MATH 250 HANDOUT 4 - UNIQUE FACTORIZATION

(1) Prove that $\log_{10} 7$ is irrational.
(2) Let $b \in \mathbb{Z}_{\geq 1}$. Prove that $\log_b 3 / \log_b 2$ is irrational.
(3) Show that if p is a prime and $p \mid a^n$, then $p^n \mid a^n$.
(4) Prove that $\sqrt{3}$ is irrational.
(5) Prove that $\sqrt[5]{5}$ is irrational.
(6) Prove that the equation
$$(x^2 - y^2)(x^2 - 4y^2) = 7$$
has no solutions with $x, y \in \mathbb{Z}$.
(7) Show that for every integer n, $n^3 + 2n$ and $n^4 + 3n^2 + 1$ have no common prime factors.
(8) Prove that there are infinitely many primes of the form $6n + 3$.
(9) Prove that there are infinitely many primes of the form $6n + 5$.
(10) Prove that there is only one positive integer n such that
$$2^8 + 2^{11} + 2^n = 48^2 + 2^n$$
is a perfect square.
(11) Prove that if n ≥ 2, then $\sqrt[n]{n}$ is irrational. Hint: use that if $n > 2$, then $2^n > n$.

Examples and counterexamples

Many of the following problems are stated as 'prove or disprove'. For such problems, work a few examples to see if you think the statement is true. Then either try to give a proof, or give a counterexample.

(1) Give an example of four positive integers such that any three of them have a common divisor greater than 1, although only ± 1 divide all four of them.
(2) Find the smallest positive integer such that $n/2$ is a square and $n/3$ is a cube.
(3) Let n be a positive integer. Prove or disprove: $n, n+2$ and $n+4$ cannot all be prime.
(4) Suppose that $a \mid b^n$. Prove or disprove: $a \mid b$.
(5) Suppose that $a \mid bc$. Prove or disprove: $a \mid bc$.
(6) Prove or disprove: let $a, b \in \mathbb{Z}_{\geq 1}$. Then $\log_a b$ is irrational.
(7) Prove or disprove: let $p_1, \ldots, p_n$ be prime numbers. Then $p_1 \cdots p_n + 1$ is prime.
(8) Prove or disprove: there are no integer solutions to the equation

$$x^2 - y^2 = 2^3$$