

Remark 38.10. This implies that there is a universal map to schemes $\mathcal{X} \to X$. Namely, you take the universal thing for each $\mathcal{X}_i$ and glue. Another way to think of it: X is the quotient of U by R in the category of ringed spaces. $\diamond$

Given $U \to Y$ étale quasi-compact (reduce to the case Y is quasi-compact), we make the fiber product

$$
\begin{array}{ccccc}
 & & \mathcal{X}_R \longrightarrow Q' \longrightarrow R \\
 & & \Vert \qquad\quad \Vert \qquad\quad \Vert \\
V \xrightarrow{\text{fin flat}} \mathcal{X}_U \rightrightarrows Q \rightrightarrows U \\
{\scriptstyle et\ qcmpt}\Big\downarrow \qquad\qquad \Big\downarrow \qquad\quad \Big\downarrow \qquad\quad \Big\downarrow \\
\operatorname{Spec} A_2 \rightrightarrows \operatorname{Spec} A_1 \longrightarrow \mathcal{X} \rightrightarrows \operatorname{Spec} A_0 \dashrightarrow Y
\end{array}
$$

Find Q and Q' by theorem. $V \to \operatorname{Spec} A_1$ étale and quasi-compact implies it is quasi-affine which implies that any finite set of points is contained in an affine.

To show: $Q \to \operatorname{Spec} A_0$ is étale surjective and $Q' \to Q \times_{\operatorname{Spec} A_0} Q$ is an isomorphism. Then $Q' \rightrightarrows Q$ is an algebraic space presentation for $\operatorname{Spec} A_0$. Then you get your arrow $\operatorname{Spec} A_0 \to Y$. In fact, if $Q \to \operatorname{Spec} A_0$ is étale, then it follows that $Q' \to Q \times_{\operatorname{Spec} A_0} Q$ is an isomorphism: $Q' \to Q \times_{\operatorname{Spec} A_0} Q$ is also étale because $Q' \to Q$ and the projections are étale. Then it is enough to check that for every algebraically closed k, the map $Q'(k) \to Q(k) \times_{(\operatorname{Spec} A_0)(k)} Q(k)$ is bijective (this is a property of étale morphisms [SGA, I.5.7]). But we know that $Q'(k) = |\mathcal{X}(k)| \times_{Y(k)} R(k) = |\mathcal{X}(k)| \times_{Y(k)} U(k) \times_{Y(k)} U(k)$, and we have that $Q(k) \times_{(\operatorname{Spec} A_0)(k)} Q(k) = (|\mathcal{X}(k)| \times_{Y(k)} U(k)) \times_{(\operatorname{Spec} A_0)(k) = |\mathcal{X}(k)|} (|\mathcal{X}(k)| \times_{Y(k)} U(k))$, which is equal to the previous guy.

Surjectivity of $Q \to \operatorname{Spec} A_0$ is clear [[★★★]], so we just need to check it is étale.

$Q \to \operatorname{Spec} A_0$ is of finite type, so it is enough to check that it is formally étale. This can be checked after base change along flat morphisms $\operatorname{Spec} A_0' \to \operatorname{Spec} A_0$. Check that everything commutes with finite flat base change on A_0. (We're using the noetherian hypothesis of Q and $\operatorname{Spec} A_0$ over S.)

Upshot: it is enough to consider the case when A_0 is complete and local with algebraically closed residue field.

Now we have $\operatorname{Spec} A_1 \to \operatorname{Spec} A_0$ is finite, so A_1 is also complete with algebraically closed residue field. Each something of A_1 surjects onto $\operatorname{Spec} A_0$, so it surjects onto $\mathcal{X}$. Thus, we may assume A_1 is complete local and $A_0 \to A_1$ induces an isomorphism on residue fields.

$$\operatorname{Spec} A_1 \longrightarrow \mathcal{X} \longrightarrow \operatorname{Spec} A_0$$

So we have

$$
\begin{array}{ccccccc}
\coprod \operatorname{Spec} k & \xrightarrow{\text{id}} & \operatorname{Spec} k & = & \operatorname{Spec} k & \xrightarrow{u} & U \\
\cap \downarrow & & \downarrow & & \downarrow {\exists!} & \nearrow {\exists!} & \downarrow \\
\coprod \operatorname{Spec} A_2^{(i)} = \operatorname{Spec} A_2 & \rightrightarrows & \operatorname{Spec} A_1 & \longrightarrow & \operatorname{Spec} A_0 & \xrightarrow{\eta} & Y
\end{array}
$$

Let $u \in U(k)$ be a lifting of $\eta|_{\operatorname{Spec} k}$. Something about a map from A_2 to U, so something. Now we are in the situation

$$
\begin{array}{ccccc}
\mathcal{X}_U & \longrightarrow & Q & \longrightarrow & U \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} A_2 \rightrightarrows \operatorname{Spec} A_1 & \longrightarrow & \mathcal{X} & \longrightarrow \operatorname{Spec} A_0 & \longrightarrow Y
\end{array}
$$

Theorem 38.11. *$\mathcal{X}/S$ finite type finite diagonal, with $U \to \mathcal{X}$ finite flat surjection such that for all $x \in U$ the orbit $s(t^1(x))$ is contained in an affine. Then there is a cover $\mathcal{X} = \bigcup \mathcal{X}_i$ such that each $\mathcal{X}_i$ admit a finite flat surjection from an affine scheme.*

Let $\mathcal{X}^{(n)} \subseteq \mathcal{X}$ be substack such that $T \to \mathcal{X}$ factors through $\mathcal{X}^{(n)}$ if and only if $T \times_{\mathcal{X}} U \to T$ has rank n. We see that $\mathcal{X} = \coprod \mathcal{X}^{(n)}$. Now we can assume that $U \to \mathcal{X}$ has constant rank.

$R = U \times_{\mathcal{X}} U \overset{s}{\underset{t}{\rightrightarrows}} U$. A subset $F \subseteq U$ is *invariant* if for all $x \in F$ and $y \in R$ such that $t(y) = x$ we have that $s(y) \in F$.

Example 38.12. If $\mathcal{X} = [U/G]$ then $R = U \times G \rightrightarrows U$. In this case, F is invariant if it really is invariant under the group action. $\diamond$

Lemma 38.13. *Let $F \subseteq U$ be a subset. then $F^{inv} = s(t^{-1}(F))$ is invariant.*

We'll omit the proof (it isn't hard).

Note: if $Z_1 \subseteq Z_2 \subseteq U$ and Z_2 is invariant, then $Z_1^{inv} \subseteq Z_2$. [[★★★ better: $Z_2^{inv} = Z_2$]]

Say $W \subseteq U$ is an open subset with complement F. Then the *saturation* of W is defined to be $W' = U \smallsetminus F^{inv} \subseteq W$. Note that this is the maximal invariant subset of W.

Idea: if $W \subseteq U$ is invariant and $\mathcal{W} \subseteq \mathcal{X}$ is its image ($T \to \mathcal{X}$ factors through $\mathcal{W}$ if and only if $W \times_{\mathcal{X}} T \to T$ is surjective), then $\mathcal{W} \times_{\mathcal{X}} U = W$.

We have to show that we can cover U by invariant affine opens. That is, we need to show that for every $x \in U$, there is an invariant affine open $W \subseteq U$ containing x. Then the image will be an open substack of $\mathcal{X}$ which admits a finite flat covering by an affine.

Why is the original problem not easy: take an affine W around x and look at its image. Well, then the map W to its image is not finite.

Let $V \subseteq U$ be any affine containing $s(t^{-1}(x))$. Then we can take $V' \subseteq V$, in which we can find a $D(f)$ in there (with $f \in \Gamma(V, \mathcal{O}_V)$, and we end up with $s(t^{-1}(x)) \subseteq D(f)' \subseteq D(f) \subseteq V' \subseteq V$. The claim is that $W = D(f)'$ is affine. It will turn out that $D(f)' = D(f \cdot Norm_S(t^*f))$

$Z(f) := V' \smallsetminus D(f)$. $s, t : V' \times_{\mathcal{X}} V' \to V'$, and $t^{-1}(Z(f))$ is the set of points where t^*f is zero. By the way, if $h : Y \to X$ is finite (flat), then $h_* \mathcal{O}_Y$ is a locally free sheaf of algebras of finite rank. If $\alpha \in h_* \mathcal{O}_Y$, then $Norm_h(\alpha) \in \mathcal{O}_X$ is $\det(\alpha)$.

We've now prove the existence of the coarse moduli space when you have a finite flat covering by a scheme.

Next time:

Theorem 38.14. *$\mathcal{X}/S$ finite type finite diagonal. Then there is an algebraic stack $\mathcal{W}/S$ and surjective separated étale morphism $\mathcal{W} \to \mathcal{X}$ which is representable by schemes and admits a finite flat surjection $Z \to \mathcal{W}$ so that Z has good proerties.*

The idea is to start with $\mathcal{X}$, find a $\mathcal{W} \to \mathcal{X}$ étale which has a coarse space W. Then hope that $\mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ has a coarse space R and that $R \rightrightarrows U$ is an étale equivalence relation. Then we can define the coarse space X as the quotient.

Proof: III

Recall that we have $\mathcal{X}$ an algebraic stack of finite type with finite diagonal over an noetherian affine base S. The goal is to get a coarse space $\mathcal{X} \to X$ with a bunch of properties. So far, we've done this if $\mathcal{X}$ admits a finite flat surjection from a quasi-projective scheme.

We need three theorems.

Theorem 38.15 (A, Zariski's main theorem for stacks?)**.** *Assume $\mathcal{X}$ admits a quasi-finite flat surjection $U \to \mathcal{X}$ with U a quasi-projective S-scheme. Then there is an algebraic stack $\mathcal{W}$ over S and a surjective separated étale morphism $\pi : \mathcal{W} \to \mathcal{X}$ which is representable by schemes (for T a scheme, $\mathcal{W} \times_{\mathcal{X}} T$ is a scheme) and a closed immersion $Z \subseteq U \times_{\mathcal{X}} \mathcal{W}$ such that $Z \to \mathcal{W}$ is a finite flat surjection and such that*

1. *for every quasi-compact open substack $\mathcal{W}' \subseteq \mathcal{W}$, the preimage $Z' \subseteq Z$ is quasi-projective over S, and*

2. *for every algebraically closed field k and $w \in \mathcal{W}(k)$, then $\underline{\mathrm{Aut}}_{\mathcal{W}(k)}(w) \to \underline{\mathrm{Aut}}_{\mathcal{X}(k)}(\pi(w))$ is an isomorphism (it is always an injection by representability of π).*

If $\mathcal{X}$ is quasi-compact, we get that $\mathcal{W}$ has a coarse moduli space W, and $\mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ will also have a coarse moduli space R, and we'd like to show that $R \rightrightarrows W$ is an étale equivalence relation. The second condition is what will force R to be étale.

Theorem 38.16 (B)**.** *Let $\mathcal{Y}' \to \mathcal{Y}$ be a separated representable étale and quasi-compact morphism of algebraic stacks of finite type over S with finite diagonals. Assume there is a finite flat covering $U \to \mathcal{Y}$ with U/S quasi-projective and for every algebraically closed*

field k and $y' \in \mathcal{Y}'(k)$ mapping to $y \in \mathcal{Y}(k)$, the map $\underline{\mathrm{Aut}}_{\mathcal{Y}'(k)}(y') \to \underline{\mathrm{Aut}}_{\mathcal{Y}(k)}(y)$ is an isomorphism. Then $\mathcal{Y}'$ admits a finite flat covering $U' \to \mathcal{Y}'$ with U'/S quasi-projective and the map on coarse spaces $Y' \to Y$ is étale.

This, together with the first theorem will say that $\mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ has a coarse space and the projections $R \rightrightarrows W$ are étale.

Theorem 38.17 (C). *Let $\mathcal{X}/S$ be an algebraic stack locally of finite type with finite diagonal. Then there is an open covering $\mathcal{X} = \bigcup \mathcal{X}_i$ such that each $\mathcal{X}_i$ admits a quasi-finite flat surjection $U_i \to \mathcal{X}_i$ with U_i/S quasi-projective. (If $\mathcal{X}$ is DM, then you could replace "flat" by "étale.")*

Now we start with a general $\mathcal{X}$. Apply Theorem C to reduce to the case where you have a quasi-finite flat surjection. Then we get $R \rightrightarrows W$ an étale equivalence relation and we win.

Let's start with theorem A.

Proof of Theorem A. Let $\mathcal{H}$ (for Hilbert) be the stack over S whose objects are triples (T, x, Z) such that T is an S-scheme, $x \in \mathcal{X}(T)$, and $Z \subseteq T \times_{x,\mathcal{X}} U$ is a closed subscheme (because it is quasi-finite, this is a quasi-finite separated T-scheme, so it is quasi-affine) flat over T. There is a natural forgetful map $\mathcal{H} \to \mathcal{X}$ given by $(T, x, Z) \mapsto (T, x)$ and we have that

$$
\begin{array}{ccc}
\mathrm{Hilb}_{T \times_{\mathcal{X}} U} & \longrightarrow & \mathcal{H} \\
\downarrow & & \downarrow \\
T & \longrightarrow & \mathcal{X}
\end{array}
$$

This implies that $\mathcal{H}$ is an algebraic stack with finite diagonal. Maybe a better notation for $\mathcal{H}$ would be $\mathrm{Hilb}_{U/\mathcal{X}}$.

