HOMEWORK FOR SPRING 2016 ALGEBRAIC TOPOLOGY

Last Modified January 18, 2016

Some notes on homework:

(1) If I modify a problem, I will leave a colored footnote with the date and a comment. [1]
(2) There won't be any official office hours. I will usually be around TuWTh, usually not around on Monday, and sometimes around on Fri.
(3) There are hints in the latex comments. (Just change the extension of this URL to .tex)
(4) I will assign lots of problems. It is best look at all of them, spend some time thinking about each problem, and pick a few to think through in a lot of detail.

Some notes on class + lectures:

(1) **Expectations**: Taken from Tony Varilly's syllabus: "In my experience as a student, most people do not follow all the details of a Math lecture in real time. During lecture, you should expect to witness the big picture of what's going on. You should pay attention to the lecturer's advice on what is important and what isn't. A lecturer spends a long time thinking on how to deliver a presentation of an immense amount of material; they do not expect you to follow every step, but they do expect you to go home and fill in the gaps in your understanding."
(2) Please do all of the following.
    (a) After each class, review all of the lecture notes.
    (b) Before the next class, briefly review again. (Make sure that you come to each lecture knowing at least the definitions and statements from the last lecture.)
    (c) Sit down at least once on your own to attempt the homework.
    (d) Meet with other students at least once to discuss homework.

[1] December 31: Like this.

1. HW 1

(1) Fill in the missing details in the proofs of basic properties of homotopy from the first two lectures.
(2) Prove that $\pi_i(X)$ is abelian for $i \geq 2$.
(3) Let G be a topological group. Prove that $\pi_1(G)$ is abelian.
(4) Prove that if $f\colon X \to Y$ is a (possibly unbased) homotopy equivalence, then the induced map $f_*\colon \pi_n(X, x) \to \pi_n(Y, f(x))$ is an isomorphism. (I.e fill in the missing detail from lecture 3.)
(5) Prove that the quasi-circle (see Hatcher 1.3, exercise 7) is weakly contractible but not contractible.
(6) Finish the (sketch of a) proof from class that $\pi_k(S^n) = 0$ for $0 < k < n$.
(7) Prove that $\pi_1(\mathbb{R}^2 - \mathbb{Q}^2)$ is uncountable.

2. HW 2

(1) Compute π_i of $\mathbb{RP}^n \backslash \{x_0\}$ for $i \leq n$ (where x_0 is any point). What familiar space is $\mathbb{RP}^n \backslash \{x_0\}$ homotopy equivalent to? What is the map $\pi_i(\mathbb{RP}^n \backslash \{x_0\}) \to \pi_i(\mathbb{RP}^n)$? Compute the action of $\pi_1(\mathbb{RP}^2)$ on $\pi_2(\mathbb{RP}^2)$.

(2) Compute π_i of $X = S^1 \vee S^n$ for $1 < i < n$. ($\vee$ means "wedge product".) Show that the inclusion $S^1 \to X$ induces an isomorphism on π_1. Compute the action of π_1 on π_n.

(3) Do each of the following problems.
   (a) Compute π_1 of the complement of a circle embedded in $\mathbb{R}^3$ as the unknot (i.e. as the usual circle in the xy plane).
   (b) Let $\Sigma_{n,m}$ be a n-holed torus with m points removed. Compute $\pi_k(\Sigma_{1,1})$. (Hint: What familiar space is $\Sigma_{1,1}$ homotopy equivalent to?)
   (c) Compute π_k of $\mathrm{SL}_2(\mathbb{R})$.
   (d) Compute the fundamental group of the subspace $S \subset \mathrm{M}_3(\mathbb{R})$ of 3×3 matrices with rank 1.

(4) **Application of topology to free groups.**
   (a) Let X_n be $\mathbb{C}$ with n points removed. Compute $\pi_1(X_n, x_0)$.
   (b) Compute the universal covering space of $S^1 \vee S^1$.
   (c) Prove that F_n (the free group on n generators) is a subgroup of F_2.
   (d) Prove that every subgroup of a free group is free.
   (e) Prove that F_n contains a subgroup that is not finitely generated.
   (f) Let $G \subset F_n$ be a subgorup of index j. G is free; how many generators does it have?

(5) Show that for every finitely generated abelian group G there is a manifold M with fundamental group G and that for every finitely generated group G there is a topological space M with fundamental group G.

(6) Let X and Y be path connected and locally path connected.
   (a) Show that if $\pi_1(X)$ is finite then every map $X \to S^1$ is null-homotopic.
   (b) Show that if X is homotopy equivalent to Y then their universal covering spaces are also homotopy equivalent.

