Diophantine and tropical geometry

David Zureick-Brown

Slides available at `http://www.math.emory.edu/~dzb/slides/`

University of Toronto Colloquium
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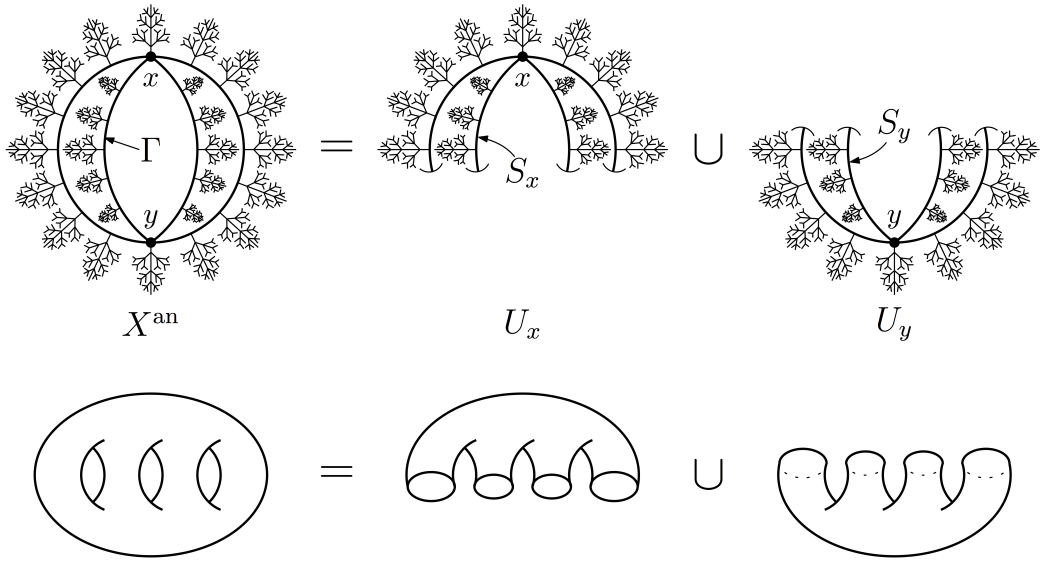

Basic Problem (Solving Diophantine Equations)

Setup

Let $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$ be polynomials.

Let R be a ring (e.g., $R = \mathbb{Z}$, $\mathbb{Q}$).

Problem

Describe the set

$$V(f_1, \ldots, f_m) = \{(a_1, \ldots, a_n) \in R^n : \forall i, f_i(a_1, \ldots, a_n) = 0\}.$$

Fact

Solving Diophantine equations is hard.

Hilbert's Tenth Problem

The ring $R = \mathbb{Z}$ is especially hard.

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Theorem (Davis–Putnam–Robinson 1961, Matijasevič 1970)

There does not exist an algorithm solving the following problem:

input: $f_1, \ldots, f_m \in \mathbb{Z}[x_1, ..., x_n]$;

output: YES / NO *according to whether the set*

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is non-empty.

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This is also *known* to be true for many rings (e.g., $R = \mathbb{C}, \mathbb{R}, \mathbb{F}_q, \mathbb{Q}_p, \mathbb{C}(t)$).

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This is also *known* to be true for many rings (e.g., $R = \mathbb{C}, \mathbb{R}, \mathbb{F}_q, \mathbb{Q}_p, \mathbb{C}(t)$).
This is *still open* for many other rings (e.g., $R = \mathbb{Q}$).

Fermat's Last Theorem

Theorem (Wiles; Taylor)

For primes $p \geq 3$ the only integer solutions to the equation

$$x^p + y^p = z^p$$

are integer multiples of the triples

$$(0,0,0), \quad (\pm 1, \mp 1, 0), \quad \pm(1,0,1), \quad \pm(0,1,1).$$

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Qualitative:

- Does there exist a solution?
- Do there exist infinitely many solutions?
- Does the set of solutions have some extra structure (e.g., geometric structure, group structure).

