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Definition. Let a and b be integers. We say that a divides b if there exists an integer k such that $b = ak$. In this case, we write $a \mid b$.

Prove that $3 \mid 4^n - 1$ for all non-negative integers n.

Definition. We say that an integer is prime if $n \neq \pm 1$ and $a \mid n$ implies that $a = \pm 1$ or $\pm n$.

Examples: 7 is prime. To prove this, just try every integer between 1 and 7, and check whether they divide 7. (This is called "brute force" or proof by computation.)

Example: 6 is not prime. To prove this, give an example of a divisor a which is not ± 1 or ± 6; $a = 2$ works.

Why isn't 1 prime? Because we want factorizations to be unique. I.e., $6 = 2 * 3 = 3 * 2$. If we allow 1 to be prime, then $6 = 1 * 2 * 3 = 1 * 1 * 2 * 3 = \ldots$

Problem: for which integers n is $n^3 - 1$ prime? If $n = 2$, $n^3 - 1 = 8 - 1 = 7$, which is prime. If $n = 3$, then $3^3 - 1 = 27 - 1 = 26$, which is not prime.

Note: if n is odd, then $n^3 - 1$ is even, so it is not prime (unless it is equal to 2, because 2 is the only even prime).