

HOMEWORK FOR SPRING 2014 ALGEBRAIC TOPOLOGY

Last Modified April 14, 2014

Some notes on homework:

- (1) Homework will be due every two weeks.
- (2) A tentative schedule is: Jan 28, Feb 11, 25, March 11, 25, April 8, 22, May 6 (“In class” take home final). This may be adjusted, and if so this will be noted below.
- (3) Each week every student should LaTeX one solution and submit it (.tex and .pdf) via dropbox, by the beginning of class on the due date. Please label them sensibly; e.g. Lastname-HW-1.4.tex. Email me whatever email address you use for dropbox and I will add you to the shared folder.
- (4) Please hand in different problems. Meet at least once as a group outside of class to coordinate this.
- (5) If you need to draw a picture, it is OK to draw it by hand. But then either scan it or take a picture of it with your phone and submit with your HW.
- (6) If I modify a problem, I will leave a colored footnote with the date and a comment.¹
- (7) There won’t be any official office hours. I will usually be around Tu-Th=F, usually not around on Monday, and sometimes around on Wed.
- (8) There are [hints](#) in the latex comments. (Just change the extension of this URL to .tex)
- (9) I will assign lots of problems. If a problem is not handed in, I might roll it forward to the next assignment.

Some notes on class + lectures:

- (1) **Expectations:** Taken from Tony Varilly’s syllabus: “In my experience as a student, most people do not follow all the details of a Math lecture in real time. During lecture, you should expect to witness the big picture of what’s going on. You should pay attention to the lecturer’s advice on what is important and what isn’t. A lecturer spends a long time thinking on how to deliver a presentation of an immense amount of material; they do not expect you to follow every step, but they do expect you to go home and fill in the gaps in your understanding.”
- (2) Please do all of the following.
 - (a) After each class, review all of the lecture notes.
 - (b) Before the next class, briefly review again. (Make sure that you come to each lecture knowing at least the definitions and statements from the last lecture.)
 - (c) Sit down at least once on your own to attempt the homework.
 - (d) Later, all of the students should meet as a group to work on the homework and to decide who writes what up.

¹December 31: Like this.

1. HW 1, DUE JANUARY 28

Review the techniques from last semester (e.g. universal covering spaces, Van Kampen theorem).

- (1) Prove that π_i is abelian for $i \geq 2$.
- (2) Let G be a topological group. Prove that π_1 is abelian.
- (3) Prove that if $f: X \rightarrow Y$ is a (possibly unbased) homotopy equivalence, then the induced map $f_*: \pi_n(X, x) \rightarrow \pi_n(Y, f(x))$ is an isomorphism.
- (4) Compute π_i of $\mathbb{R}P^n \setminus \{x_0\}$ for $i \leq n$ (where x_0 is any point). What familiar space is $\mathbb{R}P^n \setminus \{x_0\}$ homotopy equivalent to? What is the map $\pi_i(\mathbb{R}P^n \setminus \{x_0\}) \rightarrow \pi_i(\mathbb{R}P^n)$? Compute the action of $\pi_1(\mathbb{R}P^2)$ on $\pi_2(\mathbb{R}P^2)$.²
- (5) Compute π_i of $X = S^1 \vee S^n$ for $1 < i < n$. (\vee means “wedge product”.) Show that the inclusion $S^1 \rightarrow X$ induces an isomorphism on π_1 . Compute the action of π_1 on π_n .
- (6) Do each of the following problems.
 - (a) Compute π_1 of the complement of a circle embedded in \mathbb{R}^3 as the unknot (i.e. as the usual circle in the xy plane).
 - (b) Let $\Sigma_{n,m}$ be a n -holed torus with m points removed. Compute $\pi_k(\Sigma_{1,1})$. (Hint: What familiar space is $\Sigma_{1,1}$ homotopy equivalent to?)
 - (c) Compute π_k of $SL_2(\mathbb{R})$.
 - (d) Compute the fundamental group of the subspace $S \subset M_3(\mathbb{R})$ of 3×3 matrices with rank 1.
- (7) **Application of topology to free groups.**
 - (a) Let X_n be \mathbb{C} with n points removed. Compute $\pi_1(X_n, x_0)$.
 - (b) Compute the universal covering space of X_2 .
 - (c) Prove that F_n (the free group on n generators) is a subgroup of F_2 .
 - (d) Prove that every subgroup of a free group is free.
 - (e) (Extra; do not submit.) Prove that F_n contains a subgroup that is not finitely generated.
 - (f) (Extra; do not submit.) Let $G \subset F_n$ be a subgroup of index j . G is free; how many generators does it have?
- (8) Show that for every finitely generated abelian group G there is a manifold M with fundamental group G and that for every finitely generated group G there is a topological space M with fundamental group G .
- (9) Finish the (sketch of a) proof from class that $\pi_k(S^n) = 0$ for $0 < k < n$.³
- (10) Prove that the quasi-circle (see Hatcher 1.3, exercise 7) is weakly contractible but contractible.⁴
- (11) Let X and Y be path connected and locally path connected.⁵
 - (a) Show that if $\pi_1(X)$ is finite then every map $X \rightarrow S^1$ is null-homotopic.
 - (b) Show that if X is homotopy equivalent to Y then their universal covering spaces are also homotopy equivalent.