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Implicit question

- Why do equations have (or fail to have) solutions?
- Why do some have many and some have none?
- What underlying mathematical structures control this?

Example: Pythagorean triples

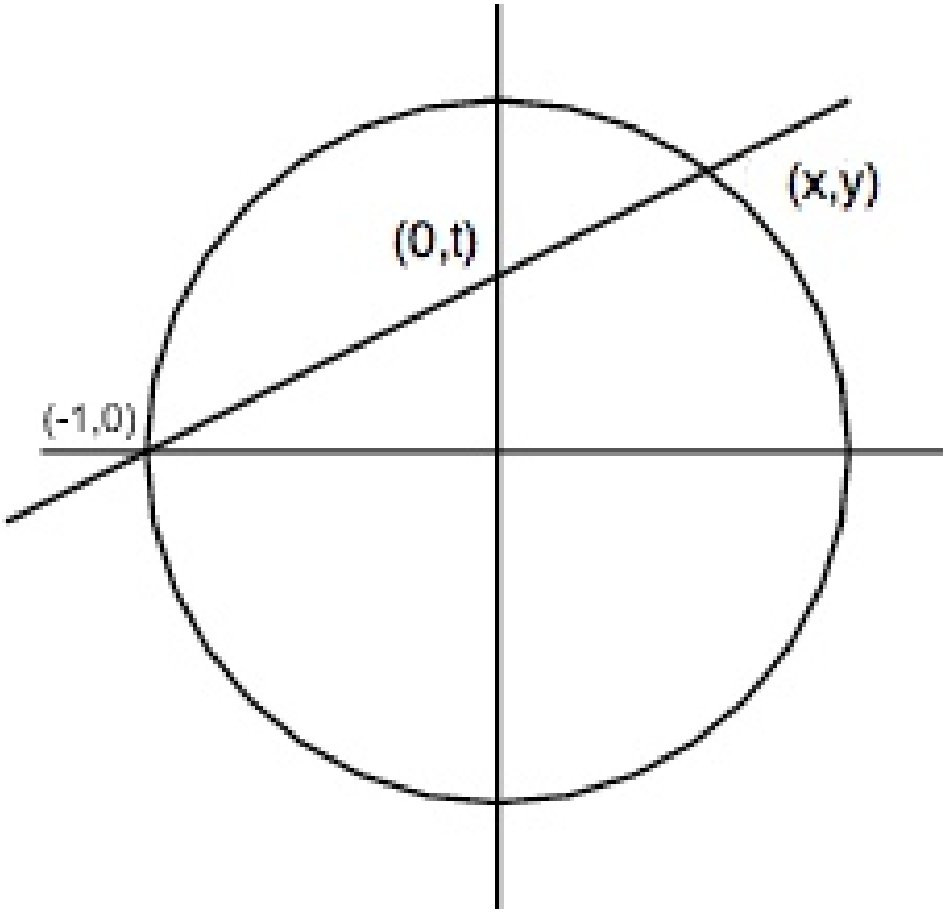
Lemma

The equation

$$x^2 + y^2 = z^2$$

has infinitely many non-zero coprime solutions.

Pythagorean triples



$$\text{Slope} = \quad t = \frac{y}{x+1}$$

$$x = \frac{1-t^2}{1+t^2}$$

$$y = \frac{2t}{1+t^2}$$

Pythagorean triples

Lemma

The solutions to

$$a^2 + b^2 = c^2$$

(with $c \neq 0$) are all multiples of the triples

$$\boxed{a = 1 - t^2} \quad \boxed{b = 2t} \quad \boxed{c = 1 + t^2}$$

The Mordell Conjecture

Example

The equation $y^2 + x^2 = 1$ has infinitely many solutions.

