PROBLEMS ABOUT COUNTABLE SETS

This is a collection of problems from class and from the homework that I would like students to be able to do. For each of these sets, either prove directly that it is countable (i.e., write down a bijection from $\mathbb{Z}_{>0}$), or use the injectivity, surjectivity, or union theorem. I may or may not ask you to prove all of your claims (i.e., I may ask you to write down a bijection, but not to prove that the map you wrote down is a bijection), but in general still expect you to justify every detail unless I request otherwise.

On the exam, I will make it clear what you are allowed to assume (i.e., "Prove that $\mathbb{Q}$ is countable. You may assume that $\mathbb{Z} \times \mathbb{Z}$ is countable."). For this worksheet though, to do any given problem, you may use the result of any previous problem.

(1) Prove that each of the following is countable.
   (a) $\mathbb{Z}$
   (b) $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$
   (c) $\mathbb{Q}$
   (d) $\mathbb{Z} \times \mathbb{Z}$
   (e) $\{e^n | n \in \mathbb{Z}\}$
   (f) $\mathbb{Q}^+ \cup \{e^n | n \in \mathbb{Z}\}$
   (g) $\mathbb{Q} \times \mathbb{Q}$
   (h) $\mathbb{Q}^n$
   (i) $\mathbb{Q}_d$ = set of polynomials of rational coefficients of degree at most d.
   (j) $\mathbb{Q}$
   (k) $P_{\mathrm{BD}}(\mathbb{Z})$
   (l) $\mathrm{Fun}_{\mathrm{BD}}(\mathbb{Z})$
   (m) $\mathrm{Seq}_{\mathrm{BD}}(\mathbb{Z}_{>0})$
   (n) $\overline{\mathbb{Q}}$

   In addition, make sure that you can do the homework problems, especially 6 and 11 of 6.1.

(2) **Hints**:

(a) $\mathbb{Z}$; Hint: write down an explicit bijection
(b) $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$; Hint: injectivity theorem.
(c) $\mathbb{Q}$; Hint: Union theorem or surjection theorem.
(d) $\mathbb{Z} \times \mathbb{Z}$; Hint: union theorem or surjection theorem.
(e) $\{e^n | n \in \mathbb{Z}\}$; Hint: write down an explicit bijection with $\mathbb{Z}$
(f) $\mathbb{Q}^+ \cup \{e^n | n \in \mathbb{Z}\}$; Hint: union theorem + injection theorem
(g) $\mathbb{Q} \times \mathbb{Q}$; Hint: union theorem
(h) $\mathbb{Q}^n$; Hint: Union theorem
(i) $\mathbb{Q}_d$; Hint: find a bijection with a set from a previous problem.
(j) $\mathbb{Q}$; Hint: Union theorem
(k) $P_{\mathrm{BD}}(\mathbb{Z})$; Hint: Union theorem and surjection theorem. This one is just like $\mathbb{Q}$.
(l) $\mathrm{Fun}_{\mathrm{BD}}(\mathbb{Z})$; Hint: Union theorem
(m) $\mathrm{Seq}_{\mathrm{BD}}(\mathbb{Z}_{>0})$; Hint: Union or surjection theorem
(n) $\overline{\mathbb{Q}}$; Hint: union theorem