MATH 250, Foundations of Mathematics
Section 003 TuTh 2:30 - 3:45

All assignments
Last updated: May 2, 2022

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Final Exam 15

Assignment 1

Topics: Introduction to the course. Mathematical reasoning.

Reading: Chapter 1, except for proof by contradiction.

Suggested problems (do not hand in)

- With answers:
  - Section 1.1, #1(adgj), 2(adji), 3(adgi), 5(ad), 6(a)
  - Section 1.2, #2(ac), 4(ac), 5(ad), 7(a), 10(a), 11(a), 12(a)
  - Section 1.3, #1(ad), 3(a), 5(ac), 7(ac)
  - Section 1.4, #1, 4(a), 6(a), 8, 12(ab), 15(a)
- Without answers: Handout 1

Assignment, due Tuesday, January 18, via Canvas:

1. Suppose that n is an even integer, and let m be any integer. Prove that nm is even.
2. Suppose that n is an odd integer. Prove that n^2 is an odd integer. (Hint: an integer n is odd if and only if there exists an integer k such that $n = 2k + 1$.)
3. Prove that if n^2 is even, then n is even. (Hint: see Section 1.4)
4. Write the negation of each of the following statements.
   (a) All triangles are isosceles.
   (b) Every door in the building was locked.
   (c) Some even numbers are multiples of three.
   (d) Every real number is less than 100.
   (e) Every integer is positive or negative.
   (f) If f is a polynomial function, then f is continuous at 0.
   (g) If $x^2 > 0$, then $x > 0$.
   (h) There exists a $y \in \mathbf{R}$ such that $xy = 1$.
   (i) $(2 > 1)$ and $(\forall x, x^2 > 0)$
   (j) $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $|x| < \delta$, then $|f(x)| < \epsilon$.

Assignment 2

Topics: “Basic” proofs and divisibility problems.

Reading:

- Finish reading chapter 1.
- Section 5.3

Suggested problems (do not hand in)

1. With answers: Section 5.3, #1(a), 4(a), 6(ac)
2. Without answers: Section 5.3, #2, 4 (without induction), 5 (without induction)
3. Handout 2

Assignment, due Tuesday, January 25, via Canvas:

1. Prove that if x is an integer, then $x^2 + 2$ is not divisible by 4. (Hint: there are two cases: x is even, x is odd. Also, feel free to use basic facts about even or odd, e.g., “odd + odd = even”, without additional proof.)
2. Prove that the product of three consecutive integers is divisible by 6. (It suffices to prove that it is divisible by 2 and 3 separately.)
3. Show that for all integers a and b,
$$a^2b^2(a^2 - b^2)$$
is divisible by 12. (It suffices to prove that it is divisible by 4 and 3 separately.)
4. Find all positive integers n such that $n^2 - 1$ is prime. Prove that your answer is correct.

Assignment 3

Topics: Proof by contradiction. Unsolvability of equations. Irrationality.

Reading:

- Section 1.4, p. 41-42 (stop at Historical Comments)
- Section 5.4

Suggested problems (do not hand in)

1. Without answers: Section 1.4 #21
2. Without answers: Section 5.4 #6, 7, 10(a), 15, 18,
3. Handout 3

Assignment, due Tuesday, February 8, via Canvas:

1. Prove that $2^{1/3}$ is irrational.
2. Prove that there are no positive integer solutions to the equation $x^2 - y^2 = 10$.
3. Let a, b, c be integers satisfying $a^2 + b^2 = c^2$. Show that abc must be even. (Harder problem, just for fun: show that a or b must be even.)
4. Suppose that a and n are integers that are both at least 2. Prove that if $a^n - 1$ is prime, then $a = 2$ and n is a prime. (Primes of the form $2^n - 1$ are called Mersenne primes.)

Assignment 4

Topics: Induction.

Reading: Section 5.2, p. 159-163

Fun Video: Vi Hart; "Doodling in Math: Spirals, Fibonacci, and Being a Plant"
https://www.youtube.com/watch?v=ahXIMUkSXX0

Suggested problems (do not hand in)

1. With answers: Section 5.2 #1(a), 4(a), 8(ad), 9(a), 29
2. Without answers: Section 5.2 #2-9, 13
3. Handout 4
4. Handout 5

Assignment, due Tuesday, February 15, via Canvas:

1. Prove that for every positive integer n,

$$1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}.$$

2. Let a_n be defined recursively by $a_1 = 1$ and $a_n = \sqrt{1 + a_{n-1}}$. Prove that for all positive integers n, $a_n < 2$.
3. Prove by induction that if $b_1, b_2, \ldots, b_n$ are even integers, then $b_1 + b_2 + \cdots + b_n$ is even.
4. Let $F_1, F_2, F_3, \ldots = 1, 1, 2, 3, 5, 8, \ldots$ be the Fibonacci sequence. Prove that $F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$.