²January 20: fixed a typo

³Added on January 20

⁴Added on January 21

⁵Added on January 22

(12) More π_1 problems. ⁶

- (a) Construct a simply-connected covering space of the space $X \subset \mathbb{R}^3$ that is the union of a sphere and a diameter. Do the same when X is a union of a sphere and a circle intersecting it in two points.
- (b) Compute π_1 of the complement of the two planes $x = y = 0$ and $z = w = 0$ in $\mathbb{R}^4 = \mathbb{C}^2$.
- (c) Use Van Kampen's theorem to calculate $\pi_1(S^n)$.
- (d) Use Van Kampen's theorem to calculate π_1 of the Klein bottle.

(13) More π_1 problems. ⁷

- (a) Compute $\pi_1(\Sigma_{2,0})$. Give an example of a degree 2 connected covering space over $\Sigma_{2,0}$. Think about (but don't hand in) $\pi_1(\Sigma_{n,m})$.
- (b) Compute π_1 of the complement of a circle embedded in \mathbb{R}^3 as the trefoil knot. Conclude that the trefoil knot is not equivalent to the unknot.

⁶Added on January 23

⁷Added on January 23

2. HW 2, DUE FEBRUARY 18

- (1) Recall that $[S^1, X]$ is $\text{Maps}(S^1, X)$ mod *unbased* homotopy. Suppose that X is path connected.
 - (a) Prove that $[S^1, X]$ has a group structure.⁸
 - (b) Prove that $[S^1, X]$ is abelian.
(More difficult is to show that $[S^1, X] \cong \pi_1^{\text{ab}}$.)
- (2) Hatcher section 0, #23. Show that this isn't true for non-CW complexes. Hatcher section 1.1, #18.
- (3) Hatcher section 4 #6.
- (4) Hatcher section 4, #8.
- (5) Hatcher section 4, #9.
- (6) Hatcher, section 1.2, #6
- (7) Show that for a (pointed) triad (X, A, B) (i.e., $x_0 \in B \subset A \subset X$), there is a long exact sequence⁹

$$\cdots \rightarrow \pi_n(A, B) \rightarrow \pi_n(X, B) \rightarrow \pi_n(X, A) \rightarrow \pi_{n-1}(A, B) \rightarrow \cdots$$

- (8) Show that $\mathbb{R}P^n$ has a CW-structure.
- (9) Hatcher section 3D, #2.
- (10) Hatcher Appendix A, #3.
- (11) Prove that $\pi_1(\mathbb{R}^2 - \mathbb{Q}^2)$ is uncountable.
- (12) (a) Prove the fundamental theorem of algebra using the Brauer fixed point theorem.
(b) Prove that $\mathbb{R}^n \cong \mathbb{R}^m$ if and only if $n = m$.
- (13) Prove the Borsuk-Ulam theorem.
- (14) Prove the ham sandwich theorem. (Warmup: prove that any region in \mathbb{R}^n can be sliced into two regions of equal area.)

Fun problems, not to be turned in.

- (1) Hatcher, section 1.2, #4. (Compute π_1 of the complement of n lines through the origin in \mathbb{R}^3 .)

⁸March 6: this problem is incorrect.

⁹March 9: fixed a typo

3. HW 3, DUE MARCH 4

Do a problem from the previous two assignments that hasn't been done yet

4. HW 4, DUE APRIL 1

Homology. Read the introduction to Chapter 2.

- (1) Prove that a short exact sequence of chain complexes induces a long exact sequence on homology.
- (2) Compute the simplicial homology of Σ_2 and $\Sigma_{1,1}$.
- (3) Compute the homology of $\mathbb{R}P^3$.
- (4) Compute the homology of S^n .
- (5) Hatcher, section 2.1, exercise 8.
- (6) Hatcher, section 2.1, exercise 11.
- (7) Hatcher, section 2.1, exercise 12. Also show that composition of chain maps induces a well-defined map on equivalence classes.
- (8) For $n \in \mathbb{Z}_{\geq 1}$, compute the simplicial homology of the space obtained by taking three copies of D_n and identifying their boundaries with each other. (You choose the Δ -structure.)
- (9) Prove carefully that if $0 \rightarrow \mathbb{Z}^{n_1} \rightarrow \mathbb{Z}^{n_2} \rightarrow \dots \rightarrow \mathbb{Z}^{n_k} \rightarrow 0$ is exact, then $\sum_{i=1}^k (-1)^i n_i = 0$. As a corollary, prove that if Γ is a graph on Σ_g , then the Euler characteristic $\chi(\Gamma) := \text{number of faces} - \text{edges} + \text{vertices}$ is $2 - 2g$.
- (10) Show that the data of a simplicial set is the same as the data of a Δ complex. Show that there is a functor from simplicial sets to topological spaces. Let Δ be the category with underlying set of objects $\mathbb{Z}_{\geq 0}$ and morphisms $\text{Hom}(i, j)$ the set of order preserving injections $\{0, \dots, i\} \rightarrow \{0, \dots, j\}$. Show that a simplicial set is the same as a functor $\Delta \rightarrow \text{Set}$. Show that $\Delta(S_\bullet)$ is a simplicial complex.
- (11) Prove the five lemma and the snake lemma. Do Hatcher section 2.1, problem 14.
- (12) If $\sigma: \Delta_n \rightarrow X$, define $\bar{\sigma}: \Delta_n \rightarrow X$ by