(7) More π_1 problems.
   (a) Construct a simply-connected covering space of the space $X \subset \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is a union of a sphere and a circle intersecting it in two points.
   (b) Compute π_1 of the complement of the two planes $x = y = 0$ and $z = w = 0$ in $\mathbb{R}^4 = \mathbb{C}^2$.
   (c) Use Van Kampen's theorem to calculate $\pi_1(S^n)$.
   (d) Use Van Kampen's theorem to calculate π_1 of the Klein bottle.

(8) More π_1 problems.
   (a) Compute $\pi_1(\Sigma_{2,0})$. Give an example of a degree 2 connected covering space over $\Sigma_{2,0}$. Harder: do the same for $\pi_1(\Sigma_{n,m})$.
   (b) Compute π_1 of the complement of a circle embedded in $\mathbb{R}^3$ as the trefoil knot. Conclude that the trefoil knot is not equivalent to the unknot.

3. HW 3

(1) Recall that $[S^1, X]$ is $\mathrm{Maps}(S^1, X)$ mod *unbased* homotopy. Suppose that X is path connected.
  (a) Prove that $[S^1, X]$ has a group structure.
  (b) Prove that $[S^1, X]$ is abelian.
  (More difficult is to show that $[S^1, X] \cong \pi_1^{\mathrm{ab}}$.)
(2) Hatcher section 0, #23: Show that a CW complex is contractible if it is the union of two contractible subcomplexes whose intersection is also contractible.. Show that this isn't true for non-CW complexes.
(3) Hatcher section 1.1, #18. (Effect of attaching cells to π_1, and some easy applications.)
(4) Hatcher, section 1.2, #6. (More on attaching cells.)
(5) Hatcher section 4 #6. (Covering spaces and pairs.)
(6) Hatcher section 4, #8. (Exactness at the tail of the homotopy LES.)
(7) Hatcher section 4, #9. (π_0 for pairs.)
(8) Show that for a (pointed) triad (X, A, B) (i.e., $x_0 \in B \subset A \subset X$), there is a long exact sequence
$$\cdots \to \pi_n(A, B) \to \pi_n(X, B) \to \pi_n(X, A) \to \pi_{n-1}(A, B) \to \cdots$$
(9) Show that $\mathbb{RP}^n$ has a CW-structure.
(10) Hatcher section 3D, #2.
(11) Hatcher Appendix A, #3.
(12) (a) Prove the fundamental theorem of algebra using the Brauer fixed point theorem.
  (b) Prove that $\mathbb{R}^n \cong \mathbb{R}^m$ if and only if $n = m$.
(13) Prove the Borsuk-Ulam theorem.
(14) Prove the ham sandwich theorem. (Warmup: prove that any region in $\mathbb{R}^n$ can be sliced into two regions of equal area.)

Fun problems, not to be turned in.

(1) Hatcher, section 1.2, #4. (Compute π_1 of the complement of n lines through the origin in $\mathbb{R}^3$.)

4. HW 4

Homology. Read the introduction to Chapter 2.