Define $\mathcal{W}' \subseteq \mathcal{H}$ to be the maximal open substack where $\mathcal{H} \to \mathcal{X}$ is étale. In general if you have a finite type morphism of schemes, the locus where the morphism is étale is an open set. You can do the same thing here.

$$
\begin{array}{ccc}
\mathcal{H}_T & \longrightarrow & \mathcal{H} \\
\downarrow & & \downarrow \\
T' \rightrightarrows T & \xrightarrow{\ sm\ } & \mathcal{X}
\end{array}
$$

Define $I_{\mathcal{X}}$, the *inertia stack*, to be

$$
\begin{array}{ccc}
I_{\mathcal{X}} & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow{\scriptstyle \Delta} \\
\mathcal{X} & \xrightarrow{\ \Delta\ } & \mathcal{X} \times \mathcal{X}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\underline{\mathrm{Aut}}_x & \longrightarrow & I_{\mathcal{X}} \\
\downarrow & & \downarrow \\
T & \xrightarrow{\ x\ } & \mathcal{X}
\end{array}
$$

An automorphism in $I_{\mathcal{X}}$ is a pair of automorphisms in $\mathcal{X}$. Now define $I'_{\mathcal{X}} = I_{\mathcal{X}} \times_{\mathcal{X}} \mathcal{W}'$. Then we have that

$$I_{\mathcal{W}'} \xrightarrow{\;j\;} I'_{\mathcal{X}}$$
$$\searrow \quad \swarrow$$
$$\mathcal{W}'$$

Lemma 38.18. *j is an open and closed immersion.*

Proof. $I_{\mathcal{W}'}$ and $I'_{\mathcal{X}}$ are both finite over $\mathcal{W}'$. The map j is also a monomorphism since $\mathcal{W}' \to \mathcal{X}$ is representable.

$$\underline{\mathrm{Aut}}_{(x,Z)} \longrightarrow \underline{\mathrm{Aut}}_x$$
$$\downarrow \qquad \qquad \swarrow$$
$$T \xleftarrow[(x,Z)]{} \mathcal{W}'$$

To prove the lemma, it is enough to show that j is étale. For this, look at the following picture.

$$
\begin{array}{ccccc}
I_{\mathcal{W}'} & \xrightarrow{et} & I'_{\mathcal{X}} & \longrightarrow & \mathcal{W} \\
\downarrow & & \downarrow & {}^{\Delta}\downarrow & \searrow \\
\mathcal{W}' & \xrightarrow{et} & \mathcal{K} & \longrightarrow & \mathcal{W} \times \mathcal{W} \quad \mathcal{X} \\
{}^{et}\searrow & & \downarrow & {}^{et}\downarrow & \searrow {}^{\Delta} \\
& & \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X}
\end{array}
$$

$\square$

Aside: if $k = \bar{k}$, $x \in \mathcal{X}(k)$, $Z \subseteq U \times_{\mathcal{X},x} k$, then $\mathrm{Aut}(x,Z)$ is the set of automorphisms $\sigma : x \to x$ in $\mathcal{X}(k)$ such that $\sigma : U \times_{\mathcal{X},x} k \to U \times_{\mathcal{X},x} k$ is an automorphism over Z.

Let $I''_{\mathcal{X}} \subseteq I'_{\mathcal{X}}$ be the complement of $I_{\mathcal{W}'}$, and let $\mathcal{W} \subseteq \mathcal{W}'$ be the complement of the image of $I''_{\mathcal{X}}$ in $\mathcal{W}'$. Here we use the fact that $I'_{\mathcal{X}}$ is proper over $\mathcal{W}'$ and so $\mathcal{I}''_{\mathcal{X}}$ is also proper over $\mathcal{W}'$. The claim is that $\mathcal{W}$ works.

$$Z \lhook\joinrel\longrightarrow U \times_{\mathcal{X}} \mathcal{W}$$
$$\searrow \qquad \downarrow$$
$$\mathcal{W}$$

We still need to show that $\mathcal{W} \to \mathcal{X}$ is surjective. Let k be an algebraically closed field and fix $x \in \mathcal{X}(k)$. Consider $w \in \mathcal{H}(k)$ corresponding to

$$U \times_{\mathcal{X}} \mathrm{Spec}\, k \xrightarrow{\;\mathrm{id}\;} U \times_{\mathcal{X},x} \mathrm{Spec}\, k$$
$$\searrow \qquad \downarrow$$
$$\mathrm{Spec}\, k$$

we have to show that something lies in the étale locus. Something about the identity not working for a *quasi*-finite morphism. Let $V \to \mathcal{X}$ be a smooth surjection, and assume that x comes from a point $v : \operatorname{Spec} k \to V$ (since $k = \bar{k}$). Then we have

$$
\begin{array}{ccccc}
 & & \operatorname{Hilb}_{P/\hat{V}} \sqcup (else) & & \\
 & & \| & & \\
w \in \operatorname{Hilb}_{U \times_{\mathcal{X}} \operatorname{Spec} k} & \hookrightarrow & \operatorname{Hilb}_{U \times_{\mathcal{X}} V} & \longrightarrow & \mathcal{H} \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Spec} k & \xrightarrow{\ v\ } & V & \longrightarrow & \mathcal{X}
\end{array}
$$

We can replaced V by a completion (strict hensilization) at v. Now look at $U \times_{\mathcal{X}} \hat{V} = P \sqcup Q$ where $P \to \hat{V}$ is finite and Q has empty closed fiber. Something with that top Hilb because of Hilbert polynomial considerations. Moreover, that Hilb is $\hat{V} \sqcup (rest)$, and our point lies in $\hat{V}$, which proves the result.   □

Outline proof of Theorem B. $\pi : \mathcal{Y}' \to \mathcal{Y}$ representable étale quasi-compact and $U \to \mathcal{Y}$ finite flat with U/S quasi-projective, and for all $y' \in \mathcal{Y}'(k)$, the map $\operatorname{Aut}(y') \to \operatorname{Aut}(\pi(y'))$ is an isomorphism.

(a) $\mathcal{Y}'$ admits a finite flat surjection $U' \to \mathcal{Y}'$ with U'/S quasi-projective. To see this,

$$
\begin{array}{ccc}
U' & \longrightarrow & \mathcal{Y}' \\
\downarrow & & \downarrow \\
U & \longrightarrow & \mathcal{Y}
\end{array}
$$

$U' \to U$ étale quasi-compact so quasi-affine and quasi-affine over quasi-projective (U) is quasi-projective. [[★★★]]

(b) $Y' \to Y$ is étale. Idea of proof: you certainly have finite type since we're in the noetherian case, so it is enough to check formally étale. Now we need to write down what the complete local rings of Y' and Y look like at a point. Fix $y \in Y(k)$ with k algebraically closed, and fix a lifting $u \in U(k)$ of y, and let $y' \in Y'(k)$ mapping to y. It takes a little argument, but you can find $u' \in U'(k)$ mapping to u. Take $\hat{\mathcal{O}}_{Y,y}^{sh}$ (sh means strict hensilization; else think closed point and completion at local rings), it is $Eq(\hat{\mathcal{O}}_{U,u}^{sh} \rightrightarrows \prod_{\xi \in R(k), s(\xi)=u=t(\xi)} \hat{\mathcal{O}}_{R,\xi}^{sh})$ with $R = U \times_{\mathcal{Y}} U$. And we get $\hat{\mathcal{O}}_{Y,y'}^{sh} = Eq(\hat{\mathcal{O}}_{U',u'}^{sh} \rightrightarrows \prod_{\xi' \in R'(k), s(\xi')=u=t(\xi')} \hat{\mathcal{O}}_{R',\xi'}^{sh})$. The map between them is induced by maps on the terms in the equalizers, the map $\hat{\mathcal{O}}_{U,y}^{sh} \to \hat{\mathcal{O}}_{U',y'}^{sh}$ is an isomorphism and something on the product terms. We want that $\{\xi' \in R'(k) | s(\xi') = t(\xi') = u'\} \to \{\xi \in R(k) | s(\xi) = t(\xi) = u\}$ to be bijective. The first set is in bijection with $\underline{\operatorname{Aut}}_{\mathcal{Y}'}(y')$ and the second is in bijection with $\underline{\operatorname{Aut}}_{\mathcal{Y}}(y)$.   □

Proof: IV

Recall that we have an Artin stack $\mathcal{X}/S$ of finite type with finite diagonal (and we were doing the case where S is noetherian). We sketched how to construct a coarse moduli space.

Theorem 38.19 (C). *$\mathcal{X} = \bigcup \mathcal{X}_i$ with each $\mathcal{X}_i$ admitting a quasi-finite flat covering.*

This is the only theorem we haven't talked about ... it is a slice argument in [SGA, 3] and we've done something similar earlier.

Then take a really nice étale cover $\mathcal{W}$ of $\mathcal{X}$ which has a coarse space, and show that $\mathcal{W} \times_{\mathcal{X}} \mathcal{W}$ has a coarse space which is an étale equivalence relation.

Today we'll talk more about the construction of the coarse space.

Theorem 38.20. *Let $\mathcal{X}/S$ finite type with finite diagonal and assume $\mathcal{X}$ is Deligne-Mumford. Then for every geometric point $\bar{x} : \operatorname{Spec} k \to \mathcal{X}$ (k separably closed), there exists an étale neighborhood $X' \to X$ of the image of $\bar{x}$ in the coarse space X such that $\mathcal{X} \times_X X' \cong [U/\Gamma_{\bar{x}}]$ where U is a finite X'-scheme and $\Gamma_{\bar{x}} = \underline{\operatorname{Aut}}_{\mathcal{X}(k)}(\bar{x})$. This is why people say DM stacks are like orbifolds ... they locally look like the quotient by the stabilizer group at that point.*

Proof. We can assume $\mathcal{X}$ is quasi-compact (everything is local on the coares space, and if the coarse space is quasi-compact, then so is $\mathcal{X}$). Choose an étale quasi-compact surjection $U \to \mathcal{X}$. Then $U \to X$ is quasi-finite, separated, and of finite type, so we can choose a base change $X' \to X$ an étale neighborhood of $\bar{x}$ (this includes a lifting of $\bar{x}$ to X') such that $U \times_X X' = P \sqcup T$ such that $P \to X'$ is finite and $T \times_{X'} \operatorname{Spec} k = \varnothing$.

Replace X by X'. Replacing $\mathcal{X}$ by the image of P, we can assume there is a finite étale covering $P \to \mathcal{X} \to X$. $\mathcal{X} \to X$ is proper surjective and $P \to X$ is proper, thne $P \to \mathcal{X}$ is proper (finite for some reason [[★★★ $P \to X'$ is finite]]). The lemma for schemes: $Z_1 \xrightarrow{f} Z_2 \xrightarrow{Z}_3$ with g separated and surjective and gf proper, then f is proper.

$$
\begin{array}{ccc}
\coprod P_{\bar{x}}^i & \longrightarrow \mathcal{X}_{sh} & \longrightarrow \operatorname{Spec} \mathcal{O}_{X,\bar{x}}^{sh} \\
\downarrow & \downarrow & \downarrow \\
P & \longrightarrow \mathcal{X} & \longrightarrow X
\end{array}
$$

with $P_{\bar{x}}^i$ strictly hensilian and local. So after further replacing X by an étale covering, we can assume that we only get one $P_{\bar{x}} := P \times_X \operatorname{Spec} \mathcal{O}_{X,\bar{x}}^{sh}$ which is strictly hensilian and local.

$$
\coprod_{\Gamma} P_{\bar{x}} = R_{\bar{x}} \rightrightarrows P_{\bar{x}} \longrightarrow \mathcal{X}_{sh} \longrightarrow \operatorname{Spec} \mathcal{O}_{X,\bar{x}}^{sh}
$$

Looking at just closed points, a closed point in $R_{\bar{x}}$ is an automorphism of a point in $P_{\bar{x}}$. You see the group structure from the groupoid structure.

The group structure on $\Gamma_{\bar{x}}$. We have

$$
\left(\coprod_{\Gamma_{\bar{x}}} P_{\bar{x}} \right) \times_{p_2, P_{\bar{x}}, p_1} \left(\coprod_{\Gamma_{\bar{x}}} P_{\bar{x}} \right) \to \left(\coprod_{\Gamma_{\bar{x}}} P_{\bar{x}} \right)
$$

The second projection $R_{\bar{x}} \xrightarrow{p_2} P_{\bar{x}}$ defines an action of $\Gamma_{\bar{x}}$ on $P_{\bar{x}}$ which doesn't have to act trivially.

So we see that $[P_{\bar{x}}/\Gamma_{\bar{x}}] = \mathcal{X}_{sh} \to \operatorname{Spec} \mathcal{O}_{X,\bar{x}}^{sh}$. Something is obtained by some limit, so somehow you get the theorem. $\qquad\square$

Remark 38.21. If $\mathcal{X} \to X$ is étale, then étale locally on X (with notation as above), $\mathcal{X} = X \times B\Gamma$ for some finite group Γ. If the map $\mathcal{X}_{sh} \to \operatorname{Spec} \mathcal{O}_{X,\bar{x}}^{sh}$ is étale, then $P_{\bar{x}} \to \operatorname{Spec} \mathcal{O}_{X,\bar{x}}^{sh}$ is an isomorphism (something about strictly henselian local rings) so the two projections must be equal. $\qquad\diamond$

Example 38.22. $\mathcal{M}_{1,1,S}$, then the coarse space is $\mathbb{A}_{j,S}^1$ (even if 6 is not invertible). Note that we haven't actually shown yet that $\mathcal{M}_{1,1,S}$ has finite diagonal, so we don't yet know that the coarse space exists. Let's first show finite diagonal. Take elliptic curves with some level structure, then it will be finite flat over $\mathcal{M}_{1,1,S}$. In the case when $3 \in \mathcal{O}_S^\times$ (then take the open subset where 2 is invertible and glue), $V = \mathcal{S}pec_S(\mathcal{O}_S[\mu,\omega][1/(\mu^3 - 1)]/(\omega^2 + \omega + 1))$. This classifies elliptic curves E with an isomorphism $\sigma : (\mathbb{Z}/3)^2 \xrightarrow{\sim} E[3]$ (the 3-torsion of E). The universal curve over it is $X^3 + Y^3 + Z^3 = 3\mu XYZ$, and the basis for the 3-torsion is $p = [1,0,-1]$, $q = [-1,\omega,0]$.