Assignment 5

Topics: Basics of set theory. Basic operations. Proofs with sets.

Reading:

1. Section 2.1, p. 49-57;
2. Section 2.2, p. 61-65 (stop at DeMorgan's laws)

Suggested problems (do not hand in)

1. With answers (many of these are calculations; do as many as you need to do to understand the definitions):
   (a) Section 2.1, #1(adg), 2(adg), 4(adg), 5(a), 7(a), 8(ae), 9(adf), 10(a), 18(acf), 19(ad), 20(ae), 21
   (b) Section 2.2, #1(adgj), 2(ad), 4(ad), 5(ad), 7(a), 9(ad), 14(a),
2. Without answers:
   (a) Section 2.1, 13, 14, 15, 16,
   (b) Section 2.2, #1-12
3. Handout 6

Assignment, due Tuesday, February 22, via Canvas:

1. Let $A = \{n \in \mathbb{Z} | n \text{ is a multiple of } 4\}$ and $B = \{n \in \mathbb{Z} | n^2 \text{ is a multiple of } 4\}$
   (a) Prove or disprove: $A \subset B$.
   (b) Prove or disprove: $B \subset A$.
2. Prove that $A \cup (A \cap B) = A$.
3. Let A, B and C be sets.
   (a) Prove that $(A \subset C) \wedge (B \subset C) \Rightarrow A \cup B \subset C$.
   (b) State the contrapositive of part (a).
   (c) State the converse of part (a). Prove or disprove it.
4. Let n and m be integers. Prove that if $n\mathbb{Z} \subset m\mathbb{Z}$ then m divides n.

Assignment 6

Topics: More proofs with sets. DeMorgan's laws. Cartesian Products. Power sets

Reading:

1. Section 2.2, p. 65-66;
2. Section 2.3, p. 72, just the part about power sets.

Suggested problems (do not hand in)

1. With answers:
   (a) Section 2.2, 13(a), 16(a)
   (b) Section 2.3, #1(a), 3, 5(adg),
2. Without answers:
   (a) Section 2.2, 14, 16-19, 21, 23-27
   (b) Section 2.3, #1(b), 2,4
3. Handout 7

Assignment, due Tuesday, March 1, via Canvas:

1. Let A and B be sets. Prove that $(A \cup B) \cap \overline{A} = B - A$.
2. Let A and B be sets. Prove that $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$.
3. Let $A = \{0, 1, 2\}$. Which of the following statements are true? (No justification is needed.)
   (a) $\{0\} \subset P(A)$;
   (b) $\{1, 2\} \in P(A)$;
   (c) $\{1, \{1\}\} \subset P(A)$.
   (d) $\{\{0, 1\}, \{1\}\} \subset P(A)$;
   (e) $\emptyset \in P(A)$;
   (f) $\emptyset \subset P(A)$;
   (g) $\{\emptyset\} \in P(A)$.
   (h) $\{\emptyset\} \subset P(A)$;
4. Let A and B be sets. Prove that if $A \subset B$, then $P(A) \subset P(B)$. State the converse of this and prove or disprove it.

Midterm

Topics: Tuesday, March 1 will be Exam review. We will not cover any new material; in class, I will answer whatever questions you have. The exam is on Thursday, March 3. **Please show up with questions**.

Content: The questions will all be either

1. homework problems,
2. suggested problems,
3. problems we worked in class, or
4. minor variations of one of these.

