

## STACKS HW2 - LIMITS, COLIMITS, GROUP OBJECTS

- (1) Let  $C$  be a category and let  $h: C \rightarrow \text{Fun}(C^{\text{op}}, \mathbf{Sets})$  be the Yoneda embedding. Show that for any arrows  $X \rightarrow Y$  and  $Z \rightarrow Y$  in  $C$ , there is a natural isomorphism

$$h_{X \times_Y Z} \rightarrow h_X \times_{h_Y} h_Z$$

of functors, where  $h_X \times_{h_Y} h_Z$  is the functor

$$h_X \times_{h_Y} h_Z: W \mapsto h_X(W) \times_{h_Y(W)} h_Z(W).$$

- (2) Let  $G$  be an object of a category  $C$ . Show that the functor of points

$$h_G: C^{\text{op}} \rightarrow \mathbf{Sets}$$

factors through the forgetful functor from groups

$$\begin{array}{ccc} & \mathbf{Groups} & \\ & \nearrow & \downarrow \\ h_G: C^{\text{op}} & \longrightarrow & \mathbf{Sets} \end{array}$$

if and only if  $G$  is a group object of  $C$ .

- (3) Let  $\star$  be one of  $\text{Spec } k$  (with  $k$  a field) or  $\text{Spec } \mathbb{Z}$ . Let

$$G = \coprod_{g \in \mathbb{Z}/2\mathbb{Z}} \star$$

be the group object corresponding to  $\mathbb{Z}/2\mathbb{Z}$ . Work out explicitly the group structure. In other words, work out the maps in terms of rings, and show that  $G$  represents the functor

$$\mathbf{Sch}^{\text{op}} \rightarrow \mathbf{Groups}, W \mapsto \mathbb{Z}/2\mathbb{Z}$$

- (4) (a) Let  $n \geq 1$  be an integer and let

$$\text{GL}_n: (\mathbf{Sch})^{\text{op}} \rightarrow \mathbf{Sets}$$

be the functor sending a scheme  $Y$  to the set  $\text{GL}_n(\Gamma(Y, \mathcal{O}_Y))$ . Prove that  $\text{GL}_n$  is a representable functor.

- (b) Let  $X$  represent the functor  $\text{GL}_n$ . Prove that the group structures on the sets  $\text{GL}_n(\Gamma(Y, \mathcal{O}_Y))$  induce the structure of a group scheme on  $X$ . (I.e. use the previous exercise, noting that there is one detail to check.)

- (5) Give an example of a category  $C$ , a subcategory  $C'$ , and a diagram  $D: I \rightarrow C'$  such that the limit (or colimit, your choice) in  $C'$  is not the limit in  $C$ .

- (6) Use the functor of points to define a map  $\mathbb{A}^1 \rightarrow \mathbb{A}^1$  given by the formula  $z \mapsto z^2$ . Compare this with how one would define such a map with locally ringed spaces.

- (7) **Monomorphisms.**

- (a) Show that a morphism  $X \rightarrow Y$  is a monomorphism if and only if for every  $T \in C$ , the map of sets  $X(T) \rightarrow Y(T)$  is injective.

- (b) Show that a map of schemes which is injective topologically may not be a monomorphism.

(c) Show that a map of schemes which is surjective topologically may not be an epimorphism.