We have $V \to \mathcal{M}_{1,1,S} = [V/GL_2(\mathbb{F}_3)]$. Now we clearly have finite diagonal. This implies that the coarse space $M_{1,1,S}$ is relative spectrum over S of the $GL_2(\mathbb{F}_3)$-invariants in $\mathcal{O}_V$. I think its pretty hard to show that this is $\mathbb{A}_{j,S}^1$ (by the way, $j = \left(3\mu(\mu^3 - 2^3)/(\mu^3 - 1)\right)^3$). Let's do it another way.

Consider the case when $S = \operatorname{Spec} k$. Then $\pi : \mathcal{M}_{1,1,k} \to M_{1,1,k}$ and we have a map $j : \mathcal{M}_{1,1,k} \to \mathbb{A}_k^1$, so we get a map $h : M_{1,1,k} \to \mathbb{A}_k^1$. We want h to be an isomorphism. We know that $M_{1,1,k}$ is a geometrically normal curve because you are taking invariants of a smooth curve by a finite group étale locally, so the quotient is normal. h induces a bijection on Ω-valued points for every algebraically closed field Ω. That implies that h is an isomorphism.

Again. (something about coarse spaces commuting with flat base change. You can see that $M_{1,1,k}$ is proper over $\mathbb{A}^1$ because V is finite over the line. We know that j and π are proper, π is surjective, which implies that h is proper, and for some reason finite. Since coarse spaces commute with flat base change, we can assume $k = \bar{k}$.

Something is bad

$$
\begin{array}{ccc}
k(j^{1/p})[\varepsilon]/\varepsilon^p & \longrightarrow & k(j^{1/p}) \\
\downarrow & & \downarrow \\
k(j^{1/p}) & \longrightarrow & k(j)
\end{array}
$$

[[★★★ somebody should explain this to me]]

Now take the case where we have an artinian local ring R with residue field k and let $S = \operatorname{Spec} R$. Say $J \subseteq R$ is a square zero ideal which is annihilated by the maximal ideal of R, and say $R_0 = R/J$. Then we have over $\mathcal{M}_{1,1,R}$

$$
J \otimes \mathcal{O}_{\mathcal{M}_{1,1,k}} \to \mathcal{O}_{\mathcal{M}_{1,1,R}} \to \mathcal{O}_{\mathcal{M}_{1,1,R_0}} \to 0
$$

But we get exactness on the left by tensoring the following with $\mathcal{O}_{\mathcal{M}_{1,1,R}}$

$$0 \to J \to R \to R_0 \to 0$$

Now on $\mathbb{A}^1_R$ we get a sequence

$$0 \to J \otimes_k j_*\mathcal{O}_{\mathcal{M}_{1,1,k}} \to j_*\mathcal{O}_{\mathcal{M}_{1,1,R}} \to j_*\mathcal{O}_{\mathcal{M}_{1,1,R_0}}$$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J \otimes_k j_*\mathcal{O}_{\mathcal{M}_{1,1,k}} & \longrightarrow & j_*\mathcal{O}_{\mathcal{M}_{1,1,R}} & \longrightarrow & j_*\mathcal{O}_{\mathcal{M}_{1,1,R_0}} & & \\
& & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & J \otimes \mathcal{O}_{\mathbb{A}^1_k} & \longrightarrow & \mathcal{O}_{\mathbb{A}^1_R} & \longrightarrow & \mathcal{O}_{\mathbb{A}^1_{R_0}} & \longrightarrow & 0
\end{array}
$$

By induction we can assume the outer two are isos so the middle one is an iso.

In general to check that $\mathcal{O}_{\mathbb{A}^1_{j,S}} \to j_*\mathcal{O}_{\mathcal{M}_{1,1,S}}$ is an isomorphism reduce to the case of artinian local.

$\diamond$

$\mathcal{M}_g$ also has finite diagonal. If R is a complete discrete valuation ring and $C_1, C_2 \in \mathcal{M}_g(R)$, then any isomorphism $\sigma_\eta : C_{1,\eta} \xrightarrow{\sim} C_{2,\eta}$ (η the generic point) over $\operatorname{Spec} k$ extends uniquely to an isomorphism $C_1 \xrightarrow{\sim} C_2$. One way to do this is to say C_1 is the minimal regular model of the generic guy.

42   Cohomological descent

The next two lectures address the following question: given a stack $\mathcal{X}$ and a sheaf F on $\mathcal{X}$, how does one compute compute $H^\bullet(\mathcal{X}, F)$? More generally, we'd like to compute $R^i f_*$. The idea is basically that of Čech cohomology: the cohomology of $\mathcal{X}$ is computable from the cohomology of a cover of $\mathcal{X}$ together with the cohomologies of intersections, triple intersections, etc.

Definition 42.1. The *simplicial category* $\tilde{\Delta}$ is the category of finite ordered sets with (weakly) order-preserving maps. This is equivalent to the category whose objects are $[n] = \{0, 1, \cdots, n\}$ with $n \geq -1$ ($[-1] = \varnothing$). Then there are $i + 1$ special order-preserving maps $[i] \to [i+1]$ and i special maps $[i+1] \to [i]$. [[★★★ these should be called d_j and δ_j, and should be explained. check out the standard notation for simplicial objects. The point is that these maps generate $\tilde{\Delta}$ with some relations (which should be written here)]]

We define $\Delta \subseteq \tilde{\Delta}$ be the full subcategory of non-empty sets, and define $\Delta^+ \subseteq \Delta$ to be the subcategory with the same objects as Δ, but only injective maps.

Let C be a category. A *simplicial object in* C is a functor $\Delta^{op} \to \mathsf{C}$.   ◇

Given an algebraic stack $\mathcal{X}$, there is a presentation $X \to \mathcal{X}$. Define $X_i = X \times_{\mathcal{X}} \cdots \times_{\mathcal{X}} X$ ($i + 1$ times). There are $i + 1$ obvious projections $X_i \to X_{i-1}$ and i obvious morphisms $X_{i-1} \to X_i$ (given by repeating one of the factors), and these satisfy the usual relations. Thus, we have a simplicial algebraic space $X_\bullet$.

We want to compute $H^\bullet(\mathcal{X}, F)$ in terms of $H^\bullet(X_i, F_{X_i})$. We need a supped up version of Čech cohomology. This is *Cohomological descent*, which you should think of as Čech cohomology on steroids.

We can talk about $\mathsf{AlgSp}/\mathcal{X}$, where objects are arrows $v : V \to \mathcal{X}$ and morphisms are morphisms over $\mathcal{X}$ (up to 2-isomorphism). Then $X_i(T \to \mathcal{X}) = \mathrm{Hom}(T, X)^{[i]}$ ($i + 1$ maps from T to X over $\mathcal{X}$). Then it is clear that if we have an order-preserving map $[i] \to [j]$, then we get a map $X_j(T \to \mathcal{X}) \to X_i(T \to \mathcal{X})$. These simplicial objects are very complicated.

Each X_i has an étale topos, and we'd like to package them all together.

Definition 42.2. Let D be a category. A D-*topos* is a functor $p : \mathsf{T} \to \mathsf{D}$ such that

1. T is fibered and cofibered over D ($\mathsf{T}^{op} \to \mathsf{D}^{op}$ is fibered),

2. for all $d \in \mathsf{D}$, the fiber T_d is a topos,

3. for each $m : d' \to d$ in D, $\mathsf{T}_{d'} \underset{m_*}{\overset{m^*}{\rightleftarrows}} \mathsf{T}_d$, there is a morphism of topoi $f : \mathsf{T}_d \to \mathsf{T}_{d'}$ such that $m^* = f_*$ and $m_* = f^*$. [[★★★ this makes me uncomfortable ... it seems like p must be a *split* fibered cofibered category for this definition to make sense]]

The *total topos* of a D-topos is $\mathcal{T}ot(\mathsf{T}) = \mathrm{HOM}_\mathsf{D}(\mathsf{D}, \mathsf{T})$, the category whose objects are data $(\{F_d\}, \{\phi^\sharp\})$ where $F_d \in \mathsf{T}_d$ and for every $\phi : d \to d'$, we have $\phi^\sharp : \phi^* F_{d'} \to F_d$ which are compatible [[★★★ write compatibility condition]]. [[★★★ how to prove that the total topos is actually a topos? If T_d is the topos of some site C_d, then $\mathcal{T}ot(\mathsf{T})$ should be the topos on some site C fibered over D whose fibers are C_d]]   ◇

[[★★★ define quasi-coherent sheaves on the total topos here]]

Roughly, a D-topos is a functor $\mathsf{T} : \mathsf{D}^{op} \to \mathsf{Topoi}$ with $d \mapsto \mathsf{T}_d$ [[★★★ if we require the spitting, then this is exactly what a D-topos is]]. We see that $X_{\bullet et} : \Delta^{op} \to Topoi$, given by $[i] \mapsto X_{i,et}$, is a Δ-topos.

Example 42.3. For $X_{\bullet et}$, the total topos consists of data $(\{F_i\}, \{\phi^\sharp_\sigma\})$, where $F_i \in X_{i,et}$ and for every $\sigma : [i] \to [j]$ (order preserving), we get a map $X(\sigma) : X_j \to X_i$, and we want $X(\sigma)^* F_i \to F_j$ in a compatible way.   ◇

Now we want to compute the cohomology of the simplicial topos and relate it to the cohomology of the stack. How do you compute the cohomology of a Δ-topos (or Δ^+-topos)?

Say we have a $\tilde{\Delta}$-topos; let $\mathsf{T}_\bullet$ be associated Δ-topos.

$$\underbrace{\cdots \mathsf{T}_2 \Lleftarrow \mathsf{T}_1 \rightrightarrows \mathsf{T}_0}_{\mathsf{T}_\bullet} \longrightarrow \mathsf{T}_\varnothing$$

Let $\varepsilon^i : \mathsf{T}_i \to \mathsf{T}_\varnothing$. There is a morphism of topoi $\varepsilon_\bullet : \mathcal{T}ot(\mathsf{T}_\bullet) \to \mathsf{T}_\varnothing$ with $\varepsilon_\bullet(\{F_i\}, \phi^\sharp_\sigma) := Eq(\varepsilon^0_* F_0 \rightrightarrows \varepsilon^1_* F_1)$, and $\varepsilon^* G = (\{\varepsilon^{n*} G\}, \mathrm{can})$. [[★★★ how does one check the adjunction?]]

Let $\mathcal{X}$ be a scheme and let $X_\bullet \to \mathcal{X}$ be a flat surjection. Then we have $\tilde{X}_\bullet : \tilde{\Delta}^{op} \to \mathsf{Sch}$, with $\tilde{X}_\varnothing = \mathcal{X}$.

Example 42.4. We could define $\mathsf{T}_\bullet$ to be the Δ-topos ($\mathsf{T}_i = X_{i,et}$) and $\mathsf{T}_\varnothing = \mathsf{Set}$ (point topos). Here, $\varepsilon_* : \mathsf{T}_\bullet \to *$ is the global section functor. [[★★★ this isn't clear]]   ◇

If we want to compute cohomology, we need to know how to compute $R^i \varepsilon_* F_\bullet$ (we want to push forward a sheaf on the total topos to $\mathsf{T}_\varnothing$).

The "standard reference" is [SGA, 4 1/2], which is based on some lectures of Deligne, which the author didn't understand. The idea isn't too bad.

So $F_\bullet \in Tot(\mathsf{T}_\bullet)$ and we're trying to compute pushforward. If you like, think of the case where the target is a point and we're trying to compute global sections.

Remark 42.5 (Aside). [[★★★ This remark should appear earlier]] We started with a stack $\mathcal{X}$ with a presentation $X_\bullet$ and a quasi-coherent sheaf $\mathcal{F}$. Then you get a simplicial sheaf $F_\bullet$ on $X_\bullet$. Later, we'll prove that $H^\bullet(\mathcal{X}, \mathcal{F}) = H^\bullet\big(\mathcal{T}ot(X_{\bullet et}, F_\bullet)\big)$ (this should properly be called "cohomological descent") [[★★★ I don't think we actually prove this. Find a ref]]. You should view this as some kind of derived version of Čech

cohomology (note that working with the total topos makes it so that you don't have to worry about taking affine covers or whatever).

For every smooth $V \to \mathcal{X}$, we have the (quasi-coherent) restriction $\mathcal{F}_V \in V_{et}$ and for every morphism $g : U \to V$ over $\mathcal{X}$ (with 2-isomorphism), we have an isomorphism $g^*\mathcal{F}_V \xrightarrow{\sim} \mathcal{F}_U$. This gives us the $F_{\bullet}$. $\diamond$

We need to understand injectives in $\mathcal{T}ot(\mathsf{T}_{\bullet})$. The answer: for every injective $I_{\bullet} \in \mathsf{Ab}\big(\mathcal{T}ot(\mathsf{T}_{\bullet})\big)$, the sheaf I_i is injective in $\mathsf{Ab}(\mathsf{T}_i)$ for all i. $\varepsilon_*I \xrightarrow{q-iso} \varepsilon_*^0 I_0 \to \varepsilon_*^1 I_1 \to \varepsilon_*^2 I_2 \to \cdots$ where the maps are $\sum (-1)^i d_i$ as in usual Čech cohomology.