Assignment 7

NO CLASS March 8, 10; spring break

Topics: Introduction to functions; images and surjectivity

Reading:

1. Section 3.1, p. 81-90 (stop at "Inverse Image");
2. Section 3.2, p. 97-100 (stop at Injective Functions).

Suggested problems (do not hand in)

1. With answers:
   (a) Section 3.1, #1(adg), 4(ace), 5(a), 8(a), 10(a), 12(1d)
   (b) Section 3.2, #1(adgj), 2(ad)
2. Without answers:
   (a) Section 3.1, 1-4,6-13
   (b) Section 3.2, 1-6
   (c) Handout 9

Assignment, due Tuesday, March 22, via Canvas:

1. Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = 6x + 5$.
   (a) Prove that $f(\mathbf{R}) = \mathbf{R}$.
   (b) Compute $f([1, 4])$. Prove your answer.
2. Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function defined by $x^4 + x^2$.
   (a) Compute the image of f. Prove that your answer is correct.
   (b) Compute $f([-1, 2])$. Prove that your answer is correct.
3. Let A and B be sets and let X and Y be subsets of A. Let $f\colon A \to B$ be a function. Prove or disprove each of the following. When giving a disproof, please give an counterexample.
   (a) $f(X \cap Y) \subset f(X) \cap f(Y)$.
   (b) $f(X \cap Y) \supset f(X) \cap f(Y)$.
   (c) $f(X) - f(Y) \subset f(X - Y)$.
   (d) $f(X) - f(Y) \supset f(X - Y)$.

Assignment 8

Topics: Inverse Image (or "Preimage").

Reading: Section 3.1, p. 90-92 (stop at the Historical Comments. Or don't.)

Suggested problems (do not hand in)

1. With answers: Section 3.1, #17(ad), #18(adg), #19(a), #21(a)
2. Without answers: 17-21
3. Handout 10

Assignment, due Tuesday, March 29, via Canvas:

1. Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = 3x + 1$.
   (a) Compute $f^{-1}(\{1, 5, 8\})$ (do not give a proof).
   (b) Compute $f^{-1}(W)$, where $W = (4, \infty)$, and give a proof that your answer is correct.
   (c) Compute $f^{-1}(\mathbf{E})$, where $\mathbf{E}$ is the set of even integers, and give a proof that your answer is correct.

2. Let $f\colon \mathbf{Z} \to \mathbf{Z}$ be the function defined by $f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ 2n + 4, & \text{if } n \text{ is odd.} \end{cases}$

   Compute $f^{-1}(\mathbf{E})$. Prove that your answer is correct. (Reminder: $\mathbf{E}$ is the set of even integers.)

3. Let A and B be sets and let X be a subset of B. Let $f\colon A \to B$ be a function. Prove or disprove the following. (For a disproof, please give an explicit counterexample.)
   (a) $X \subset f(f^{-1}(X))$.
   (b) $X \supset f(f^{-1}(X))$.

4. Let A and B be sets. Let $S \subset A$ and let $T \subset B$. Let $f\colon A \to B$ be a function. Prove or disprove the following. (For a disproof, please give an explicit counterexample.)
   (a) $f(S) \subset T \Rightarrow S \subset f^{-1}(T)$.
   (b) $S \subset f^{-1}(T) \Rightarrow f(S) \subset T$.

Assignment 9

Topics: Injectivity.

Reading: Section 3.2, p. 100-105

Suggested problems (do not hand in)

1. With answers: 3.2, #12(adg), #13(bd)
2. Without answers: 3.2 #9-14, 19(abc)
3. Handout 11

Assignment, due Tuesday, April 5, via Canvas:

1. Which of the following functions $f\colon \mathbf{R} \to \mathbf{R}$ are injective? If the function is injective, give a proof. If it is not injective, give a counterexample.
   (a) $f(x) = x^4 + x^2$;
   (b) $f(x) = x^3 + x^2$;
   (c) $f(x) = \begin{cases} -x-1, & \text{if } x > 0 \\ x^2, & \text{if } x \leq 0. \end{cases}$
2. Let A and B be sets and let X and Y be subsets of A. Let $f\colon A \to B$ be an injective function. Prove that $f(X \cap Y) = f(X) \cap f(Y)$.
3. Let $f\colon A \to B$ be a function. Which of the followings statements are equivalent to the statement 'f is injective'? (No proof necessary.)
   (a) $f(a) = f(b)$ if $a = b$;
   (b) $f(a) = f(b)$ and $a = b$ for all $a, b \in A$;
   (c) If a and b are in A and $f(a) = f(b)$, then $a = b$;
   (d) If a and b are in A and $a = b$, then $f(a) = f(b)$;
   (e) If a and b are in A and $f(a) \neq f(b)$, then $a \neq b$;
   (f) If a and b are in A and $a \neq b$, then $f(a) \neq f(b)$.
4. We define a function $f\colon [a, b] \to \mathbf{R}$ to be **decreasing** if for all $x_1, x_2 \in [a, b]$, if $x_1 < x_2$, then $f(x_1) > f(x_2)$.
   (a) Negate the definition of decreasing.
   (b) Prove that a decreasing function is injective.

