If you have **questions** you are welcome to unmute and interrupt, or ask in chat (publicly or privately is fine!).

If there are any **technical difficulties** (for example, I am writing offscreen) please let me know right away!

My **website** is http://www.math.emory.edu/~dzb/

The **course website** is https://www.math.emory.edu/~dzb/teaching/250Spring2022/

The **syllabus** is available here https://www.math.emory.edu/~dzb/teaching/250Spring2022/syllabus-math-250-spring-2022.pdf

This **Miro** board is available at https://miro.com/app/board/uXjVOXdsj88=/

The **Overleaf** board is available at https://www.overleaf.com/read/qmgpmvmbyzpw

Office hours are Mondays, 4:30-5:30 via Zoom. Link is on the syllabus. If you cannot make it to these office hours, please email me to set up an alternative time. (Preferrably email me 24 hours in advance, and please suggest a big list of times that you are available.)

"root of 3" rule

if "root of 3" of

$b, c, b \pm c$

are divisible by a, then so is the 3rd!

FE

- $a \mid b \wedge a \mid c \Rightarrow a \mid b+c$
- $a \mid b \wedge a \mid b+c \Rightarrow a \mid c$

$\vdots$

Prove: $3 \mid 4^n - 1$ for all integers $n \geq 0$.

Experiment:

$3 \mid 0$

$n = 0$ $\quad 4^n - 1 = 4^0 - 1 = 1 - 1 = 0$ ✓

1 $\quad 4 - 1 = 3$ ✓

2 $\quad 4^2 - 1^2 = 4^2 - 1 = 15 = 3 \cdot 5$

$(4 - 1)(4 + 1)$

3 $\quad 4^3 - 1 = 64 - 1 = 63$

$\vdots$

$$4^n - 1 = 4^n - 1^n$$

$$= (4-1)(4^{n-1} + 4^{n-2} + \dots + 1)$$

$$= 3 \cdot (\text{AN INTEGER})$$

FG

$$1 + a + a^2 + \dots + a^{n-1} = \frac{a^n - 1}{a - 1}$$

Algebra identity:

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \dots + y^{n-1})$$

Proof: "Just multiply", i.e., compute the RHS (Right hand side)

$$(x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \dots + y^{n-1})$$

$$= x^n + \cancel{x^{n-1}y} - \cancel{yx^{n-1}} + \cancel{y^2x^{n-2}} - \cancel{yx^{n-2}y} + \dots$$

$$- y^n. \quad \boxed{\text{QED}}$$

"Proof by computation".

Setting $x=4$, $y=1$

$$4^n - 1 = (4-1)\left(\overbrace{\qquad\qquad}^{N}\right)$$

$$= 3 \cdot N$$

Thus $3 \mid 4^n - 1$.

Also, $6 \mid 7^n - 1$ b/c $7-1=6$

If n is an odd positive integer, then $5 \mid 4^n + 1$.

$n = 1 \quad 4^n + 1 = 4+1 = 5$

$= 3 \quad 4^3 + 1 = 64+1 = 65$

$= 2 \quad 4^2 + 1 = 17, \ 5 \nmid 17$

$$\underline{\text{Fact:}}\ (-1)^n = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd.} \end{cases}$$

$$4^n + 1 = 4^n - (-1)^n \quad \text{since } n \text{ is odd}$$

$$\begin{aligned} &x = 4 \\ &y = -1 \end{aligned}$$

$$4^n - (-1) \quad \left(4 - (-1)\right)\left(4^{n-1} - 4^{n-2} + \cdots\right)$$

$$= 5 \cdot (\text{AN Integer}).$$

For all positive integers n,

$$5 \mid 4^n + 1.$$

This is false.

To give a "disproof" of a "for all" statement, give a "counterexample"

i.e. an example demonstrating that it is false.

When $n=2$, $4^n+1=17$,

which is not divisible by 5. $\boxed{QED}$

For all positive integers n,
n^2+1 is never divisible by 4.

Proof: If n is even, then n^2+1 is odd, and thus not divisible by 4.

If n is odd, then $n=2k+1$ for some integer k. Then $(2k+1)^2+1 = 4k^2+4k+1+1$
$= 4(k^2+k)+2.$

By the division algorithm, the remainder when dividing by 4 is 2. Thus n^2+1 is <u>not</u> divisible by 4. $\blacksquare$

(3) Suppose that n is an integer such that $5 \mid (n+2)$. Which of the following are divisible by 5?