How to compute $\varepsilon_* F_{\bullet}$: choose an injective resolution $F_{\bullet} \to I_{\bullet}^{\bullet}$, so we have

$$
\begin{array}{c}
F_2 \\
\uparrow \\
F_1 \longrightarrow I_1^0 \longrightarrow I_1^1 \longrightarrow I_1^2 \longrightarrow \\
\uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow \\
F_0 \longrightarrow I_0^0 \longrightarrow I_0^1 \longrightarrow I_0^2 \longrightarrow
\end{array}
\qquad\qquad
\begin{array}{c}
\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow \\
\varepsilon_*I_1^0 \longrightarrow \varepsilon_*I_1^1 \longrightarrow \varepsilon_*I_1^2 \longrightarrow \\
\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow \\
\varepsilon_*I_0^0 \longrightarrow \varepsilon_*I_0^1 \longrightarrow \varepsilon_*I_0^2 \longrightarrow \\
\uparrow \qquad\qquad \uparrow \qquad\qquad \uparrow \\
K^0 \longrightarrow K^1 \longrightarrow K^2 \longrightarrow
\end{array}
$$

the vertical maps are alternating sums of the d_i. We're supposed to apply ε_* to the whole complex (without the left column) and get the kernels $K^{\bullet}$. We claim that the columns are exact after applying ε_*. We have a quasi-isomorphism $K^{\bullet} \to \text{Tot}(\varepsilon_*I^{\bullet})$, given by

$$
\begin{array}{c}
\varepsilon_*I_0^0 \longrightarrow \varepsilon_*I_0^1 \oplus \varepsilon_*I_1^0 \longrightarrow \cdots \\
\uparrow \qquad\qquad\qquad \uparrow \\
K^0 \longrightarrow \qquad K^1 \longrightarrow
\end{array}
$$

The upshow is that you can forget about the mysterious equalizer, you're just computing the total cohomology of the bicomplex.

$R^i\varepsilon_*F = \mathcal{H}^i\big(\text{Tot}(\varepsilon_*I_{\bullet}^{\bullet})\big)$. Whenever you have a bicomplex, you get a filtration $Fil^{\bullet}$ on $\text{Tot}(\varepsilon_*I_{\bullet}^{\bullet})$.

$$
Fil^k = \bigoplus_{i+j=n, j\geq k} \varepsilon_*I_j^i \subseteq \bigoplus_{i+j=n} \varepsilon_*I_j^i
$$

You allow things that don't go too far vertical. Whenever you have a filtration, you have some spectral sequence which relates the cohomology of the filtered pieces to the cohomology of the whole thing. [Lan02, XX.9.3]

The claim is that $Fil^k/Fil^{k+1} = I_k^{\bullet}$. You find that there is a spectral sequence $E_1^{p,q} = R^q\varepsilon_*^p F_p \Rightarrow R^{p+q}\varepsilon_*F$. Note that the first thing is some cohomlogy on one of the spaces.

More Cohomological Descent

Recall the setup. We have $\mathsf{T}_\bullet \xrightarrow{\varepsilon} \mathsf{T}_\varnothing$, and we want to compute $R^i \varepsilon_* F_\bullet$.

If $I \in Ab(\mathsf{T}_\bullet)$, then (1) each $I_k \in Ab(\mathsf{T}_k)$ is injective, and (2) $\varepsilon_* I$ (concentrated in degree zero) is quasi-isomorphic to $\varepsilon_*^0 I_0 \to \varepsilon_*^1 I_1 \to \cdots$, so there is no higher cohomology. By Lang, we have $E_1^{pq} = R^q \varepsilon_*^p F_p \Rightarrow R^{p+q} \varepsilon_* F$.

Example 42.6 (Čech cohomology)**.** What happens in the Čech cohomology situation. Let X be a quasi-compact separated scheme, and let $X = \bigcup U_i$ a finite covering with each U_i affine. Let $\mathcal{F}$ be quasi-coherent on X. Then the simplicial scheme we get is $\coprod U_i \to X$. The next step is $\coprod U_i \cap U_j$, then triple intersections (allowing $i = j$, by the way; this is how you get the sections); call this thing $U_\bullet$. We're supposed to have a spectral sequence $E_1^{pq} = H^q(\coprod_{i_0 \dots i_p} U_{i_0 \dots i_p}, \mathcal{F}) \Rightarrow H^{p+q}(U_\bullet \mathcal{F})$. The picture is

[[★★★ pictures]]

For $q > 0$, the whole thing is zero because you have an affine scheme. The bottom row is the Čech complex, except for this $i < j$ issue. You get the complex by taking alternating sums. This simplicial abelian group (of 0th cohomologies) can be handled in 2 ways; one of them (the normalized complex) is the usual Čech complex, and the other is what we have. There is some result that says that they are the same thing (give the same cohomlogy). ◇

Let's say why injective objects look like what we've claimed they are. If you have an abelian sheaf in $Ab(\mathsf{T}_\bullet)$. Then for every n, we have a restriction map $r_n : Ab(\mathsf{T}_\bullet) \to Ab(\mathsf{T}_n)$. This is an exact functor (this is how you define exact sequences in $Ab(\mathsf{T}_\bullet)$). It has a right adjoint $e_n : Ab(\mathsf{T}_n) \to Ab(\mathsf{T}_\bullet)$, given by $(e_n F)_k = \prod_{\rho \in \mathrm{Hom}_\Delta([k],[n])} \rho_* F$ (ρ gives a map $\mathsf{T}_n \to \mathsf{T}_k$). Any functor with an exact left adjoint takes injectives to injectives, so e_n does so. Thus, e_n of an injective is injective.

Given $F_\bullet \in Ab(\mathsf{T}_\bullet)$, choose for every n and inclusion $F_n \hookrightarrow I_n$ where $I_n \in Ab(\mathsf{T}_n)$ is injective. Then $F_\bullet \hookrightarrow \prod_n e_n(f_n(F_\bullet)) \hookrightarrow \prod_n e_n(I_n)$. A corollary of this is that every injective sheaf in $Ab(\mathsf{T}_\bullet)$ is a direct summand of a sheaf of the form $\prod_n e_n(I_n)$, with the I_n injective for each n.

ρ_* takes injectives to injectives (because exact left adjoint). Thus, each $e_n(I_n)$ is injective at each level, and since each injective is a summand of one of these, we've checked the first point. Also, it is possible to show this by showing that r_n has an exact left adjoint, so it takes injectives to injectives.

To check the second point for a direct sum, it is equivalent to check it for each summand. To check that $\varepsilon_*^0 I_0 \to \varepsilon_*^1 I_1 \to \cdots$ has no higher cohomology, it is enough to consider $I_\bullet = \prod_n e_n(I_n)$. Let $T_n : \Delta \to Ab$ be given by $[p] \mapsto \prod_{\mathrm{Hom}_\Delta([p],[n])} \mathbb{Z}$. Let $\tilde{T}_n$ be the associated total complex. In this case, the sequence with the ε_*^is is $(\varepsilon_*^n I_n) \otimes_\mathbb{Z} \tilde{T}$; you first push down to $[n]$ and then push down to $\varnothing$. But $\tilde{T}_n$ is a complex of abelian groups, it computes cellular homology of the standard n-simplex, which is zero, so the tensor product is isomorphic to $\varepsilon_*^n I_n$.

Let's now take $\mathcal{X}$ to be an algebraic stack. Let $X \to \mathcal{X}$ be a smooth surjection with X and algebraic space. Then we get $X_\bullet$, which is a simplicial algebraic space.

Now you have to be a little careful. Remember we're thinking about $\mathcal{X}_{lis\text{-}et}$. We'd like to say we get an augmentation $X_{\bullet\,lis\text{-}et} \to \mathcal{X}_{lis\text{-}et}$, but the lisse-étale topos is not functorial; for example $\Delta : X \to X \times_{\mathcal{X}} X$ doesn't give a morphism of topoi. However, we do get $X^+_{\bullet\,lis\text{-}et} \to \mathcal{X}_{lis\text{-}et}$ (the + means restrict to Δ^+); somehow all these maps are smooth, so we're ok. We also have morphisms of topoi $X^+_{\bullet\,lis\text{-}et} \to X^+_{\bullet\,et}$ and $X_{\bullet\,et} \to X^+_{\bullet\,et}$. A sheaf of $\mathcal{O}_{\mathcal{X}_\bullet}$-modules in $X_{\bullet\,et}$ (for each i, the sheaf on X_i is an $\mathcal{O}_{X_i}$-modules in a compatible way) is quasi-coherent if each F_i on X_i is quasi-coherent and the transition maps $X(\sigma)^\star F_i \to F_j$ is an isomorphism for every σ a morphism in Δ. Similarly, we can talk about quasi-coherent sheaves on the other topoi.

Theorem 42.7. $\mathsf{Qcoh}(X^+_{\bullet\,lis\text{-}et})$, $\mathsf{Qcoh}(\mathcal{X})$, $\mathsf{Qcoh}(X^+_{\bullet\,et})$, and $\mathsf{Qcoh}(X_{\bullet\,et})$ are all equivalent (by the maps above), and the map $\mathsf{Qcoh}(\mathcal{X}) \to \mathsf{Qcoh}(X_{\bullet\,et})$ is the natural restriction.

Note that at least for this subject, you don't need to worry about the degeneracy maps.

The good statement is that there is an equivalence of derived categories $\mathsf{D}^+_{qcoh}(\mathcal{X}) \to \mathsf{D}^+_{qco}(X_{\bullet\,et})$. Concretely, you can always choose presentations

$$
\begin{array}{ccc}
X_\bullet \xrightarrow{f_\bullet} Y_\bullet & \qquad & \mathsf{D}(X_{\bullet\,et}) \xrightarrow{Rf_{\bullet\,*}} \mathsf{D}(Y_{\bullet\,et}) \\
\downarrow \qquad \downarrow & & \downarrow{\wr} \qquad\qquad \downarrow{\wr} \\
\mathcal{X} \xrightarrow{\;f\;} \mathcal{Y} & & \mathsf{D}^+_{qcoh}(\mathcal{X}) \xrightarrow{Rf_*} \mathsf{D}^+_{qcoh}(\mathcal{Y})
\end{array}
$$

The upshot is (cohomological descent) that you can compute the cohomology of a stack by a spectral sequence $E_1^{pq} = H^q(X_p, F|_{X_p}) \Rightarrow H^{p+q}(\mathcal{X}, F)$. This is the general tool for computing cohomology of artin stacks.

Example 42.8. Let $\mathcal{X} = B_k G$ (let's say G is a finite group). Then we have

$$
G \times G \rightrightarrows G \rightrightarrows \mathrm{Spec}\, k \longrightarrow B_k G
$$

$$
Map(G \times G, F) \Longleftarrow Map(G, F) \Longleftarrow F
$$

(and the other arrows). If F is a sheaf on BG (i.e. a representation of G), then we get the second row. That bottom row is the standard complex computing group cohomology. The derived functors of invariants are computed by this second row.   ◇

So cohomology of a stack is a mixture of group cohomology and regular old cohomology.

Example 42.9 (special case)**.** Take $G = \mathbb{Z}/p$ and $k = \mathbb{F}_p$. Then $H^i(G, k) \cong \mathbb{F}_p$ for each $i \geq 0$. So we see that $H^*(BG, \mathcal{O}_{BG})$ is unbounded.   ◇

Definition 42.10. Let $\mathcal{X}$ be a Deligne-Mumford stack over some noetherian S with finite diagonal. We call $\mathcal{X}$ *tame* if for every algebraically closed field k and $x \in \mathcal{X}(k)$, the order of $\underline{\mathrm{Aut}}_{\mathcal{X}(k)}(x)$ is invertible in k. $\diamond$

Think of $\mathcal{M}_{1,1}$ outside of characteristic 2 and 3.
In this case, look at the coarse space $\pi : \mathcal{X} \to X$.

Proposition 42.11. *For any quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{X}$, $R^i \pi_* \mathcal{F} = 0$ for $i > 0$*

Corollary 42.12. $H^p(\mathcal{X}, \mathcal{F}) = H^p(X, \pi_* \mathcal{F})$.

In particular, for an algebraic space, there is an integer so that cohomology vanishes after a certain point [[★★★ we proved it by taking a dense open scheme and do some kind of induction. . . there should be some assumption like finite type over a field]]

Proof of Proposition. The result is étale local on X. We can assume $\mathcal{X} = [U/\Gamma]$ where $U \to \mathcal{X}$ is finite and Γ is the stabilizer group of some point. The assumption that it is tame means that Γ has order which is invertible in $\mathcal{X}$ (again taking an étale map if needed). Then a quasi-coherent sheaf on $\mathcal{X}$ is the same thing as an $\mathcal{O}_U$-module F with a lifting of the Γ action to F.

We have $(M, \text{action}) \mapsto M^\Gamma$, which is an A-module. So we're really just computing group cohomology. But group cohomology over a field where the order of the group is invertible is zero for higher stuff. $\frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \gamma$ is a projector from M to M^Γ. $\square$

Often when you do algebraic geometry, you'll see stamements like "quotient singularities are as good as smooth". Quotient singularity means you're locally the quotient of a smooth thing by a finite group. If X has quotient singularities, it means that it looks like the coarse space of a smooth DM stack. It is clear that the coarse space of a smooth DM stack has only quotient singularities. If you have only quotient singularities and you're over $\mathbb{Q}$, then there is some DM stack whose coarse space is the thing you started with. You can prove lots of things about $\mathcal{X}$ from its coarse space. In characteristic p, this is unknown; bummer.

The problem is: given an X/k with k even algebraically closed of characteristic p, with quotient singularities, produce a smooth DM stack with X as its coarse space. This would be very interesting if somebody solves it.

44   Brauer Groups and Gabber's Theorem

Today let's go back to gerbes.

Brauer groups and quotient stacks. Let X be a scheme. An *Azumaya algebra* on X is a locally free sheaf of (non-commutative) $\mathcal{O}_X$-algebras $\mathcal{A}$ of finite rank such that étale locally on X, $\mathcal{A} \cong End(\mathcal{V})$ for some vector bundle $\mathcal{V}$ on X (this implies that the center is $\mathcal{O}_X$). $\mathcal{A} \cong \mathcal{A}'$ if there exist vector bundles $\mathcal{V}$ and $\mathcal{V}'$ such that $\mathcal{A} \otimes_{\mathcal{O}_X} End(\mathcal{V}) \cong \mathcal{A}' \otimes_{\mathcal{O}_X} End\mathcal{V}'$. Giving a group structure of tensor product over $\mathcal{O}_X$, we get the Brauer group $Br(X)$.