$$\bar{\sigma}(t_0, \dots, t_n) := \sigma(t_n, \dots, t_0).$$

Define $T: C_n(X) \rightarrow C_n(X)$ by $T(\sigma) := (-1)^{n(n+1)/2} \bar{\sigma}$.

- (a) Show that T is a chain map.
 - (b) Show (without constructing it explicitly) that there exists a chain homotopy from T to the identity.
- (13) Let X and Y be compact surfaces, and let $f: X \rightarrow Y$ be a degree d branched cover. Let r denote the sum of the orders of the ramification points in X . Prove the *Riemann-Hurwitz formula*

$$\chi(X) = d\chi(Y) - r$$

(Branched cover means that locally f looks like a power map; i.e., for every $p \in X$, there are identifications of neighborhoods of p and $f(p)$ with \mathbb{C} identifying p and $f(p)$ with 0, such that under these identifications, $f(z) = z^n$ near the origin for some positive integer n . So if $n = 1$ then f is a local homeomorphism at p , while if $n > 1$ then p is called a ramification point of order $n - 1$, and $f(p)$ is a branch point. Over the complement of the branch points, f is a covering, and d is its degree. Ordinarily one requires X and Y to be oriented and the above identifications to be orientation-preserving, but that it is not necessary for this problem.)

5. HW 5, DUE APRIL 8

Relative homology and Degree.

- (1) Hatcher section 2.1, problems 14, 17(b), 18, 27.

Mayer-Vietoris sequence.

- (1) Compute the homology of the space obtained by taking three copies of D_n and identifying their boundaries with each other.
- (2) Compute the homology of a genus g surface with n disjoint discs removed.

- (3) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; this induces a self-homeomorphism ϕ_A of $\mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$. Let Y be the 3-manifold obtained by taking two copies of $S^1 \times D^2$ and identifying the boundary tori via A . Compute the homology of Y , in terms of A .

For example, if A is the identity matrix, then $Y = S^1 \times S^2$; if $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ then $Y \cong S^3$.

6. HW 6, DUE MAY 12

In some of the following problems, you are asked to compute homology and cohomology with coefficients in a general commutative ring A . If you aren't comfortable doing that then do separate computations for $A = \mathbb{Z}$, $A = \mathbb{Q}$, and $A = \mathbb{F}_2$. (And if you aren't comfortable doing that then just do the computation for $A = \mathbb{Z}$.)

- (1) Hatcher section 3.2 exercises 4, 10, 11.
- (2) Hatcher section 3.3 exercises 5, 7, 3.
- (3) Let Σ_g denote the compact orientable surface of genus g . Show that if $g < h$, then any map $f: \Sigma_g \rightarrow \Sigma_h$ has degree zero.
- (4) Let A be an $n \times n$ matrix with integer entries. Then A induces a map $\mathbb{R}^n/\mathbb{Z}^n \rightarrow \mathbb{R}^n/\mathbb{Z}^n$.
 - (a) Show that under the obvious identification $H^1(T^n; \mathbb{Z}) \cong \mathbb{Z}^n$, the pullback
$$\phi^*: H^1(T^n; \mathbb{Z}) \rightarrow H^1(T^n; \mathbb{Z})$$
is equal to the transpose of A .
 - (b) Show that the degree of ϕ equals the determinant of A .
- (5) Let X be the oriented surface of genus 2 (the 2-holed torus). Compute the homology and cohomology of X with coefficients in a commutative ring A .
- (6) Compute the homology and cohomology of $\mathbb{R}P^3$ with coefficients in a commutative ring A .
- (7) Compute the homology and cohomology of S^3 with coefficients in a commutative rings A .
- (8) Suppose that X is a retract of Y .
 - (a) (Hatcher, 2.1, #11) Show that $H_n(X, A) \rightarrow H_n(Y, A)$ is injective.
 - (b) Show that $H^n(Y, A) \rightarrow H^n(X, A)$ is surjective.
- (9) Compute the cellular homology of $\mathbb{R}P^n$ and $\mathbb{R}P^n/\mathbb{R}P^m$ with coefficients in \mathbb{Z} .
- (10) Hatcher, 2.2, #10
- (11) Hatcher, 2.2, #11