Assignment 10

Topics: Composition of functions.

Reading: Section 3.3, p. 110-113

Suggested problems (do not hand in)

1. With answers: 3.3, #1(a), 2(a), 3(ad), 7(a)
2. Without answers: 3.3 #1-7, 9
3. Handout 12

Assignment, due Tuesday, April 12, via Canvas:

1. Let A, B and C be sets and let $f\colon A \to B$ and $g\colon B \to C$ be functions. Prove or disprove each of the following.
   (a) If $g \circ f$ is an injection, then g is an injection.
   (b) If $g \circ f$ is a surjection, then f is a surjection.
   (c) If $g \circ f$ is a surjection, then g is a surjection.
2. Let A and B be sets and let $f\colon A \to B$ and $g\colon B \to A$ be functions. Prove that if $g \circ f$ and $f \circ g$ are bijective, then so are f and g.
3. Let $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ be functions. Suppose that f and g are both decreasing. Prove that $g \circ f$ is increasing.

Assignment 11

Topics: Inverse functions.

Reading: Section 3.3, p. 114-116

Suggested problems (do not hand in)

1. With answers: 3.3 #10(adgj), 11(a)
2. Without answers: 3.3 #10, 12, 14, 15, 17, 18, 19, 22
3. Handout 13

Assignment, due Tuesday, April 19, via Canvas:

1. Define $f\colon \mathbf{R}-\{1\} \to \mathbf{R}-\{1\}$ by $f(x) = \frac{x+1}{x-1}$. Prove that f is a bijection. Find a formula for the inverse $f^{-1}(x)$, and prove that it is correct.
2. Let A, B and C be sets and let $f\colon A \to B$ and $g\colon B \to C$ be functions. Prove that if f and g are invertible, then so is $g \circ f$, and prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.
3. Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function $f(x) = x^3 + x$. Prove that f is invertible without finding a formula for f^{-1}.
4. Let A and B be sets and let $f\colon A \to B$ be a function. Suppose that f has a *left inverse* g; that is, suppose that there exists a function $g\colon B \to A$ such that $g \circ f = id_A$. Prove that f is injective.

Assignment 12

Topics: Relations.

Reading: Section 4.2, p. 139-144 (stop at the proof of Theorem 4.2.6)

Suggested problems (do not hand in)

1. With answers: Section 4.2 #1(a), 3(ad), 4(a), 5(a), 12(a)
2. Without answers: Section 4.2 #1, 3, 4
3. Handout 14

Assignment, due Monday, April 25, 11:59PM, via canvas:

1. Let $A = \{1, 2, 3\}$ and define a relation on A by $a \sim b$ if $a + b \neq 3$. Determine whether this relation is reflexive, symmetric, transitive, or antisymmetric.
2. Define a relation on $\mathbf{Z}$ given by $a \sim b$ if $a - b$ is divisible by 4.
   (a) Prove that this is an equivalence relation.
   (b) What integers are in the equivalence class of 18? (No proof necessary.)
   (c) What integers are in the equivalence class of 31? (No proof necessary.)
   (d) How many distinct equivalence classes are there? What are they? (No proof necessary.)
3. Define a relation on $\mathbf{Z}$ given by $a \sim b$ if $a^2 - b^2$ is divisible by 4.
   (a) Prove that this is an equivalence relation.
   (b) How many distinct equivalence classes are there? What are they? (No proof necessary.)
4. Let A be a set, and let $P(A)$ be the power set of A. Assume that A is not the empty set. Define a relation on $P(A)$ by $X \sim Y$ if $X \subset Y$. Is this relation reflexive, symmetric, and/or transitive? In each case give a proof, or disprove with a counterexample. (For a counterexample, give an example of A, X, and Y that disproves the statement.)

Please note the unusual due date of the homework assignment.

Final Exam

Final exam is Tuesday May 3, 3:00pm - 5:30pm

The **last day of class** is April 21.

There will be **office hours** on Monday, May 2, 10:30-11:30 and 6-7, via Zoom.