If you have an Azumaya algebra $\mathcal{A}$ of rank n^2 (n is the rank of that local vector bundle), then you can define $P_{\mathcal{A}} = \underline{Isom}(\mathcal{A}, M_{n \times n}(\mathcal{O}_X))$. This is a functor on X-schemes, it gives you $(\mathsf{Sch}/X)^{op} \to \mathsf{Set}$, given by $(f : T \to X) \mapsto$ the set of isomorphism $f^*\mathcal{A} \xrightarrow{\sim} M_{n \times n}(\mathcal{O}_T)$. This $P_{\mathcal{A}}$ is a PGL_n-torsor (the Scholem-Noether theorem tells you that all the automorphisms of $M_{n \times n}$ are given by conjugation by some matrix). From a PGL_n-torsor, we get the sequence

$$1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$$

from which we get a boundary map $H^1(X, PGL_n) \to H^2(X, \mathbb{G}_m)$. This sends a torsor to the stack $[P_{\mathcal{A}}/GL_n]$, which is $\mathbb{G}_m$-gerbe. In fact, this map lands in the torsion part of $H^2(X, \mathbb{G}_m)$. The reason is

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{G}_m & \longrightarrow & GL_n & \longrightarrow & PGL_n & \longrightarrow & 1 \\
& & \uparrow & & \uparrow & & \| & & \\
1 & \longrightarrow & \mu_n & \longrightarrow & SL_n & \longrightarrow & PGL_n & \longrightarrow & 1
\end{array}
$$

So the map factors through $H^2(X, \mu_n)$ which is torsion.

We get a map $Br(X) \to H^2(X, \mathbb{G}_m)_{tors} =: Br'(X)$ (cohomological Brauer group) given by $\mathcal{A} \mapsto [P_{\mathcal{A}}/GL_n]$. We should check that this is well-defined and a homomorphism, but we won't. It is not too hard to check that this is injective. Under some circumstances, there is no torsion [[★★★ normal and something?]]

Theorem 44.1 (Gabber). *If X has an ample sheaf (a little more general than quasi-projective), then this map is an isomorphism.*

Consider the piece of the long exact sequence

$$H^1(X, GL_n) \to H^1(X, PGL_n) \to H^2(X, \mathbb{G}_m)$$

The first map is $\mathcal{V} \mapsto P_{End(\mathcal{V})}$. From that we see that the kernel of the second map consists of [[★★★]], which is the statement of injectivity of the map $Br(X) \to Br'(X)$ (you also have to say that if two Azumaya algebras are equivalent, then the torsors $P_{\mathcal{A}}$ are isomorphic: if $\mathcal{A}$ and $\mathcal{A}'$ with $P_{\mathcal{A}} \cong P_{\mathcal{A}'}$, then we look at $P_{\mathcal{A}}^{-1} \wedge P_{\mathcal{A}'}$ and somehow

show that $\mathcal{A} \simeq \mathcal{A}'$; this shouldn't be hard). The second map is $P_\mathcal{A} \mapsto \mathcal{G}_\mathcal{A}$.

$$
\begin{array}{ccc}
 & & \mathcal{G}_\mathcal{A} \\
 & \overset{\tilde{f}}{\nearrow} & \downarrow \\
T & \underset{f}{\longrightarrow} & X
\end{array}
$$

$\mathcal{G}_\mathcal{A}(T)$ is the category of pairs $(\mathcal{V}_T, \iota)$ with $\mathcal{V}_T$ a locally free rank n on T and $\iota : f^* \xrightarrow{\sim} End(\mathcal{V}_T)$. This is a $\mathbb{G}_m$-gerbe.

This means that there is a tautological locally free sheaf of rank n on this gerbe. $\mathcal{G}_\mathcal{A}$ comes equipped with a canonical locally free sheaf $\mathcal{V}_\mathcal{A}$. To give a locally free sheaf on a stack is to give a locally free sheaf for every morphism in and for every diagram, an isomorphism of the two pull-backs. So take $\mathcal{V}_T$; if you have a morphism $T \to T'$, then you automatically get an isomorphism between $\mathcal{V}_T$ and the pullback of $\mathcal{V}_{T'}$.

Let $\mathcal{G} \to X$ be a $\mathbb{G}_m$-gerbe, and let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{G}$.

Proposition 44.2. *$\mathcal{F}$ has a canonical decomposition $\mathcal{F} \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{F}^{(n)}$.*

If $f : T \to \mathcal{G}$, then you get a sheaf $\mathcal{F}_T := f^*\mathcal{F}$. This sheaf has an action of $\underline{\mathrm{Aut}}_f = \mathbb{G}_m$, given by $u \in \mathbb{G}_m$ acts by the 2-morphism given by μ over $\mathrm{id}_T : T \to T$ over $\mathcal{G}$. Then we get a decomposition $\mathcal{F}_T \cong \bigoplus_{n \in \mathbb{Z}} \mathcal{F}_T^{(n)}$, with $u \in \mathbb{G}_m$ acts on $\mathcal{F}_T^{(n)}$ by multiplication by u^n.

Definition 44.3. A *twisted sheaf on $\mathcal{G}$* (the standard terminology is "a $\mathcal{G}$-twisted sheaf on X") is a quasi-coherent sheaf $\mathcal{F}$ such that $\mathcal{F} = \mathcal{F}^{(1)}$. You should really specify the character ... since we're working with $\mathbb{G}_m$, we have the canonical character (1).   ◇

Remark 44.4. $\mathcal{V}_\mathcal{A}$ on $\mathcal{G}_\mathcal{A}$ is twisted. Because we defined the action as just multiplication.   ◇

Proposition 44.5. *Let $\alpha \in H^2(X, \mathbb{G}_m)$ with associated gerbe $\pi : \mathcal{G}_\alpha \to X$. Then $\alpha \in Br(X) \subseteq H^2(X, \mathbb{G}_m)$ if and only if $\mathcal{G}_\alpha$ admits a twisted locally free sheaf.*

Reason: ($\Rightarrow$) is already done by remark. ($\Leftarrow$) if $\mathcal{F}_1 = \mathcal{F}_1^{(n)}$ and $\mathcal{F}_2 = \mathcal{F}_2^{(m)}$, then $\mathcal{F}_1 \otimes \mathcal{F}_2 = (\mathcal{F} \otimes \mathcal{F})^{(n+m)}$ (if u acts by u^n and u^m, then it acts on the tensor product by u^{n+m}). Also, $\mathcal{F} = \mathcal{F}^{(1)}$ if and only if $\pi^*\pi_* F \to \mathcal{F}$ is an isomorphism (so the ones that are untwisted are just the sheaves on X).

Say $\mathcal{V}$ on $\mathcal{G}$ is twisted. Then $\mathcal{V} \otimes \check{\mathcal{V}} = (\mathcal{V} \otimes \check{\mathcal{V}})^{(0)}$ (because u acts my multiplication by the inverse on $\check{\mathcal{V}}$), so $\pi_* End(\mathcal{V}) = \mathcal{A}$ is a locally free sheaf on $\mathcal{O}_X$-algebras on X of finite rank. You check that if the gerbe is trival, then this gives you endomorphisms of the vector bundle, so this is an Azumaya algebra. You go through the definitions and check that $\mathcal{G}_\mathcal{A} \cong \mathcal{G}_\alpha$.

Remark 44.6 (Aside)**.** Given an $\mathcal{A}$, you get $End(\mathcal{V}_\mathcal{A}) \cong \pi^*\mathcal{A}$ as part of the data. If you start with $\pi : \mathcal{G}_\mathcal{A} \to X$, then you check that $\pi_* End(\mathcal{V}_\mathcal{A}) \cong \mathcal{A}$.   ◇

To understand this theorem of Gabber, the best thing to do is read de Jong's proof.
    Idea of Gabber's Theorem:
- Gabber's thesis is the case where X is affine. If you search for Gabber on MathSciNet, then there aren't many choices, so you should find it easily.
- Given X quasi-projective over S and $\alpha \in H^2(X, \mathbb{G}_m)_{tors}$ you get $\mathcal{G}_\alpha \to X$. Zariski locally on X there is a twisted vector bundle on $\mathcal{G}_\alpha$ (by the affine case). We want to piece them together to get a global twisted sheaf. The proof of this is on deJong's web page. We don't have time to do that.
    Another way to think about this problem. What does it mean to have such a vector bundle.

Definition 44.7. A stack $\mathcal{X}$ over S is a *quotient stack* if $\mathcal{X} \cong [Z/G]$ where Z is an algebraic space over S and $G \subseteq GL_{n,S}$ is a subgroup scheme flat over S.   ◇

Proposition 44.8. *The following are equivalent for a stack $\mathcal{X}$.*

1. *$\mathcal{X}$ is a quotient stack.*

2. *there is a vector bundle $\mathcal{V}$ on $\mathcal{X}$ so that for every point $x : \operatorname{Spec} k \to \mathcal{X}$, the action of the group scheme $\underline{\operatorname{Aut}}_x$ on the fiber $\mathcal{V}_x$ is faithful.*

Restatement: if $\alpha \in H^2(X, \mathbb{G}_m)$, then $\alpha \in Br(X)$ if and only if the gerbe $\mathcal{G}_\alpha$ is a quotient stack. A twisted sheaf is an example of something where the automorphism group acts faithfully: $\mathcal{V}_x = \bigoplus \mathcal{V}_{\bar{x}}^{(a_i)}$ and the $gcd(a_i) = 1$. Certainly a twisted sheaf gives such a vector bundle, but the other direction is also true.
    The issue then is whether something is a quotient stack or not.

Theorem 44.9. *Let k be a field, and let $\mathcal{X}$ over k be a smooth Deligne-Mumford stack with finite diagonal, and generic stabilizer is trivial (there is an open set with no stabilizer). Then $\mathcal{X}$ is a quotient stack.*

This doesn't treat the problem because for a gerbe you often don't have a generic stabilizer.
    Idea of proof: you want to produce a vector bundle and there is not much you can do. Let's look at the case where k is characteristic zero and algebraically closed. Let $T_\mathcal{X}$ be the dual of $\Omega^1_\mathcal{X}$. We claim first that it satisfies this proposition. You have the coarse space $\mathcal{X} \to X$. Then we have

$$
\begin{array}{ccc}
[U/\Gamma] & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \\
X' & \longrightarrow & X
\end{array}
$$

where Γ is the stabilizer group of a point $x \in \mathcal{X}$. $\hat{U} = \operatorname{Spec} k[[t_1, \ldots, t_r]]$ has an action of Γ on it. $T_{\mathcal{X}, \hat{U}'_x} = \mathfrak{m}/\mathfrak{m}^2$. How do you get a decomposition when you have a smooth thing: look at $\hat{\mathcal{O}}_{U,x} \to \hat{\mathcal{O}}_{U,x}/\mathfrak{m}^2$ and find a section for $\mathfrak{m}/\mathfrak{m}^2$. Since we are in characteristic zero, something is semi-simple representation of Γ. So you get $k[[\mathfrak{m}/\mathfrak{m}^2]]$ and something is trival because the action is fiathful.

Appendix

[[★★★ This is where I'll put stuff that I'd like written down, but is too long-winded to insert elsewhere without disturbing the flow of ideas.]] [[★★★ I guess I'll also put common arguments here if they don't fit well elsewhere... I'd really rather this kind of thing be in the main text]]

A1   Verification of the adjunctions $f^* \vdash f_*$ and $f^\star \vdash f_*$

I feel like maybe these calculations should be done out. [[★★★ which way is that adjunction symbol supposed to be written?]]

A2   Extending properties

If C is a site in which coproducts exist, then we can replace the topology on C by the finest topology which produces the same topos. Then any cover can be replaced by a big coproduct. [[★★★ and the explanation]]

[[★★★ do the following]] $\mathcal{P}$ (objects) *descends along covers*, *is stable*; $\mathcal{P}$ (morphisms) *descends along covers*, *is stable*, *is local on domain.* These should probably be done in the manner of lecture 10. Note in particular that the axioms of a Grothendieck topology imply that "is a cover" is stable.

Let C be a full subcategory of D_0. Then define what it means for $X \in \mathsf{D}_0$ to be C-representable and what it means for a morphism in D_0 to be C-representable.

If $\mathcal{P}$ is a stable property of morphisms, a representable morphism $f : F \to G$ in D_0 *has* $\mathcal{P}$ if for every morphism $X \to G$ with $X \in \mathsf{C}$, $F \times_G X \to X$ (which is a morphism in C since f is representable) has $\mathcal{P}$. In particular, since "is a cover" is stable, we now know when a representable morphism in D is a cover.

A3   Effective Descent Classes

Let C be a site. [[★★★ I think]] A property $\mathcal{P}$ of morphisms in C is an *effective descent class* if for any $\mathcal{P}$ morphism $F \to T$ from a sheaf F on C to an object $T \in \mathsf{C}$ (in the sense of A2), we get that $F \in \mathsf{C}$.

For example, "closed immersion" is an effective descent class in Sch_{fppf}.

[[★★★ I'd like to have a long list of effective descent classes in $\mathsf{Sch}_{??}$ here]]

– closed immersion

– open immersion

– quasi-affine

A4 Descent for Algebraic Spaces

Lemma A4.1. *Let $F : C^{op} \to$ Set be a presheaf on a category C in which fiber products are representable, and let $F \to U$ be a morphism from F to an object in C (thought of as a presheaf via the Yoneda embedding). Then the diagonal morphism $F \to F \times F$ is representable if and only if the diagonal morphism $F \to F \times_U F$ is representable.*

Proof. ($\Rightarrow$) Let $T \in C$ and let $T \to F \times_U F$ be a morphism. By composing with the canonical morphism $F \times_U F \to F \times F$, we get a morphism $T \to F \times F$. Then $T \times_{F \times_U F, \Delta} F = T \times_{F \times F, \Delta} F$ is a sheaf by the hypothesis that $\Delta : F \to F \times F$ is representable.

