MATH 250 HANDOUT 11 - INDEXED UNIONS AND INTERSECTIONS, PROOFS

(1) **Proofs.**
   (a) Let d be an integer. Compute $\bigcup_{i=1}^{\infty} d^i\mathbb{Z}$ and $\bigcap_{i=1}^{\infty} d^i\mathbb{Z}$; give a proof that your computation is correct.
   (b) Compute $\bigcup_{i=1}^{\infty}(1, i^2)$ and $\bigcap_{i=1}^{\infty}(1, i^2)$; give a proof that your computation is correct.
   (c) Compute $\bigcup_{i=1}^{\infty}(-i, i)$ and $\bigcap_{i=1}^{\infty}(-i, i)$; give a proof that your computation is correct.
   (d) Prove that
$$\overline{\bigcup_{i\in\mathbb{Z}_{\geq 1}} A_i} = \bigcap_{i\in\mathbb{Z}_{\geq 1}} \overline{A}_i$$
   (e) Prove that
$$A \cup \bigcap_{i\in\mathbb{Z}_{\geq 1}} B_i = \bigcap_{i\in\mathbb{Z}_{\geq 1}} (A \cup B_i).$$
   (f) Suppose that for every i, $A \subset A_i$. Prove that
$$A \subset \bigcap_{i=1}^{\infty} A_i.$$
   (g) Let $\mathbb{Q}_{\leq d} = \{a/b : a, b \in \{-d, \ldots, d\} \text{ and } b \neq 0\}$. Prove that $\mathbb{Q} = \bigcup_{d=1}^{\infty} \mathbb{Q}_{\leq d}$.
   (h) Let $\mathbb{Q}[x]$ be the set of polynomials with rational coefficients, and let $\mathbb{Q}[x]_d$ be the set of polynomials with rational coefficients of degree at most d. Prove that $\mathbb{Q}[x] = \bigcup_{d=0}^{\infty} \mathbb{Q}[x]_d$.

(2) **Other Index Sets**
   (a) Compute $\bigcap_{S\in\mathbb{P}(A)-\{\emptyset\}} S$ and $\bigcup_{S\in\mathbb{P}(A)} S$; give a proof that your computation is correct.
   (b) Prove that $\bigcup_{n\in\mathbb{Z}} (\mathbb{Z} \times \{n\}) = \mathbb{Z} \times \mathbb{Z}$.

Unrelated Challenge problem: Let $S_n = \{1, \ldots, n\}$ and let $S_0 = \emptyset$. Prove that $|P(S_n)| = 2^n$.