DAVID ZUREICK-BROWN, INSTRUCTOR

Math 250 ~ Foundations of Math
Emory University
Dept. of Math and CS
Fall 2012

Midterm I

Name: Key

Instructions:

- No calculators or extraneous scratch paper please!
- If you need more scratch paper, please raise your hand.
- Please make sure that your answers are in complete sentences!
- *Good luck!*

Problem	Points	Score
1	20	
2	30	
3	50	
4	50	
5	50	
6	50	
7	50	
Total	300	

1. (20 points) Negate the following, and simplify (i.e., don't just add "it is not true that..." to the beginning):
   (a) There exists $x \in \mathbb{Z}$ such that for all $y \in \mathbb{Z}$, $xy \neq 1$.
   (b) If for all $x \in \mathbb{Z}$, $y(x-1) > 0$, then $y^2 = 0$ and $y \neq 1$.

$\forall x \in \mathbb{Z}$, $\exists y \in \mathbb{Z}$ s.t. $xy = 1$

$\forall x \in \mathbb{Z}$, $y(x-1) > 0$ and $y^2 \neq 0$ or $y = 1$

2. (30 points) Let a, b, c, d be integers. Prove that if $a|b$ and $a|c$, then $a|(b + cd)$.

Suppose $a|b$ and $a|c$.

Then $\exists\ y, z$ s.t. $b = ya$ and $c = za$.

Then $b + cd = ya + zad$

$= a(y + zd)$

So $a \mid b + cd$. $\square$

3. (50 points) Consider the Fibonacci sequence defined by $F_1 = 1, F_2 = 1, F_{n+1} = F_{n+1} + F_n$ for $n \geq 1$. Prove that $F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$ for all integers $n \geq 1$.

see other key

4. (50 points) Prove that there are no positive integer solutions to the equation $x^2 - y^2 = -2$.

Suppose $x, y \in \mathbb{Z}_{>0}$ and $x^2 - y^2 = -2$.

Then $(x-y)(x+y) = -2$.

There are 4 cases:

$x-y$	$= -1$	1	-2	2
$x+y$	$= 2$	-2	1	-1.

Since $x, y > 0$, $x+y > 0$, so $x+y \neq -2$ or -1.

Sps $x+y = 2$. then $x-y = -1$, so $2x = 1$ contradicting $x \in \mathbb{Z}$.

Sps $x+y = 1$. ~~Then both~~

Since $x, y > 0$, $x+y \geq 1+1 = 2$, a contradiction. $\square$

5. (50 points) Prove that for all integers $a \in \mathbb{Z}$, 15 divides $a(a^2-1)(a^2-4)$.

See other Key

6. (50 points) Prove that if $n\mathbb{Z} \subset m\mathbb{Z}$, then m divides n.

Proof by contrapositive

Suppose $m \nmid n$. Then $n \notin m\mathbb{Z}$,

~~so $\mathbb{Z}$ n~~

Since $n \in n\mathbb{Z}$, $n\mathbb{Z} \not\subset m\mathbb{Z}$. $\square$

7. (50 points) Let $r = \frac{1+\sqrt{5}}{2}$ (which is approximately 1.62). Consider the Fibonacci sequence defined by $F_1 = 1, F_2 = 1, F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. Prove that for every integer $n > 0$, $F_n \geq r^{n-2}$. (Hint: you are allowed to use that $r^2 = r + 1$.)

P(1) is true, since $1 \geq (1.62)^{-1}$

P(2) ———— $1 \geq (1.62)^0 = 1$.

Assume P(n), i.e., $F_n \geq r^{n-2}$.

And assume P(n-1), ———— $F_{n-1} \geq r^{n-3}$.

Then

$$F_{n+1} = F_n + F_{n-1} \geq r^{n-2} + r^{n-3} = r^{n-3}(r+1)$$

$$= r^{n-3} \cdot r^2$$

$$= r^{n-1}.$$

$\boxtimes$

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Total	300	

1. (20 points) Negate the following, and simplify (i.e., don't just add "it is not true that..." to the beginning):

   (a) For all $x \in \mathbb{Z}$, there exists $y \in \mathbb{Z}$ such that $xy = 1$.