($\Leftarrow$) Let $T \in C$ and let $T \to F \times F$ be a morphism. In the diagram below, verify that all squares are cartesian.

$$
\begin{array}{ccccc}
T \times_{F \times F, \Delta} F & \longrightarrow & T \times_{U \times U, \Delta} U & \longrightarrow & T \\
\downarrow & & \downarrow & & \downarrow \\
F & \xrightarrow{\;\Delta\;} & F \times_U F & \hookrightarrow & F \times F \\
& & \downarrow & & \downarrow \\
& & U & \xrightarrow{\;\Delta\;} & U \times U
\end{array}
$$

Since fiber products are representable in C, we have that $T \times_{U \times U, \Delta} U$ is in C, so $T \times_{F \times F, \Delta} F$ is in C by the hypothesis that $F \to F \times_U F$ is representable. $\qquad\square$

Definition A4.2. For a scheme U, let $\mathsf{AlgSp}_{nice}(U)$ be the category of algebraic F over U whose diagonal morphism $F \to F \times_U F$ belongs to some effective descent class. $\quad\diamond$

Supposedly, $F \to F \times_U F$ almost always belongs to some effective descent class. In particular, the diagonal is almost always quasi-affine. [[★★★ I guess]]

Theorem A4.3. *If $V \to U$ is an étale cover of schemes, then the pullback functor $\mathsf{AlgSp}_{nice}(U) \to \mathsf{AlgSp}_{nice}(V \to U)$ is an equivalence of categories.*

Proof. In light of Theorem 7.5 (descent for sheaves in a site) and Exercise 2.4 (the topos of C/X is equivalent to the category of morphism to X in the topos of C), it is enough to prove that if F is a sheaf with a morphism to U, and $X = F \times_U V$ is an algebraic space (whose diagonal is in some effective descent class), then F is an algebraic space (whose diagonal is in some effective descent class)

(1) F is an étale sheaf already.

(2) By the lemma, it is enough to show that $F \to F \times_U F$ is representable. Let $T \to F \times_U F$ be a a morphism from a scheme, and let $P = T \times_{F \times_U F} F$. Define T' and

P' so that all the squares in the following diagram are cartesian.
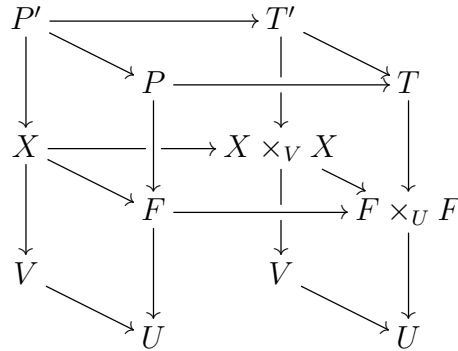
All the down-right arrows in the diagram are étale surjections. Since $T' = T \times_U V$, T' is a scheme. By the hypothesis that X is an algebraic space, P' is a scheme, and $P' \to T'$ belongs to the effective descent class that $X \to X \times_V X$ is in, so P is a scheme.
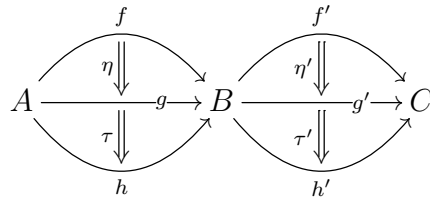
(3) If $W \to X$ is an étale cover of X, then $W \to X \to F$ is an étale cover of F. $\square$

A5   2-Categories

[[★★★ this is where I'll put relevant stuff about 2-categories. **Watch out, I make most of this stuff up, but hopefully the obvious definitions are the right ones.**]]

[[★★★ definitions about 2-categories go here. It may or not be worth it to talk about non-strict 2-categories]]

Definition A5.1. A *strict 2-category* C is a category in which the Hom sets are categories. The morphisms in Hom sets are called *2-morphisms*. If $f, g, h \in \mathrm{Hom}_{\mathsf{C}}(A, B)$, and $f \overset{\eta}{\Rightarrow} g \overset{\tau}{\Rightarrow} h$ are two 2-morphisms, then we denote the composition by $\tau \cdot \eta : f \Rightarrow h$. This is referred to as *vertical composition*. Additionally, we require that there is an associative *horizontal composition*: if $f, g \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ with $\eta : f \Rightarrow g$ and $f', g' \in \mathrm{Hom}_{\mathsf{C}}(B, C)$ with $\eta' : f' \Rightarrow g'$, then we get a 2-morphism $\eta' \circ \eta : f' \circ f \Rightarrow g' \circ g$. Finally, we impose the following compatibility relation: if $f, g, h \in \mathrm{Hom}_C(A, B)$, $f', g', h' \in \mathrm{Hom}_{\mathsf{C}}(B, C)$, with $f \overset{\eta}{\Rightarrow} g \overset{\tau}{\Rightarrow} h$ and $f' \overset{\eta'}{\Rightarrow} g' \overset{\tau'}{\Rightarrow} h'$, then $(\tau' \circ \tau) \cdot (\eta' \circ \eta) = (\tau' \cdot \eta') \circ (\tau \cdot \eta)$.


That is, we require the diagram above to be an unambiguous 2-morphism $f' \circ f \Rightarrow h' \circ h$. $\diamond$

Example A5.2. The category Cat is a 2-category in which the objects are categories, the 1-morphisms are functors, and the 2-morphisms are natural transformations. ◇

Example A5.3. Just as any set can be thought of as a category in which all morphisms are identities, any category can be thought of as a 2-category in which all 2-morphisms are identities. ◇

Commutative diagrams in 2-categories look like hallow 2-dimensional polytopes, with 1-morphisms along the edges and 2-morphisms along the faces. Since these are hard to draw, I'll draw them by cutting them open and saying that the 2-morphism represented by one half is equal to the 2-morphism represented by the other half (if your 2-morphisms are not isomorphisms, then you have to be careful about where you cut, but we will not have such troubles).

Definition A5.4. Let C and D be 2-categories, then a *lax 2-functor* $F : \mathsf{C} \to \mathsf{D}$ associates to each object $A \in \mathsf{C}$ an object $FA \in \mathsf{D}$, to each morphism $f : A \to B$ in C a morphism $Ff : FA \to FC$ in D, and to each pair of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in C a 2-morphism $F_{g,f} : Fg \circ Ff \Rightarrow F(gf)$ such that for every triple of morphisms $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ in C, we have that $F_{h,gf} \cdot (\mathrm{id}_h \circ F_{g,f}) = F_{hg,f} \cdot (F_{h,g} \circ \mathrm{id}_f)$.
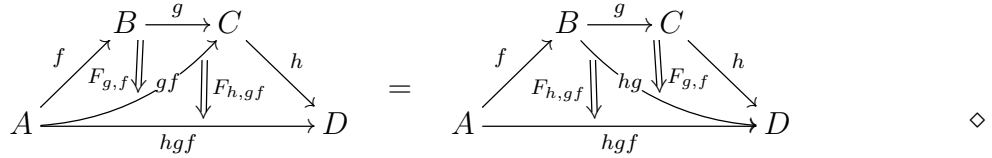
Definition A5.5. Let $F, G : \mathsf{C} \to \mathsf{D}$ be lax 2-functors, then a *lax natural transformation* $\eta : F \to G$ consists of a 1-morphism $\eta_A : FA \to GA$ for each $A \in \mathsf{C}$ and a 2-isomorphism $\eta_f : Gf \circ \eta_A \Rightarrow \eta_B \circ Ff$ for each $f : A \to B$ in C so that for any pair of morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ in C, we have that $\eta_{gf} \cdot (G_{g,f} \circ \mathrm{id}_{\eta_A}) = (\mathrm{id}_{\eta_C} \circ F_{g,f}) \cdot (\eta_g \circ \mathrm{id}_{Ff} \circ \mathrm{id}_{F(gf)}) \cdot (\eta_f \circ \mathrm{id}_{F(gf)})$.
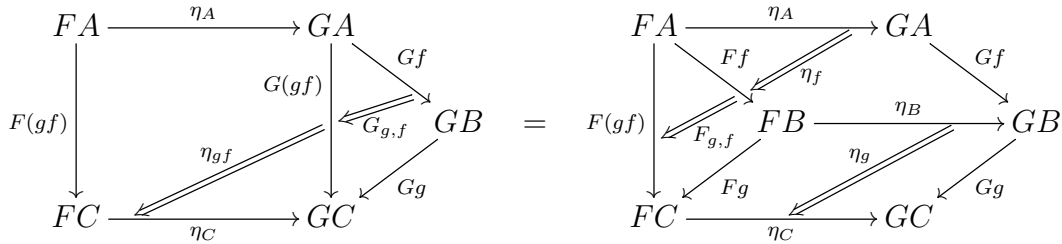
If $\eta, \tau : F \to G$ are lax natural transformations, a morphism between them $\alpha : \eta \to \tau$ is a 2-morphism $\alpha_A : \eta_A \Rightarrow \tau_A$ for each $A \in \mathsf{C}$ such that $\tau_f \cdot (\mathrm{id}_{Gf} \circ \alpha_A) = (\alpha_B \circ \mathrm{id}_{Ff}) \cdot \eta_f$ for each $f : A \to B$ in C. This makes the set of lax natural transformations from F to G into a category, which we'll denote $\mathrm{NAT}(F, G)$. A natural transformation $\eta : F \to G$ is a *natural equivalence* if there is a $\tau : G \to F$ such that $\eta \circ \tau \cong \mathrm{id}_G$ in $\mathrm{NAT}(G, G)$ and $\tau \circ \eta \cong \mathrm{id}_F$ in $\mathrm{NAT}(F, F)$. ◇

Definition A5.6. If C is a 2-category and $X \in \mathsf{C}$, $h_X : \mathsf{C} \to \mathsf{Cat}$ is the functor given by $Y \mapsto \mathrm{Hom}_{\mathsf{C}}(Y, X)$. A lax 2-functor $F : \mathsf{C} \to \mathsf{Cat}$ is *representable* if is naturally equivalent to a functor of the form h_X for some $X \in \mathsf{C}$. ◇

Theorem A5.7 (2-Yoneda Lemma)**.** *If $F : \mathsf{C}^{op} \to \mathsf{Cat}$ is a lax 2-functor and $X \in \mathsf{C}$, then the "evaluation functor" $e_X : \mathrm{NAT}(h_X, F) \to F(X)$, given by $\eta \mapsto \eta_X(\mathrm{id}_X) \in F(X)$ and $(\alpha : \eta \to \tau) \mapsto \alpha_{\mathrm{id}_X}$ (which is a morphism in $F(X)$), is an equivalence of categories, natural in F and X.*

Proof. We need to construct an inverse functor $\eta : F(X) \to \mathrm{NAT}(h_X, F)$. Given $a \in F(X)$, we define $\eta^a : h_X \to F$ by $\eta^a_Y : h_X(Y) \ni f \mapsto Ff(a) \in F(Y)$ for each $Y \in \mathsf{C}$. If $g : Z \to Y$ is a morphism in C and $f : Y \to X$ is a morphism in C, then we define $\eta^a_g(f) : Fg(Ff(a)) = Fg(\eta^a_Y(f)) \to \eta^a_Z(fg) = F(gf)(a)$ to be $F_{g,f}(a)$. One can check that this is a lax natural transformation.

One can check that a morphism in FX yields a morphism of lax natural transformations and that η is inverse to e_X [[★★★ I think the easiest way to do this is to check that η is fully faithful and essentially surjective]]. □

Definition A5.8 (Limits)**.** If I and C are 2-categories and $F : I \to \mathsf{C}$ is a lax 2-functor, then we define $\varprojlim F : \mathsf{C}^{op} \to \mathsf{Cat}$ to be the functor $X \mapsto \mathrm{NAT}(k_X, F)$, where k_X is the functor which sends all objects, morphisms, and 2-morphisms of I to X, id_X, and the identity 2-morphism of id_X, respectively. Similarly, we define $\varinjlim F : \mathsf{C}^{op} \to \mathsf{Cat}$ by $X \mapsto \mathrm{NAT}(F, k_X)$. ◇

Example A5.9. Taking $I = (\cdot \rightrightarrows \cdot)$, a set, or $(\cdot \to \cdot \leftarrow \cdot)$ (with no non-identity 2-morphisms), we get equalizers, products, and fiber products, respectively. ◇

[[★★★ I think the proof of Lemma 3.9 carries over to give the following result]]

Lemma A5.10. *Let C be a 2-category, then the following are equivalent.*

1. *Projective limits (resp. finite projective limits) in C are representable.*

2. *Products (resp. finite products) and equalizers are representable.*

3. *Products and fiber products (resp. finite products and fiber products) are representable.*

Lemma A5.11. *Finite projective limits are representable in Cat.*

Proof. Since Cat has a terminal object (an object that represents the 2-functor which sends every object to the trivial category and all (2-)morphisms to identities), it is enough to show that fiber products are representable.

