

MATH 250 HANDOUT 2 - DIVISIBILITY

- (1) Show that if $d \neq 0$ and $d \mid a$, then $d \mid (-a)$ and $-d \mid a$.
- (2) Show that if $a \mid b$ and $b \mid a$, then $a = b$ or $a = -b$.
- (3) Suppose that n is an integer such that $5 \mid (n+2)$. Which of the following are divisible by 5?
 - (a) $n^2 - 4$
 - (b) $n^2 + 8n + 7$
 - (c) $n^4 - 1$
 - (d) $n^2 - 2n$
- (4) Prove that the square of any integer of the form $5k+1$ for $k \in \mathbb{Z}$ is of the form $5k'+1$ for some $k' \in \mathbb{Z}$.
- (5) Show that if $ac \mid bc$ and $c \neq 0$, then $a \mid b$.
- (6)
 - (a) Prove that the product of three consecutive integers is divisible by 6.
 - (b) Prove that the product of four consecutive integers is divisible by 24.
 - (c) Prove that the product of n consecutive integers is divisible by $n(n-1)$.
 - (d) (Challenge problem) Prove that the product of n consecutive integers is divisible by $n!$.
- (7) Find all integers $n \geq 1$ so that $n^3 - 1$ is prime. Hint: $n^3 - 1 = (n^2 + n + 1)(n - 1)$.
- (8) Show that for all integers a and b ,

$$ab(a^2 - b^2)(a^2 + b^2)$$

is divisible by 30.

- (9) Suppose that a is an integer greater than 1 and that n is a positive integer. Prove that if $a^n + 1$ is prime, then a is even and n is a power of 2. Primes of the form $2^{2^k} + 1$ are called Fermat primes.
- (10) Suppose that a is an integer greater than 1 and that n is a positive integer. Prove that if $a^n - 1$ is prime other than 2, then $a = 2$ and n is a prime. Primes of the form $2^n - 1$ are called Mersenne primes.
- (11) Let n be an integer greater than 1. Prove that if one of the numbers $2^n - 1, 2^n + 1$ is prime, then the other is composite.
- (12) Show that every integer of the form $4 \cdot 14^k + 1$, $k \geq 1$ is composite. Hint: show that there is a factor of 3 when k is odd and a factor of 5 when k is even.

- (13) Can you find an integer $n > 1$ such that the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

is an integer?