

## PROBLEMS ABOUT COUNTABLE SETS

This is a collection of problems from class and from the homework that I would like students to be able to do. For each of these sets, either prove directly that it is countable (i.e., write down a bijection from  $\mathbb{Z}_{>0}$ ), or use the injectivity, surjectivity, or union theorem. I may or may not ask you to prove all of your claims (i.e., I may ask you to write down a bijection, but not to prove that the map you wrote down is a bijection), but in general still expect you to justify every detail unless I request otherwise.

On the exam, I will make it clear what you are allowed to assume (i.e., “Prove that  $\mathbb{Q}$  is countable. You may assume that  $\mathbb{Z} \times \mathbb{Z}$  is countable.”). For this worksheet though, to do any given problem, you may use the result of any previous problem.

(1) Prove that each of the following is countable.

- (a)  $\mathbb{Z}$
- (b)  $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$
- (c)  $\mathbb{Q}$
- (d)  $\mathbb{Z} \times \mathbb{Z}$
- (e)  $\{e^n \mid n \in \mathbb{Z}\}$
- (f)  $\mathbb{Q}^+ \cup \{e^n \mid n \in \mathbb{Z}\}$
- (g)  $\mathbb{Q} \times \mathbb{Q}$
- (h)  $\mathbb{Q}^n$
- (i)  $\mathbb{Q}_d =$  set of polynomials of rational coefficients of degree at most  $d$ .
- (j)  $\mathbb{Q}$
- (k)  $P_{\text{BD}}(\mathbb{Z})$
- (l)  $\text{Fun}_{\text{BD}}(\mathbb{Z})$
- (m)  $\text{Seq}_{\text{BD}}(\mathbb{Z}_{>0})$
- (n)  $\overline{\mathbb{Q}}$

In addition, make sure that you can do the homework problems, especially 6 and 11 of 6.1.

(2) **Hints:**

- (a)  $\mathbb{Z}$ ; Hint: write down an explicit bijection
- (b)  $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ ; Hint: injectivity theorem.
- (c)  $\mathbb{Q}$ ; Hint: Union theorem or surjection theorem.
- (d)  $\mathbb{Z} \times \mathbb{Z}$ ; Hint: union theorem or surjection theorem.
- (e)  $\{e^n | n \in \mathbb{Z}\}$ ; Hint: write down an explicit bijection with  $\mathbb{Z}$
- (f)  $\mathbb{Q}^+ \cup \{e^n | n \in \mathbb{Z}\}$ ; Hint: union theorem + injection theorem
- (g)  $\mathbb{Q} \times \mathbb{Q}$ ; Hint: union theorem
- (h)  $\mathbb{Q}^n$ ; Hint: Union theorem
- (i)  $\mathbb{Q}_d$ ; Hint: find a bijection with a set from a previous problem.
- (j)  $\mathbb{Q}$ ; Hint: Union theorem
- (k)  $P_{\text{BD}}(\mathbb{Z})$ ; Hint: Union theorem and surjection theorem. This one is just like  $\mathbb{Q}$ .
- (l)  $\text{Fun}_{\text{BD}}(\mathbb{Z})$ ; Hint: Union theorem
- (m)  $\text{Seq}_{\text{BD}}(\mathbb{Z}_{>0})$ ; Hint: Union or surjection theorem
- (n)  $\overline{\mathbb{Q}}$ ; Hint: union theorem