Let $F(\cdot \to \cdot \leftarrow \cdot) = \mathsf{C}_1 \xrightarrow{a} \mathsf{C}_2 \xleftarrow{b} \mathsf{C}_3$ be a diagram of categories. Define $\mathsf{C}_1 \times_{\mathsf{C}_2} \mathsf{C}_3$ to have objects triples (x_1, x_3, σ) where $x_i \in \mathsf{C}_i$ and $\sigma : a(x_1) \xrightarrow{\sim} b(x_3)$. A morphism

$(x'_1, x'_3, \sigma') \to (x_1, x_3, \sigma)$ is a pair of morphisms $f_i : x'_i \to x_i$ such that the diagram on the left commutes.

$$
\begin{array}{ccc}
a(x'_1) & \xrightarrow{a(f_1)} & a(x_1) \\
{\scriptstyle \sigma'} \downarrow & & \downarrow {\scriptstyle \sigma} \\
b(x'_3) & \xrightarrow{b(f_3)} & b(x_3)
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathsf{C}_1 \times_{\mathsf{C}_2} \mathsf{C}_3 & \xrightarrow{x_3} & \mathsf{C}_3 \\
{\scriptstyle x_1} \downarrow & {\scriptstyle \sigma} & \downarrow {\scriptstyle b} \\
\mathsf{C}_1 & \xrightarrow{a} & \mathsf{C}_2
\end{array}
$$

$\mathsf{C}_1 \times_{\mathsf{C}_2} \mathsf{C}_3$ comes with the obvious maps to C_1 and C_3, and a 2-isomorphism σ (the union of all the σ's) between bx_3 and ax_1.

Given a category D, the category $\mathrm{NAT}(k_\mathsf{D}, F)$, which consists of triples $(g_1 : \mathsf{D} \to \mathsf{C}_1, g_3 : \mathsf{D} \to \mathsf{C}_3, \eta : bg_3 \xrightarrow{\sim} ag_1)$, is obviously equivalent (in fact, isomorphic) to the category $\mathrm{Hom}_\mathsf{Cat}(\mathsf{D}, \mathsf{C}_1 \times_{\mathsf{C}_2} \mathsf{C}_3)$. Thus, $\mathsf{C}_1 \times_{\mathsf{C}_2} \mathsf{C}_3$ represents the fiber product. $\qquad\square$

E   Exercises and solutions

Exercise set 1

Exercise 1.1. *If you have never done so, prove Yoneda's lemma: The association* $X \to h_X$ *defines a fully faithful functor from* C *to the category of functors* $\mathsf{C} \to \mathsf{Set}$.

Solution. Before we prove the theorem, let's state it slightly more generality.

Theorem (Yoneda's Lemma). *For any functor* $F : \mathsf{C}^\circ \to \mathsf{Set}$, *there is a natural bijection* $\mathrm{Nat}(h_X, F) \cong F(X)$. *In particular, taking* $F = h_Y$, *we see that the functor* $h_- : \mathsf{C} \to \mathsf{Fun}(\mathsf{C}^\circ, \mathsf{Set})$ *is a fully faithful embedding of categories.*

Proof. Given $\eta \in \mathrm{Nat}(h_X, F)$, we have $\eta(X) : \mathrm{Hom}(X, X) \to F(X)$, so we get an element $a = \eta(X)(\mathrm{id}_X) \in F(X)$. Conversely, given $a \in F(X)$, we define a natural transformation η by taking $f \in h_X(Y) = \mathrm{Hom}(Y, X)$ to $\eta(Y)(f) := (Ff)(a)$. Check that these are inverses, and that the bijection is natural in F and X. The following diagram should help:

$$
\begin{array}{ccccc}
\mathrm{id}_X & \mathrm{Hom}(X,X) & & F(X) & a \\
\uparrow & \downarrow{\scriptstyle -\circ f} & & \downarrow{\scriptstyle Ff} & \uparrow \\
 & \mathrm{Hom}(Y,X) \xrightarrow{\ \eta(Y)\ } & & F(Y) & \\
f & \longmapsto & & (Ff)(a) &
\end{array}
$$
$\square$

[[★★★ maybe this should be done in two steps: (1) We say a functor $G \colon \mathsf{C}^\circ \to \mathsf{Set}$ has a *universal point* $y \in G(Y)$ for some object Y if for any $x \in G(X)$, there is a unique morphism $f \colon X \to Y$ such that $Gf(y) = x$. If $y \in G(Y)$ is a universal point, then for any functor F, $\mathrm{Nat}(G, F) \cong F(Y)$. (2) h_X has the universal point $\mathrm{id}_X \in h_X(X)$]]

Exercise 1.2. *(a) Let* $n \geq 1$ *be an integer and let* $GL_n : \mathsf{Sch}^\circ \to \mathsf{Set}$ *be the functor sending a scheme* Y *to the set* $GL_n\big(\Gamma(Y, \mathcal{O}_Y)\big)$. *Prove that* GL_n *is a representable functor.*
(b) Let X *represent* GL_n. *Prove that the group structure on* $GL_n\big(\Gamma(Y, \mathcal{O}_Y)\big)$ *induces morphisms*
$$m : X \times X \to X, \quad i : X \to X, \quad e : \mathrm{Spec}\,\mathbb{Z} \to X$$
such that the following diagrams commute:

$$
\begin{array}{ccc}
\mathrm{Spec}\,\mathbb{Z} \times X \xrightarrow{e \times \mathrm{id}} X \times X & \quad X \times \mathrm{Spec}\,\mathbb{Z} \xrightarrow{\mathrm{id}\times e} X \times X & \quad X \times X \times X \xrightarrow{m \times \mathrm{id}} X \times X \\
\Big\Vert \qquad \qquad \downarrow{\scriptstyle m} & \Big\Vert \qquad \qquad \downarrow{\scriptstyle m} & {\scriptstyle \mathrm{id}\times m}\downarrow \qquad \qquad \downarrow{\scriptstyle m} \\
X & X & X \times X \xrightarrow{\ m\ } X
\end{array}
$$

$$
\begin{array}{cc}
X \xrightarrow{i \times \mathrm{id}} X \times X & \qquad X \xrightarrow{\mathrm{id}\times i} X \times X \\
\downarrow \qquad\qquad \downarrow{\scriptstyle m} & \qquad \downarrow \qquad\qquad \downarrow{\scriptstyle m} \\
\mathrm{Spec}\,\mathbb{Z} \xrightarrow{\ e\ } X & \qquad \mathrm{Spec}\,\mathbb{Z} \xrightarrow{\ e\ } X
\end{array}
$$

Solution. (a) Let $Z = \mathrm{Spec}\big(\mathbb{Z}[x_{11}, \ldots, x_{nn}, y]/(1 - y \det X)\big)$, where X is the matrix with (i, j)-th entry x_{ij}. The claim is that Z represents GL_n. To see this, note that

$$\mathrm{Hom}_{\mathsf{Sch}}(Y, Z) \cong \mathrm{Hom}_{\mathsf{Ring}}\Big(\frac{\mathbb{Z}[x_{11}, \ldots, x_{nn}, y]}{(1 - y \det X)}, \Gamma(Y, \mathcal{O}_Y)\Big) \cong GL_n\big(\Gamma(Y, \mathcal{O}_Y)\big).$$

(b) The functor $GL_n(-)$ factors through Gp, so $GL_n(-) \simeq \mathrm{Hom}(-, X)$ is a group object in the category $\mathsf{Fun}(\mathsf{Sch}^\circ, \mathsf{Set})$, i.e. it has maps like m, i, and e, satisfying the diagrams above, with $\mathrm{Spec}\,\mathbb{Z}$ replaced by the final object, $\mathrm{Hom}(-, \mathrm{Spec}\,\mathbb{Z})$. Since the Yoneda embedding is a fully faithful, we have that X is a group object in Sch, as desired.

Exercise 1.3. *(a) Let $\mathbb{A}^n \smallsetminus \{0\} : \mathsf{Sch}^\circ \to \mathsf{Set}$ be the functor sending a scheme Y to the set of n-tuples $(y_1, \ldots, y_n)$ of sections $y_i \in \Gamma(Y, \mathcal{O}_Y)$ such that for every point $y \in Y$ the images of the y_i in $k(y)$ are not all zero. Show that $\mathbb{A}^n \smallsetminus \{0\}$ is representable.*
(b) Let $(\mathbb{A}^n \smallsetminus \{0\})/\mathbb{G}_m : \mathsf{Sch}^\circ \to \mathsf{Set}$ be the functor sending a scheme Y to the quotient of the set $(\mathbb{A}^n \smallsetminus \{0\})(Y)$ by the equivalence relation $(y_1, \ldots, y_n) \sim (y_1', \ldots, y_n')$ if there exists a unit $u \in \Gamma(Y, \mathcal{O}_Y^\times)$ such that $y_j = u y_j'$ for all j. Show that $(\mathbb{A}^n \smallsetminus \{0\})/\mathbb{G}_m$ is not representable.

Solution. (a) Something like "an open subfunctor of a representable functor is representable". You can check explicitly that this is represented by the open subscheme of $\mathbb{A}^n$ obtained by removing the closure of the point $(x_1, \ldots, x_n)$ (note that this point isn't closed!).

(b) Call the functor in question F. A representable functor is a sheaf on the Zariski site because morphisms glue and morphisms which agree locally agree globally. We will show that F is not representable by showing it is not a sheaf.

Let k be a field, and let $U = \mathrm{Spec}\,k[x]$ and $V = \mathrm{Spec}\,k[1/x]$ be the usual open sets in $\mathbb{P}_k^1$. We have that $\Gamma(\mathbb{P}_k^1, \mathcal{O}_{\mathbb{P}_k^1}) = k$, so $F(\mathbb{P}_k^1) = (k^n \smallsetminus \{0\})/k^\times$. We have sections $[x : 1 : \cdots : 1] \in F(U)$ and $[1 : 1/x : \cdots : 1/x] \in F(V)$ which are not restrictions of global sections on $\mathbb{P}_k^1$ (because every global section can be represented by an n-tuple in k). However, on the intersection $\mathrm{Spec}\,k[x, 1/x]$, the two sections agree. Therefore, F is not a sheaf, so it is not representable.

Ishai's solution (sketch): $\mathbb{A}^n \smallsetminus \{0\}$ is the functor $Y \mapsto \{\mathcal{O}_Y^n \twoheadrightarrow \mathcal{L}, \varphi : \mathcal{L} \cong \mathcal{O}_Y\}$. The functor F is $Y \mapsto \{\mathcal{O}_Y^n \twoheadrightarrow \mathcal{L}, \text{ with } \mathcal{L} \cong \mathcal{O}_Y, \text{ but you don't care how}\}$. Since there are sheaves which are locally trivial but not globally trivial, this functor is not a sheaf. The sheafification is represented by $\mathbb{P}^{n-1}$, and it is $Y \mapsto \{\mathcal{O}_Y^n \twoheadrightarrow \mathcal{L}, \text{ with } \mathcal{L} \text{ invertible}\}$ ... this approach is discussed in Hartshorne's section on morphisms to projective space.

Exercise 1.4. *Let Top be the category of topological spaces with morphisms being continuous maps. Let $F : \mathsf{Top}^\circ \to \mathsf{Set}$ be the functor sending a topological space S to the collection $F(S)$ of all its open sets.*
(a) Endow $\{0, 1\}$ with the coarsest topology in which the subset $\{1\} \subset \{0, 1\}$ is closed. Show that the open sets in this topology are $\varnothing$, $\{0\}$, and $\{0, 1\}$.

(b) Show that $\{0,1\}$ with the above topology represents F.
(c) Let HausTop $\subseteq$ Top *denote the full subcategory of Hausdorff topological spaces.*
Show that the restriction $F|_{\mathsf{HausTop}} : \mathsf{HausTop}^\circ \to \mathsf{Set}$ *is not representable.*

Solution. (a) $\varnothing$ and $\{0,1\}$ will be open in any topology, and $\{0\}$ must be open for $\{1\}$ to be closed, so these three sets must be open. On the other hand, these three sets form a topology, so this is the coarsest such topology.

(b) Given any continuous map $f : S \to \{0,1\}$, we get an open set $f^{-1}(0) \subseteq S$. Conversely, given any open set $U \subseteq S$, the function f_U given by $f_U(U) = 0$ and $f_U(S \smallsetminus U) = 1$ is continuous, so $F(S) \cong \mathrm{Hom}(S, \{0,1\})$. It wasn't specified what F did on morphisms, but the most obvious thing is to pull back open sets along morphisms, as is done by $\mathrm{Hom}(-, \{0,1\})$.

(c) Assume F_{HausTop} is represented by a Hausdorff space X. Since the 1-point Hausdorff space $*$ has two open sets, $|\mathrm{Hom}(*, X)| = |X| = 2$. Since X is Hausdorff, it must be a 2-point set with the discrete topology. Let I be the unit interval, then there are only two elements of $\mathrm{Hom}(I, X)$ since I is connected, but I has an uncountable number of open sets, a contradiction. Thus, F_{HausTop} is not representable.

Exercise 1.5. *Another proof that $\mathcal{M}_{1,1}$ is not representable.*
(a) Let k be a field and $k \hookrightarrow \bar{k}$ an algebraic closure. Show that if F is a representable functor then the map $F(\mathrm{Spec}\, k) \to F(\mathrm{Spec}\, \bar{k})$ is injective.
(b) Let $D > 1$ be an integer. Show that the two elliptic curves over $\mathbb{Q}$

$$E_1 : y^2 = x^3 + x, \quad E_2 : y^2 = x^3 + Dx$$

are isomorphic over $\overline{\mathbb{Q}}$, but not over $\mathbb{Q}$. From this and (a), conclude that the functor $\mathcal{M}_{1,1}$ defined in class is not representable.

Solution. (a) Let $f : \mathrm{Spec}\, \bar{k} \to \mathrm{Spec}\, k$ be the obvious map. Assume $F \simeq \mathrm{Hom}(-, X)$ and that $g, h : \mathrm{Spec}\, k \to X$ are two morphisms such that $g \circ f = h \circ f$, i.e. two elements of $F(\mathrm{Spec}\, k)$ that map to the same element of $F(\mathrm{Spec}\, \bar{k})$. Let $\mathrm{Spec}\, R$ be an open affine neighborhood of the image of $g \circ f = h \circ f$. Then the images of g and h lie in $\mathrm{Spec}\, R$, and we have that $f^\sharp \circ g^\sharp = f^\sharp \circ h^\sharp : R \to k \xrightarrow{f^\sharp} \bar{k}$. Since $f^\sharp$ is an injection, we must have that $g^\sharp = h^\sharp$, so $g = h$. Thus, $F(\mathrm{Spec}\, k) \to F(\mathrm{Spec}\, \bar{k})$ is injective.