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Theorem (Faltings)

For $n \geq 5$, the equation

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Theorem (Faltings)

For $n \geq 5$, the equation

$$y^2 = f(x)$$

has only finitely many solutions if $f(x)$ is squarefree, with degree > 4.

Fermat Curves

Question

Why is Fermat's last theorem believable?

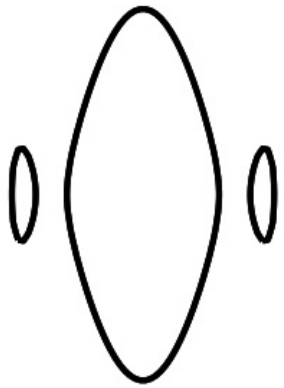
1. $x^n + y^n - z^n = 0$ looks like a surface (3 variables)
2. $x^n + y^n - 1 = 0$ looks like a curve (2 variables)

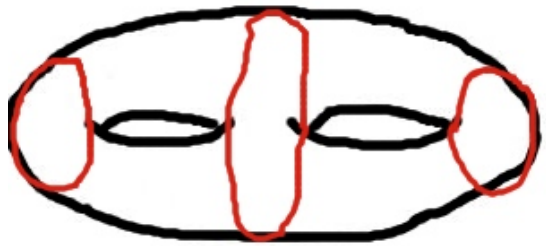
Mordell Conjecture

Example

$$y^2 = -(x^2 - 1)(x^2 - 2)(x^2 - 3)$$

This is a cross section of a two holed torus.





The **genus** is the number of holes.

Conjecture (Mordell, 1922)

A curve of genus $g \geq 2$ has only finitely many rational solutions.

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2. $x^n + y^n - 1 = 0$ looks like a curve (2 variables)
3. and has genus

$$(n-1)(n-2)/2$$

which is ≥ 2 iff $n \geq 4$.

Fermat Curves

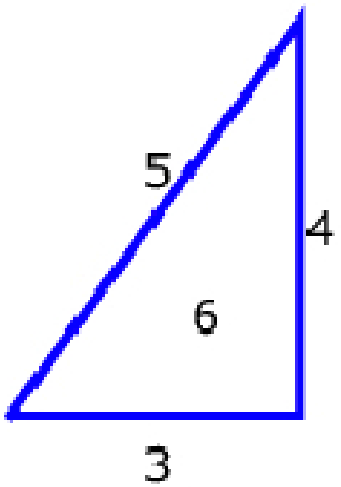
Question

What if $n = 3$?

1. $x^3 + y^3 - 1 = 0$ is a curve of genus $(3-1)(3-2)/2 = 1$.
2. We were lucky; $Ax^3 + By^3 = Cz^3$ can have infinitely many solutions.

Congruent number problem

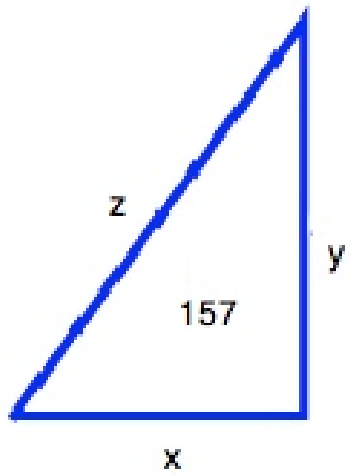
$$x^2 + y^2 = z^2,\ xy = 2 \cdot 6$$



$$3^2 + 4^2 = 5^2,\quad 3 \cdot 4 = 2 \cdot 6$$

Congruent number problem

$$x^2 + y^2 = z^2,\ xy = 2 \cdot 157$$



Assume the Birch–Swinnerton-Dyer conjectures

If you assume \$1,000,000 worth of conjectures, then the equations

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"Next" soluton has **176 digits**!

Back of the envelope calculation (as of 2011)

$$\boxed{x^2 + y^2 = z^2,\ xy = 2 \cdot 157}$$

- Num, $\mathrm{den}(x, y, z) \leq 10 \sim 10^6$ many, **1 min** on Emory's computers.

