Overconvergent de Rham-Witt Cohomology for Algebraic Stacks

David Zureick-Brown (Emory University)
Christopher Davis (University of Copenhagen)

Slides available at `http://www.mathcs.emory.edu/~dzb/slides/`

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Weil Conjectures

Throughout, p is a prime and $q = p^n$.

Definition

The **zeta function** of a variety X over $\mathbb{F}_q$ is the series

$$\zeta_X(T) = \exp\left(\sum_{n=1}^{\infty} \#X(\mathbb{F}_{q^n})\frac{T^n}{n}\right).$$

Rationality: For X smooth and proper of dimension d

$$\zeta_X(T) = \frac{P_1(T)\cdots P_{2d-1}(T)}{P_0(T)\cdots P_{2d}(T)}$$

Cohomological description: For any Weil cohomology H^i,

$$P_i(T) = \det(1 - T\,\mathrm{Frob}_q, H^i(X)).$$

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Consequences for point counting:

$$\#X(\mathbb{F}_{q^n}) = \sum_{r=0}^{2d}(-1)^r \sum_{i=1}^{b_r} \alpha_{i,r}^n$$

Riemann hypothesis (Deligne):

$$P_i(T) \in 1 + T\mathbb{Z}[T], \text{ and the } \mathbb{C}\text{-roots } \alpha_{i,r} \text{ of } P_i(T) \text{ have norm } q^{i/2}.$$

Independence of de Rham cohomology

Fact

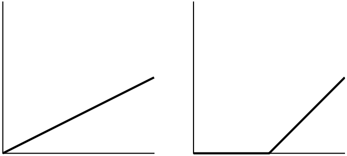
For a prime p, the condition that two proper varieties X and X' over $\mathbb{Z}_p$ with good reduction at p have the *same* reduction at p implies that their Betti numbers agree.

Explanation

$$H^i_{\mathrm{cris}}(X_p/\mathbb{Z}_p) \cong H^i_{\mathrm{dR}}(X, \mathbb{Z}_p)$$

Newton above Hodge

1. The **Newton Polygon** of X is the lower converx hull of $(i, v_p(a_i))$.
2. The **Hodge Polygon** of X is the polygon whose slope i segment has width
$$h^{i,\dim(X)-i} := H^i(X, \Omega_X^{\dim(X)-i}).$$
3. **Example**: E supersingular elliptic curve.



Newton **Hodge**

Weil cohomologies

Modern:

(Étale)	$H^i_{\text{et}}(\overline{X}, \mathbb{Q}_\ell)$
(Crystalline)	$H^i_{\text{cris}}(X/W)$
(Rigid/overconvergent)	$H^i_{\text{rig}}(X)$

Variants, preludes, and complements:

(Monsky-Washnitzer)	$H^i_{MW}(X)$
(de Rham-Witt)	$H^i(X, W\Omega^\bullet_X)$
(overconvergent dRW)	$H^i(X, W^\dagger\Omega^\bullet_X)$

de Rham-Witt

Let X be a smooth variety over $\mathbb{F}_q$.

Theorem (Illusie, 1975)

There exists a complex $W\Omega_X^\bullet$ of sheaves on the Zariski site of X whose (hyper)cohomology computes the crystalline cohomology of X.

1. Main points
   1. Sheaf cohomology on **Zariski** rather than the **crystalline** site.
   2. Complex is independent of choices (compare with Monsky-Washnitzer).
   3. Somewhat explicit.

de Rham-Witt

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1. Applications - easy proofs of
   1. Finite generation.
   2. Torsion-free case of Newton above Hodge
2. Generalizations
   1. Langer-Zink (relative case).
   2. Hesselholt (big Witt vectors).

Definition of $W\Omega^\bullet_X$

1. It is a particular quotient of $\Omega^\bullet_{W(X)/W(\mathbb{F}_p)}$.
2. **Recall**: if A is a perfect ring of char p,

$$W(A) = \prod A \ni \sum a_i p^i \quad (a_i \in A)$$

What is $W(k[x])$?

1. $W(k[x]) \subset W(k[x]^{\mathrm{perf}}) \ni \sum_{k \in \mathbb{Z}[1/p]} a_k x^k$
2. $f = \sum_{k \in \mathbb{Z}[1/p]} a_k x^k \in W(k[x])$ if f is V-adically convergent, i.e.,
3. $v_p(a_k k) \geq 0$.
4. (I.e., $V(x) = p x^{\frac{1}{p}} \in W(k[x])$, but $x^{\frac{1}{p}} \notin W(k[x])$.)

Definition of $W\Omega^\bullet_X$

1. For X a general scheme (or stack), one can glue this construction.
2. $W\Omega^\bullet_X$ is an initial V-pro-complex.