(a) $n^2 - 4$
(b) $n^2 + 8n + 7$
(c) $n^4 - 1$
(d) $n^2 - 2n$

a) Note that

$$n^2 - 4 = n^2 - 2^2 = (n-2)(n+2).$$

By our hypothesis, $5 \mid n+2$.

By transitivity, $5 \mid (n-2)(n+2)$ $\square$

b) Note that $n^2 + 8n + 7 = (n+7)(n+1)$.

Since $n+7 = n+2+5$, and since $5 \mid n+2$ and $5 \mid 5$, by additivity, $5 \mid n+7$. Thus by transitivity, $5 \mid (n+7)(n+1)$. $\square$

c) $n^4 - 1 = n^4 - 1^4 = (n^2)^2 - (1^2)^2$

$= (n^2-1)(n^2+1) = (n-1)(n+1)(n^2+1)$

$n=3$, $5 \mid n+2$ ✓

does $5 \mid n^4 - 1$? $\quad 3^4 - 1 = 81 - 1 = 80$

$5 \mid 80$ ✓

$n^2+1 = n^2 - 4 + 5$

and by part (a),

$5 \mid n-2 \Rightarrow 5 \mid n^2 - 4$

$$n^2 - 2n = n(n-2)$$

False: a counterexample is

$n = 3$, since $5 \mid n+2$,

but $5 \nmid 3(3-2)$.

Note: Hyp is true, Conc. is F.

$$n^3 - 1 = (n-1)(n^2+n+1)$$

Suppose $n^3 - 1$ is prime.

Since $n-1 \mid n^3-1$, $n-1 = \pm 1$ or

$n-1 = \pm(n^3-1)$.

(Switch to a proof by cases)

Case 1: $n-1 = 1$.

In this case, $n = 2$.

Then $2^3 - 1 = 7$, which is prime

Case 2: $n-1 = -1$. In this case

$n = 0$. Then $0^3 - 1 = -1$,

which is <u>not</u> prime.

Case 3: $n \cancel{+1} = n^3 \cancel{+1}$.

$$\Rightarrow n^3 - n = 0$$

$$\Rightarrow n(n^2-1) = n(n-1)(n+1) = 0$$

$$\Rightarrow n = 0 \text{ or } \pm 1$$

$$\Rightarrow n^3 - 1 = -1, 0, -2.$$

Only -2 is prime.

Case 4: $n-1 = -(n^3-1)$

$$\Rightarrow n^3+n-2=0$$

$$\Rightarrow (n-1)(n+2)=0 \Rightarrow$$

$$n = 1 \text{ or } -2$$

$$\Rightarrow n^3-1 = 0 \text{ or } -9$$

neither are prime.

We conclude that if

$n^3 - 1$ is prime, then

$n^3 - 1 = 7$ or -2,

and $n = 2$ or -1 ∎

$$n^3+1 = n^3 - (-1)^3$$

$$= (n - (-1))(n^2 - n + 1)$$

$$\Rightarrow n+1 \mid n^3+1$$

if n^3+1 is prime, then

$n+1 = \pm 1$ or n^3+1 or $-(n^3+1)$.

When is a^n+1 prime?

Want a even (else $2 \mid a^n+1$)

$2^n+1 = 2^n-(-1) = 2^n-(-1)^n$

if n is odd $= (2-1)(- - - -)$

If $n = ab$ where a is odd
and $a, b \neq 1$

then

$$2^n + 1 = 2^{ab} + 1$$

$$= \left(2^b\right)^a - (-1)^a$$

$$= \left(2^b - 1\right)(\text{integer})$$

If 2^n+1 is prime,

then $n = 2^k$ for some k

$F_k = 2^{2^k} + 1$ Fermat #s

$k = 0$ $2^{2^0} + 1 = 2^1 + 1 = 3$

$2^{2^1} + 1 = 2^2 + 1 = 5$

$2^{2^2} + 1 = 2^4 + 1 = 16 + 1 = 17$

$$2^{2^3}+1 = 2^8+1 = 256+1 = 257$$

$$2^{2^4}+1 = 65537$$