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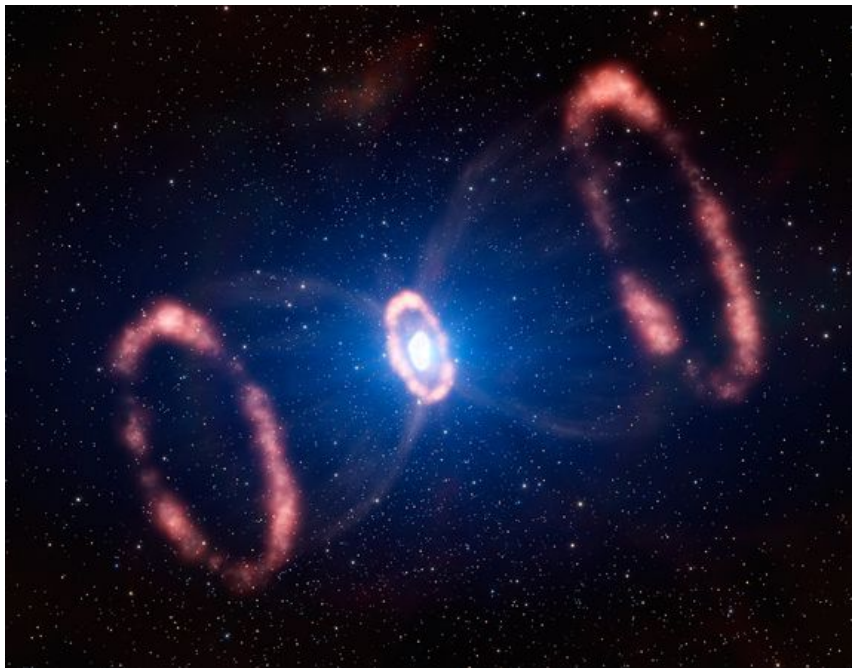
- Expected time until 'heat death' of universe – $\mathbf{10^{100}}$ **years**.