Definition of $W^\dagger\Omega^\bullet_X$

Theorem (Davis, Langer, and Zink)

There is a subcomplex

$$W^\dagger\Omega^\bullet_X \subset W\Omega^\bullet_X$$

such that if X is a smooth scheme, $H^i(X, W^\dagger\Omega^\bullet_X) \otimes \mathbb{Q} \cong H^i_{\mathrm{rig}}(X)$.

Note well:

1. Left hand side is Zariski cohomology.
2. (Right hand side is cohomology of a complex on an associated rigid space.)
3. The complex $W^\dagger\Omega^\bullet_X$ is **independent of choices** and functorial.

Étale Cohomology for stacks

$$
\begin{aligned}
\# B\mathbb{G}_m(\mathbb{F}_p) &= \sum_{x \in |B\mathbb{G}_m(\mathbb{F}_p)|} \frac{1}{\# \mathrm{Aut}_x(\mathbb{F}_p)} = \frac{1}{p-1} \\
&= \sum_{i=-\infty}^{i=\infty} (-1)^i \operatorname{Tr} \operatorname{Frob} H^i_{c,\text{ét}}(B\mathbb{G}_m, \overline{\mathbb{Q}_\ell}) \\
&= \sum_{i=1}^{\infty} (-1)^2 p^{-i} = \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots
\end{aligned}
$$

Example

Étale cohomology and Weil conjectures for stacks are used in Ngô's proof of the fundamental lemma.

Crystalline cohomology for stacks

1. Book by Martin Olsson "*Crystalline cohomology of algebraic stacks and Hyodo-Kato cohomology*".
2. **Main application** – new proof of C_{st} conjecture in p-adic Hodge theory.
3. Key insight –
$$H^i_{\text{log-cris}}(X, M) \cong H^i_{\text{cris}}(\mathcal{L}og_{(X,M)}).$$
4. One technical ingredient – generalizations of de Rham-Witt complex to stacks. (Needed, e.g., to prove finiteness.)

Rigid cohomology for stacks

Original motivation: **Geometric Langlands** for $\mathrm{GL}_n(\mathbb{F}_p(C))$:

1. Lafforgue constructs a 'compactified moduli stack of shtukas' $\mathcal{X}$ (actually a compactification of a stratification of a moduli stack of shtukas).
2. The ℓ-adic étale cohomology of étale sheaves on $\mathcal{X}$ **realize a Langlands correspondence** between certain **Galois** and **automorphic** representations.

Other motivation: applications to **log-rigid** cohomology.

Rigid cohomology for stacks

Theorem (ZB, thesis)

1. **Definition** *of rigid cohomology for stacks (via le Stum's overconvergent site)*
2. *Define variants with* **supports in a closed subscheme**,
3. *show they* **agree** *with the classical constructions.*
4. **Cohomological descent** *on the overconvergent site.*

Rigid cohomology for stacks

In progress (ZB)

1. Duality.
2. Compactly supported cohomology.
3. Full Weil formalism.
4. Applications.

Main theorems

Theorem (Davis-ZB; in preparation)

Let $\mathcal{X}$ be a smooth Artin stack of finite type over $\mathbb{F}_q$. Then there exists a functorial complex $W^\dagger\Omega^\bullet_{\mathcal{X}}$ whose cohomology agrees with the rigid cohomology of $\mathcal{X}$.

Theorem (in preparation)

Let X be a smooth affine scheme over $\mathbb{F}_q$. Then the étale cohomology $H^i_{ét}(X, W^\dagger\Omega^j_X) = 0$ for $i > 0$.

Theorem (Accepted; MRL)

Integral *MW-cohomology agrees with overconvergent cohomology (for $i < p$).*

Main technical details

Remark

In the classical case, once can write

$$W\Omega^i = \varprojlim_n W_n\Omega^i.$$

The sheaves $W_n\Omega^i$ are coherent.

Remark

In the overconvergent case,

$$W^\dagger\Omega^i = \varinjlim_\epsilon W^\epsilon\Omega^i.$$

Main technical details

Tools used in the proof

1. Limit Čech cohomology.
2. Topological (in the Grothendieck sense) unwinding lemmas.
3. Structure theorem for étale morphisms.
4. (Stein property)

$$U \subset X \text{ and } \mathcal{O}(U) = \mathcal{O}(X) \Rightarrow W^{\dagger}\Omega^{i}(U) \cong W^{\dagger}\Omega^{i}(X)$$

5. Nisnevish Devissage.
6. Brutal direct computations: need surjectivity of

$$W^{\dagger}\Omega^{i}(U) \oplus W^{\dagger}\Omega^{i}(X') \to W^{\dagger}\Omega^{i}(X)$$

for a standard Nisnevich cover $U \coprod X' \to X$.