(b) If D is a fourth power, then you can do some change of coordinate operations to show that $E_1 \cong E_2$. Since everything is a fourth power in $\overline{\mathbb{Q}}$, the two are isomorphic over $\overline{\mathbb{Q}}$.

Something shows that they are not isomorphic over $\mathbb{Q}$, like looking at torsion points, perhaps after reducing at some prime dividing D.

Exercise set 2

Exercise 2.1. *For a set S, let $\mathrm{Aut}(S)$ denote the group of bijections $S \to S$. An action of a group G on S is defined to be a homomorphism $G \to \mathrm{Aut}(S)$. Let G be a*

group and define BG to be the category with one object $*$ and with $\mathrm{Hom}(*, *) = G$.
(a) Show that if F is a presheaf on BG then $F(*)$ admits a natural action of G.
(b) Show that the induced functor (presheaves on BG) $\to$ (sets with G-action), given by $F \mapsto F(*)$, is an equivalence of categories.

Exercise 2.2. *(a)*
(b)
(c)
(d)

Exercise 2.3. *(a)*
(b)
(c)
(d)

Exercise 2.4. *Let C be a site with associated topos T, and let $X \in \mathsf{C}$ be an object. Assume that the functor of points h_X is a sheaf.*
(a) Show that the topology on C induces a topology on C/X.
(b) Show that the category of sheaves on C/X is equivalent to T/h_X. In particular, T/h_X is a topos.
(c) Define $j^* : \mathsf{T} \to \mathsf{T}/h_X$ by sending F to $F \times h_X$ with the projection to h_X. Show that j^* commutes with finite projective limits and has a right adjoint j_*, given by $(j_*G)(Y) = \mathrm{Hom}_{\mathsf{T}/h_X}(h_Y^a \times h_X, G)$. In particular, there is a morphism of topoi $j : \mathsf{T}/h_X \to \mathsf{T}$.

Solution. (a) Let C be a site, let D be category, and let $F : \mathsf{D} \to \mathsf{C}$ be a functor which respects fiber products, then we can declare $\{Y_i \to Y\}$ to be a covering of an object $Y \in \mathsf{D}$ if $\{FY_i \to FY\}$ is a covering of the object $FY \in \mathsf{C}$. It is immediate to verify that this satisfies the axioms of a Grothendieck topology. In our case, we take F to be the forgetful functor $\mathsf{C}/X \to \mathsf{C}$.

(b) Given a sheaf $F : \mathsf{C}^\circ \to \mathsf{Set}$, with a morphism of sheaves $\eta : F \to h_X$, we can define a sheaf $\tilde{F}$ on C/X by $(f : U \to X) \mapsto \eta(U)^{-1}(f)$. To make sense of this, note that $\eta(U) : F(U) \to h_X(U)$, and $f \in h_X(U)$, so $\tilde{F}(f : U \to X)$ is a subset of $F(U)$. One can then check that $\tilde{F}$ is a presheaf (functor) and satisfies the sheaf axiom.

Conversely, if $G : (\mathsf{C}/X)^\circ \to \mathsf{Set}$ is a sheaf on C/X, then we can define a sheaf $G' : \mathsf{C} \to \mathsf{Set}$ by $U \mapsto \coprod_{f \in h_X(U)} G(f : U \to X)$, with the natural projection to $h_X(U)$. Again, one checks that G' is a sheaf and $G' \to h_X$ is a morphism of sheaves.

I've swept it under the rug, but you can define what these two operations do to morphisms of sheaves (it is the obvious thing), and they are inverse (up to isomorphism), so we get the desired equivalence of categories.

(c) Note that for an object $(G \to h_X) \in \mathsf{T}/h_X$ and a sheaf $F \in \mathsf{T}$, we have that $\mathrm{Hom}_{\mathsf{T}/h_X}(G \to h_X, j^*F) = \mathrm{Hom}_{\mathsf{T}}(G, F)$. That is, j^* is right adjoint to the forgetful functor $\mathsf{C}/X \to \mathsf{C}$, so it commutes with all projective limits.

[[★★★ prove that j_* is right adjoint to j^*]]

Exercise set 3

Exercise 3.1. *For a scheme S let $\mathcal{M}_0(S)$ denote the category whose objects are smooth proper morphisms $C \to S$ all of whose geometric fibers are smooth connected curves of genus 0. For an fppf and quasi-compact cover $X \to Y$ define the category $\mathcal{M}_0(X \to Y)$ as in class. Show that the pullback functor $\mathcal{M}_0(Y) \to \mathcal{M}_0(X \to Y)$ is an equivalence of categories (hint: consider the dual of the canonical sheaf).*

Solution. The dual of the canonical bundle is a functorial choice of relatively very ample sheaf. Then the proof of descent for $\mathcal{M}_g$ ($g \geq 2$) works word for word.

Exercise 3.2. *Let $f : X \to Y$ be a faithfully flat morphism of locally noetherian schemes.*
(a) Show that a quasi-coherent sheaf F on Y is coherent if and only if $f^\star F$ on X is coherent.
(b) Show that a quasi-coherent sheaf F on Y is locally free of finite rank if and only if $f^\star F$ is locally free of finite rank.
(c) Prove that the pullback functor $\mathsf{Qcoh}(Y) \to \mathsf{Qcoh}(X \to Y)$ identifies the category of coherent (resp. locally free of finite rank) sheaves on Y with the fully subcategory of $\mathsf{Qcoh}(X \to Y)$ consisting of pairs (E_X, σ), where E_X is coherent (resp. locally free of finite rank).

Solution. Being coherent (resp. locally free of finite rank) is local, so we may take open affines $V = \operatorname{Spec} A \subseteq Y$ and $U = \operatorname{Spec} B \subseteq f^{-1}(V) \subseteq X$.

If F is coherent (resp. locally free of finite rank), then it is $\tilde{M}$ for some A-module M which is finitely generated (resp. free of finite rank). Then it is clear that $f^\star F = (M \otimes_A B)^\sim$ is coherent (resp. locally free of finite rank).

If $F = \tilde{M}$ is quasi-coherent on V, and $f^\star F = (M \otimes_A B)^\sim$ is coherent (resp. locally free of finite rank) on U. Then $M \otimes_A B$ is finitely generated (resp. free of finite rank). Let $\{\sum_j m_{ij} \otimes b_{ij}\}_i$ be a finite (resp. finite free) set of generators for $M \otimes_A B$. It is enough to check locally that M is finitely generated [[★★★ is this true? No! Take the $k[x]$-module $M = \bigoplus_{a \in k} k[x]/(x - a)$ for some $k = \bar{k}$]] (resp. free of finite rank (Ex. II.5.7)). Let $\mathfrak{p} \in \operatorname{Spec} A$ and let $\mathfrak{P} \in \operatorname{Spec} B$ lie over $\mathfrak{p}$. Then $B_{\mathfrak{P}}$ is faithfully flat over $A_{\mathfrak{p}}$ [[★★★ is this true?]].Thus, we may assume that B is faithfully flat over A.

(coherence) The map $A^N \to M$, given by $1_{ij} \mapsto m_{ij}$ is surjective after tensoring with B, so it is surjective, so M is finitely generated over A.

(loc. free)

(c) Let $\mathcal{P}$ be any property of quasi-coherent sheaves so that for any fppf cover f, F has $\mathcal{P}$ if and only if $f^\star F$ does.

A $\mathcal{P}$ sheaf on Y pulls back to some E_X which is $\mathcal{P}$ on X. Conversely, assume E_X is $\mathcal{P}$. By the equivalence of categories, it is $f^\star F$ for some F on Y. Then F (the image of E_X under the equivalence) is also $\mathcal{P}$.

Exercise 3.3. *For a scheme Y let $\mathsf{Aff}(Y)$ denote the category whose objects are affine Y-schemes and whose morphisms are Y-morphisms. For a morphism of schemes $X \to$*

Y let $\mathsf{Aff}(X \to Y)$ *denote the category whose objects are pairs* (Z_X, σ), *where* $Z_X \in \mathsf{Aff}(X)$ *and*

$$\sigma : Z_X \times_{X, pr_1} (X \times_Y X) \to (X \times_Y X) \times_{pr_2, X} Z_X$$

is an isomorphism of $X \times_Y X$-*schemes satisfying a cocycle condition on* $X \times_Y X \times_Y X$.
(a) Write out precisely the cocycle condition in this case.
(b) Prove that if $X \to Y$ *is a quasi-compact fppf cover then the pullback functor* $\mathsf{Aff}(Y) \to \mathsf{Aff}(X \to Y)$ *is an equivalence of categories.*

Solution. (a) To save ink, we use the usual notation: write $Z_X \times_{X, \mathrm{pr}_1} (X \times X)$ as $p_1^* Z_X$, and similarly define $p_2^* Z_X$. Then σ is an isomorphism $p_1^* Z_X \to p_2^* Z_X$. The cocycle condition is that following diagram must commute.

$$
\begin{array}{ccc}
& p_{13}^* p_1^* Z_X \xrightarrow{p_{13}^* \sigma} p_{13}^* p_2^* Z_X & \\
p_{12}^* p_1^* Z_X & & p_{23}^* p_2^* Z_X \\
& p_{12}^* p_2^* Z_X = p_{23}^* p_1^* Z_X &
\end{array}
$$

(b) An affine X-scheme is $\mathcal{S}pec\, \mathcal{A}$ for some quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}$. Use descent for quasi-coherent sheaves of algebras.

(4) *Let B be a base scheme, $d \geq 1$ an integer, and fix a projective B-scheme P with an ample line bundle L. For a B-scheme Y, let $\mathcal{M}_g(P, d)(Y)$ denote the category whose objects are pairs (C, f), where $C \to Y$ is an object of $\mathcal{M}_g(Y)$ and $f : C \to P$ is a morphism such that the restriction of $f^* L$ to each geometric fiber of $C \to Y$ is an ample sheaf of degree d. A morphism $(C', f') \to (C, f)$ is a morphism $\varepsilon : C' \to C$ in $\mathcal{M}_g(Y)$ (so an isomorphism $C' \to C$ over Y) such that $f \circ \varepsilon = f'$.*

(4a) *For a morphism of schemes $X \to Y$ define the category $\mathcal{M}_g(P, d)(X \to Y)$.*

Objects in $\mathcal{M}_g(P, d)(X \to Y)$ are pairs $\big((C, f), \sigma\big)$, where $(C, f) \in \mathcal{M}_g(P, d)(X)$ and σ is an isomorphism $p_2^*(C, f) \xrightarrow{\sim} p_1^*(C, f)$ satisfying the usual cocycle condition. A morphism $\big((C, f), \sigma\big) \to \big((C', f'), \sigma'\big)$ is a morphism $\varepsilon : (C, f) \to (C', f')$ such that the following diagram commutes.

$$
\begin{array}{ccc}
p_2^*(C, f) & \xrightarrow{p_2^* \varepsilon} & p_2^*(C', f') \\
\sigma \downarrow & & \downarrow \sigma' \\
p_1^*(C, f) & \xrightarrow{p_1^* \varepsilon} & p_1^*(C', f')
\end{array}
$$

(4b) *Prove that for an fppf quasi-compact cover $X \to Y$ pullback defines an equivalence $\mathcal{M}_g(P, d)(Y) \to \mathcal{M}_g(P, d)(X \to Y)$.*

If $f^* L$ some fixed tensor power of it had to be relatively *very* ample, then the proof of descent for $\mathcal{M}_g$ would carry over.

Exercise set 4

Exercise set 5

Exercise set 6

Exercise set 7

References

[AM69]   M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969. 151

[Con]   Brian Conrad. The Keel-Mori theorem via stacks. Available at www.math.lsa.umich.edu/~bdconrad/.

[FGI⁺05]   Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry*, volume 123 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, 2005. Grothendieck's FGA explained.

[EGA]   Alexander Grothendieck and Jean-Alexandre-Eugène Dieudonné. *Éléments de géométrie algébrique.* Numbers 4, 8, 11, 17, 20, 24, 28, and 32. 1960-1967. Institut des Hautes Études Scientifiques. Publications Mathématiques. 21, 41, 54, 56, 69, 70, 71, 74, 115, 135

[SGA]   Alexander Grothendieck, Michèle Raynaud, Michel Demazure, Michael Artin, Jean-Louis Verdier, Pierre Deligne, Pierre Berthelot, Luc Illusie, and Nicholas Katz. *Séminaire de Géométrie Algébrique du Bois-Marie.* Springer-Verlag, Berlin, 1970-1973. Lecture Notes in Mathematics, Volumes 151, 152, 153, 224, 225, 269, 270, 288, 305, 340, 569, and 589. 9, 116, 149, 154, 160, 164

[Har77]   Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52. 31, 33, 42, 67, 111

[KM97]   Seán Keel and Shigefumi Mori. Quotients by groupoids. *Ann. of Math. (2)*, 145(1):193–213, 1997.

[Knu71]   Donald Knutson. *Algebraic spaces.* Springer-Verlag, Berlin, 1971. Lecture Notes in Mathematics, Vol. 203. 43

[Lan02]   Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, third edition, 2002. 77, 165

[RG71]   Michel Raynaud and Laurent Gruson. Critères de platitude et de projectivité. Techniques de "platification" d'un module. *Inventiones Mathematicae*, 13:1–89, 1971. 74

[Vis05]   Angelo Vistoli. Notes on Grothendieck topologies, fibered categories and descent theory. arxiv.org/math.AG/0412512. This appears as the first part of [FGI⁺05]. 54, 56, 74, 84