Fermat Surfaces

Conjecture

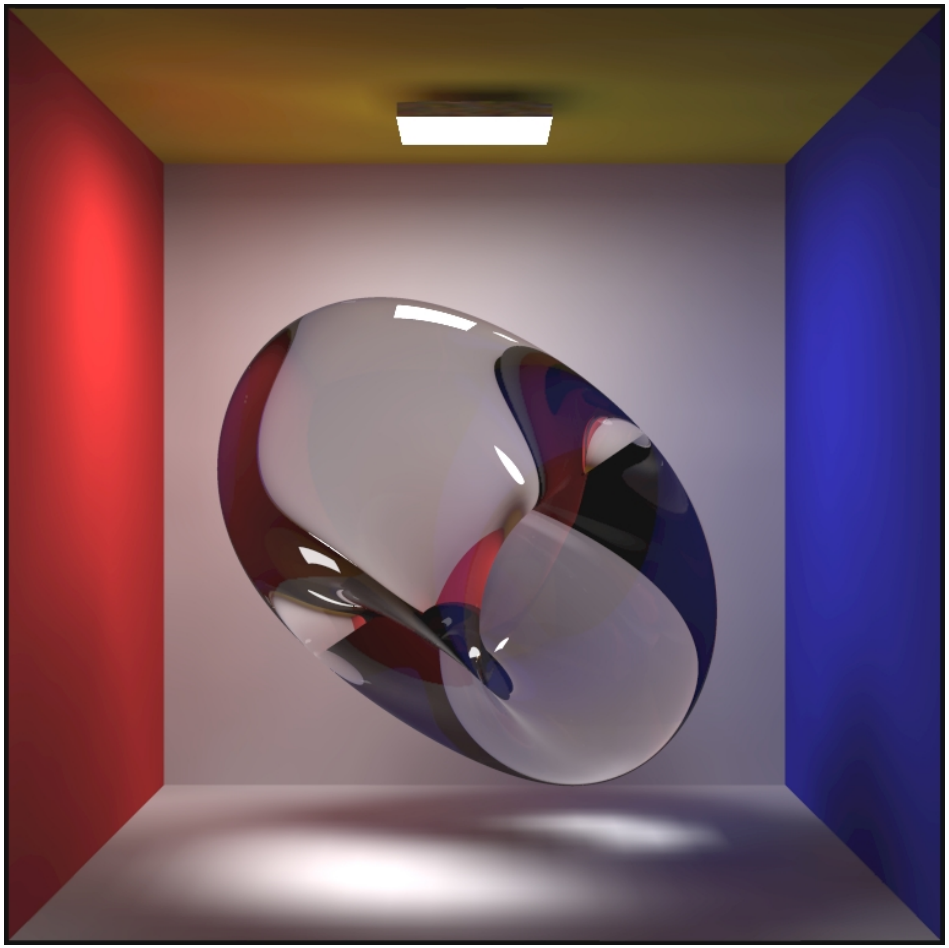
The only solutions to the equation

$$x^n + y^n = z^n + w^n, n \geq 5$$

satisfy $xyzw = 0$ or lie on the lines 'lines' $x = z$, $y = w$ (and permutations).

The Swinnerton-Dyer K3 surface

$$x^4 + 2y^4 = 1 + 4z^4$$



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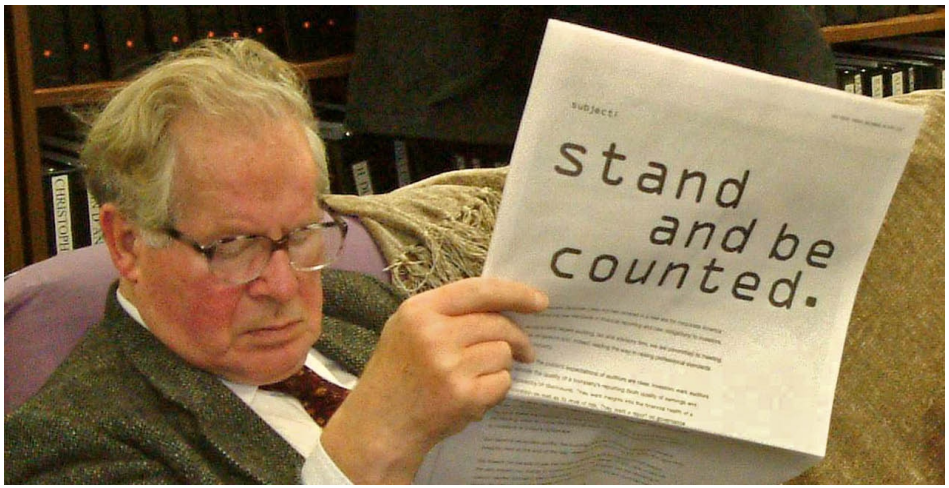

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- Two 'obvious' solutions – $(\pm 1 : 0 : 0)$.
- The next smallest solutions are $\left(\pm\frac{1484801}{1169407}, \pm\frac{1203120}{1169407}, \pm\frac{1157520}{1169407}\right)$.

Problem

Find another solution. (Probably impossible.)

Back of envelope calcluation

1. $\mathbf{10^{16}}$ **years** to find via brute force.
2. Age of the universe – $\mathbf{13.75 \pm .11}$ **billion years** (roughly $\mathbf{10^{10}}$).

"Generalized" Fermat equations

Theorem (Poonen, Schaefer, Stoll)

The coprime integer solutions to $x^2 + y^3 = z^7$ *are the* 16 *triples*

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Generalized Fermat Equations

Problem

What are the solutions to the equation $x^a + y^b = z^c$ *?*

Generalized Fermat Equations

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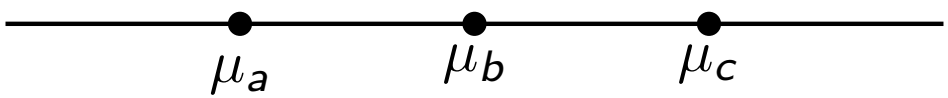
What are the solutions to the equation $x^a + y^b = z^c$?