   (b) If there exists an $x \in \mathbb{Z}$ such that $x(x-1) = 0$, then $x = 0$ or $x = 1$.

a) $\exists\, x \in \mathbb{Z}$ s.t. $\forall y \in \mathbb{Z}$, $xy \neq 1$

b) $\exists\, x \in \mathbb{Z}$ s.t. $x(x-1) = 0$ and $x \neq 0$ and $x \neq 1$.

2. (30 points) Prove that if $x, y \in 5\mathbb{Z}$, then $x + y \in 5\mathbb{Z}$ and $xy \in 5\mathbb{Z}$.

Suppose $x, y \in 5\mathbb{Z}$.

Then $\exists\, z, w$ s.t. $x = 5z$, $y = 5w$.

Then $x+y = 5z + 5w = 5(z+w)$,

so $x+y \in 5\mathbb{Z}$.

Also, $xy = 5z \cdot 5w$, ~~5~~ $= 5(z \cdot 5 \cdot w)$

So $xy \in 5\mathbb{Z}$. $\square$

3. (50 points) Prove that $x^3 = 8n + 2$ has no solutions with $x, n \in \mathbb{Z}$.

Proof by contradiction.

Suppose $n, x \in \mathbb{Z}$ and $x^3 = 8n+2$.

Then since $2 \mid 8n+2$, $2 \mid x^3$.

By FTA, $2 \mid x$.

Thus $8 \mid x^3$, so $8 \mid 8n+2$.

But $8 \mid 8n$, so $8 \mid 2$, a contradiction.

4. (50 points) Consider the Fibonacci sequence defined by $F_1 = 1, F_2 = 1, F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$. Prove that $F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$ for all integers $n \geq 1$.

Induction

Base case: $P(1)$: $F_1^2 = F_1 \cdot F_2$

$1 = 1$ ✓.

Assume $P(n)$, i.e. $F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$.

Then $F_1^2 + \cdots + F_n^2 + F_{n+1}^2 = F_n F_{n+1} + F_{n+1}^2$

$= F_{n+1}(F_n + F_{n+1})$

$= F_{n+1}(F_{n+2})$.

$\blacksquare$

5. (50 points) Prove that for all integers $a \in \mathbb{Z}$, 15 divides $a(a^2-1)(a^2-4)$.

Enough to show it for 3 & 5 separately.

~~If $3 \mid a$, done.~~

~~If $3 \nmid a$~~

But $a(a^2-1)(a^2-4) = (a-2)(a-1)(a)(a+1)(a+2)$

which is 5 consecutive integers.

Thus it is divisible by both 3 and 5. $\square$

6. (50 points) Prove that for all positive integers n, 5 divides $(4^n+1)(4^n-1)$.

$$\|$$

$$4^{2n}-1 = 16^{n}-1$$

$$= (16-1)(\,-\,).$$

$\square$

7. (50 points) Prove that for $n > 1$, the equation $x^n - y^n = \cancel{1}\ -1$ has no solutions in positive integers.

Suppose $x^n - y^n = -1$, $x, y \in \mathbb{Z}_{>0}$.

Then $(x-y)(m) = -1$, where $m = x^{n-1} + x^{n-2}y + \dots + y^{n-1}$.

In particular, since $x, y > 0$, $m > 0$.

By FTA, either $x - y = +1$, $m = 1$

or $x - y = -1$, $m = -1$.

Since $m > 0$, the 2nd case can't happen.

~~In the 1st case, $x - y = +1$, so $x = 1$~~

Since

Thus $m = 1$. But $m = x^{n-1} + x^{n-2}y + \dots + y^{n-1} \geq n$

since there are n terms, all ≥ 1.

this contradicts $n > 1$. ∎