MATH 250 HANDOUT 6 - INDUCTION

(1) Prove that for any integer $n \geq 1$,
$$2^0 + 2^1 + ... + 2^{n-1} = 2^n - 1.$$
(2) Prove that for any integer $n \geq 1$, n^2 is the sum of the first n odd integers. (For example, $3^2 = 1 + 3 + 5$.)
(3) Show that $7^n - 1$ is divisible by 6 for all integers $n \geq 0$.
(4) Prove that $n^5 - 5n^3 + 4n$ is divisible by 120 for all integers $n \geq 1$.
(5) Prove that
$$n^9 - 6n^7 + 9n^5 - 4n^3$$
is divisible by 8640 for all integers n ≥ 1.
(6) Prove that
$$n^2 \mid ((n+1)^n - 1)$$
for all integers n ≥ 1.
(7) Show that
$$(x - y) \mid (x^n - y^n)$$
for all integers $n \geq 1$.
(8) Use the result of the previous problem to show that for all integers $n \geq 1$
$$8767^{2345} - 8101^{2345}$$
is divisible by 666.
(9) Show that
$$2903^n - 803^n - 464^n + 261^n$$
is divisible by 1897 for all integers $n \geq 1$.
(10) Prove that if n is an even natural number, then the number $13^n + 6$ is divisible by 7.
(11) Prove that for every $n \in \mathbb{Z}_{\geq 2}$, $n^3 - n$ is a multiple of 6.
(12) Prove that $n! \geq 3^n$ for all integers $n \geq 7$.
(13) Prove that $2^n \geq n^2$ for all integers $n \geq 4$.
(14) Consider the sequence defined by $a_1 = 1$ and $a_n = \sqrt{2a_{n-1}}$. Prove that $a_n < 2$ for all integers $n \geq 1$.
(15) Prove that the equation $x^2 + y^2 = z^n$ has a solution in positive integers x, y, z for all integers $n \geq 1$.
(16) Prove that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 3 for all integers $n \geq 1$.
(17) Prove that
$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{3n+1} > 1$$
for all integers $n \geq 1$.
(18) Prove that
$$\frac{4^n}{n+1} \leq \frac{(2n)!}{(n!)^2}$$
for all integers $n \geq 1$.

(19) Consider the Fibonacci sequence $\{F_n\}$ defined by $F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}, n \geq 1$. Prove that each of the following statements is true for all integers $n \geq 1$.

(a) $F_1 + F_3 + F_5 + ... + F_{2n-1} = F_{2n}$
(b) $F_2 + F_4 + F_6 + ... + F_{2n} = F_{2n+1} - 1$
(c) $F_n < 2^n$
(d) $F_{n-1}F_{n+1} = F_n^2 + (-1)^{n+1}$.
(e) Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Prove that $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$. (Hint: first prove by, for example, direct calculation that α and β are solutions of the equation $x^2 - x - 1 = 0$.)
(f) Prove that $F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$.
(g) Find a formula for $F_1 + \cdots + F_n$ and prove it via induction.

(20) Prove that $n! > 2^n$ for all integers $n \geq 4$.

(21) Prove that the expression $3^{3n+3} - 26n - 27$ is a multiple of 169 for all positive integers n.

(22) Prove that if k is odd, then 2^{n+2} divides

$$k^{2^n} - 1$$

for all positive integers n.

(23) Prove that

$$\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$$

is an integer for all integers $n \geq 0$.

(24) Let a_n be the sequence defined by $a_1 := 1$, $a_n := na_{n-1}$. Prove that $a_n = n!$.

(25) Let a_n be the sequence defined by $a_1 := 2$, $a_n := 2a_{n-1}$. Prove that $a_n = 2^n$.

(26) Prove that 3^n is odd for every non-negative integer.

(27) Prove that $n(n-1)$ is even for every positive integer n.

(28) Prove that $n^3 + 2n$ is a multiple of 3 for every positive integer n.

(29) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)n} = \frac{n-1}{n}$

(30) Prove that $1^2 + 4^2 + 7^2 + \cdots + (3n-2)^2 = \frac{1}{2}n(6n^2 - 3n - 1)$ for $n \in \mathbb{Z}_{\geq 1}$.

(31) Prove that $2^2 + 5^2 + 8^2 + \cdots + (3n-1)^2 = \frac{1}{2}n(6n^2 + 3n - 1)$ for $n \in \mathbb{Z}_{\geq 1}$.

(32) Prove that $1^3+2^3+3^3+\cdots+n^3 = (1+2+3+\cdots+n)^2$ (Hint: use $1+2+3+\cdots+k = \frac{k(k+1)}{2}$.)

(33) Let $n \in \mathbb{Z}_{\geq 0}$. Prove that $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$.

(34) Let $n \in \mathbb{Z}_{\geq 0}$. Prove that $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$.

(35) Let $n \in \mathbb{Z}_{\geq 0}$. Prove that $\sum_{i=1}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$.

(36) Let $n \in \mathbb{Z}_{\geq 0}$. Find a formula for $\sum_{i=1}^{n} i^4$. Prove that your formula is correct.

(37) Prove that $2^n > n^2$ for $n \geq 5$ for $n \in \mathbb{Z}_{\geq 1}$.

In the following problems, let $\binom{n}{k} = \frac{n!}{(n-k)!k!}$.

(38) Let x and y be variables. Prove that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

(39) Prove that

$$\binom{n}{k} = \binom{n-1}{k} - \binom{n-1}{k-1}.$$