(1) Prove that a short exact sequence of chain complexes induces a long exact sequence on homology.
(2) Compute the simplicial homology of Σ_2 and $\Sigma_{1,1}$.
(3) Compute the homology of $\mathbb{RP}^3$.
(4) Compute the homology of S^n.
(5) Hatcher, section 2.1, exercise 8. (Simplicial Homology of tetrahedra arrangements.)
(6) Hatcher, section 2.1, exercise 11. (Retract implies injectivitiy on H_n.)
(7) Hatcher, section 2.1, exercise 12. Show that chain homotopy of chain maps is an equivalence relation. Also show that composition of chain maps induces a well-defined map on equivalence classes.
(8) For $n \in \mathbb{Z}_{\geq 1}$, compute the simplicial homology of the space obtained by taking three copies of D_n and identifying their boundaries with each other. (You choose the Δ-structure.)
(9) Prove carefully that if $0 \to \mathbb{Z}^{n_1} \to \mathbb{Z}^{n_2} \to \; \to \mathbb{Z}^{n_k} \to 0$ is exact, then $\sum_{i=1}^{k}(-1)^i n_i = 0$. As a corollary, prove that if Γ is a graph on Σ_g, then the Euler characteristic $\chi(\Gamma) :=$ number of faces - edges + vertices is $2-2g$.
(10) Show that the data of a simplicial set is the same as the data of a Δ complex. Show that there is a functor from simplicial sets to topological spaces. Let Δ be the category with underlying set of objects $\mathbb{Z}_{\geq 0}$ and morphisms $\mathrm{Hom}(i,j)$ the set of order preserving injections $\{0,\ldots,i\} \to \{0,\ldots,j\}$. Show that a simplicial set is the same as a functor $\Delta \to \mathrm{Set}$. Show that $\Delta(S_\bullet)$ is a simplicial complex.
(11) Prove the five lemma and the snake lemma. Do Hatcher section 2.1, problem 14.
(12) If $\sigma\colon \Delta_n \to X$, define $\overline{\sigma}\colon \Delta_n \to X$ by
$$\overline{\sigma}(t_0,\cdots,t_n) := \sigma(t_n,\cdots,t_0).$$
Define $T\colon C_n(X) \to C_n(X)$ by $T(\sigma) := (-1)^{n(n+1)/2}\overline{\sigma}$.
 (a) Show that T is a chain map.
 (b) Show (without constructing it explicitly) that there exists a chain homotopy from T to the identity.
(13) Let X and Y be compact surfaces, and let $f\colon X \to Y$ be a degree d branched cover. Let r denote the sum of the orders of the ramification points in X. Prove the *Riemann-Hurwitz formula*
$$\chi(X) = d\chi(Y) - r$$
(Branched cover means that locally f looks like a power map; i.e., for every $p \in X$, there are identifications of neighborhoods of p and $f(p)$ with $\mathbb{C}$ identifying p and $f(p)$ with 0, such that under these identifications, $f(z) = z^n$ near the origin for some positive integer n. So if $n = 1$ then f is a local homeomorphism at p, while if $n > 1$ then p is called a ramification point of order $n-1$, and $f(p)$ is a branch point. Over the complement of the branch points, f is a covering, and d is its degree. Ordinarily one requires X and Y to be oriented and the above identifications to be orientation-preserving, but that it is not necessary for this problem.)

5. HW 5

Relative homology and Degree.

(1) Hatcher section 2.1, problems 14, 17(b), 18, 27.

Mayer-Vietoris sequence.

(1) Compute the homology of the space obtained by taking three copies of D_n and identifying their boundaries with each other.
(2) Compute the homology of a genus g surface with n disjoint discs removed.
(3) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; this induces a self-homeomorphism ϕ_A of $\mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$. Let Y be the 3-manifold obtained by taking two copies of $S^1 \times D^2$ and identifying the boundary tori via A . Compute the homology of Y , in terms of A.

For example, if A is the identity matrix, then $Y = S^1 \times S^2$; if $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ then $Y \cong S^3$.

6. HW 6

In some of the following prboblems, you are asked to compute homology and cohomology with coefficients in a general commutative ring A. If you aren't comfortable doing that then do separate computations for $A = \mathbb{Z}$, $A = \mathbb{Q}$, and $A = \mathbb{F}_2$. (And if you aren't comfortable doing that then just do the computation for $A = \mathbb{Z}$.)

(1) Hatcher section 3.2 exercises 4, 10, 11.
(2) Hatcher section 3.3 exercises 5, 7. 3.
(3) Let Σ_g denote the compact orientable surface of genus g . Show that if $g < h$, then any map $f\colon \Sigma_g \to \Sigma_h$ has degree zero.
(4) Let A be an $n \times n$ matrix with integer entries. Then A induces a map $\mathbb{R}^n/\mathbb{Z}^n \to \mathbb{R}^n/\mathbb{Z}^n$.
   (a) Show that under the obvious identification $H^1(T^n;\mathbb{Z}) \cong \mathbb{Z}^n$, the pullback
   $$\phi^*\colon H^1(T^n;\mathbb{Z}) \to H^1(T^n;\mathbb{Z})$$
   is equal to the transpose of A.
   (b) Show that the degree of ϕ equals the determinant of A.
(5) Let X be the oriented surface of genus 2 (the 2-holed torus). Compute the homology and cohomology of X with coefficients in a commutative ring A.
(6) Compute the homology and cohomology of $\mathbb{RP}^3$ with coefficients in a commutative ring A.
(7) Compute the homology and cohomology of S^3 with coefficients in a commutative rings A.
(8) Suppose that X is a retract of Y.
   (a) (Hatcher, 2.1, #11) Show that $H_n(X, A) \to H_n(Y, A)$ is injective.
   (b) Show that $H^n(Y, A) \to H^n(X, A)$ is surjective.
(9) Compute the cellular homology of $\mathbb{RP}^n$ and $\mathbb{RP}^n/\mathbb{RP}^m$ with coefficients in $\mathbb{Z}$.
(10) Hatcher, 2.2, #10
(11) Hatcher, 2.2, #11