Theorem (Darmon and Granville)

Fix $a, b, c \geq 2$. Then the equation $x^a + y^b = z^c$ has only finitely many coprime integer solutions iff $\chi = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 \leq 0$.



Known Solutions to $x^a + y^b = z^c$ with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$

$$1^p + 2^3 = 3^2, \quad 2^5 + 7^2 = 3^4$$

$$7^3 + 13^2 = 2^9, \quad 2^7 + 17^3 = 71^2$$

$$3^5 + 11^4 = 122^2$$

$$17^7 + 76271^3 = 21063928^2$$

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Problem (Beal's conjecture)

These are all solutions with $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 < 0$.

Generalized Fermat Equations – Known Solutions

Conjecture (Beal, Granville, Tijdeman–Zagier)

This is a complete list of coprime non-zero solutions such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1 < 0$.

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Examples of Generalized Fermat Equations

Theorem (Poonen, Schaefer, Stoll)

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$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} - 1 = -\frac{1}{42} < 0$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6} - 1 = 0$$

Examples of Generalized Fermat Equations

Theorem (Darmon, Merel)

Any pairwise coprime solution to the equation

$$x^n + y^n = z^2, n > 4$$

satisfies $xyz = 0$.

$$\frac{1}{n} + \frac{1}{n} + \frac{1}{2} - 1 = \frac{2}{n} - \frac{1}{2} < \frac{2}{4} - \frac{1}{2} = 0$$

Other applications of the modular method

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Theorem (Bugeaud, Mignotte, Siksek; 2006)

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$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

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Theorem (Silliman–Vogt; 2013 REU)

0 and 1 are the only perfect powers in the Lucas sequence

$$L_1 = 0, L_2 = 1, \quad L_n = 3L_{n-1} - 2L_{n-2}.$$

$$0, 1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, 2047, 4095, 8191, \ldots, 2^n - 1, \ldots$$

Examples of Generalized Fermat Equations

Theorem (Klein, Zagier, Beukers, Edwards, others)

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$$(T/2)^2 + H^3 + (f/12^3)^5$$

1. $f = st(t^{10} - 11t^5s^5 - s^{10})$,
2. $H =$ Hessian of f,
3. $T =$ a degree 3 covariant of the dodecahedron.

(a, b, c) such that $\chi < 0$ and the solutions to $x^a + y^b = z^c$ have been determined.

$\{n, n, n\}$	Wiles,Taylor–Wiles, building on work of many others
$\{2, n, n\}$	Darmon–Merel, others for small n
$\{3, n, n\}$	Darmon–Merel, others for small n
$\{5, 2n, 2n\}$	Bennett
$(2, 4, n)$	Ellenberg, Bruin, Ghioca $n \geq 4$
$(2, n, 4)$	Bennett–Skinner; $n \geq 4$
$\{2, 3, n\}$	Poonen–Shaefer–Stoll, Bruin. $6 \leq n \leq 9$
$\{2, 2\ell, 3\}$	Chen, Dahmen, Siksek; primes $7 < \ell < 1000$ with $\ell \neq 31$
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$(2, 3, 10)$	**ZB**

Faltings' theorem / Mordell's conjecture

Theorem (Faltings, Vojta, Bombieri)

Let X be a smooth curve with genus at least 2. Then $\#X(\mathbb{Q}) < \infty$.

Example

For $g \geq 2$, $y^2 = x^{2g+1} + 1$ has only finitely many solutions with $x, y \in \mathbb{Q}$.

Conjecture (Lang, Vojta)

Let X be a variety of general type. Then $X(\mathbb{Q})$ is not (Zariski) dense.

Uniformity

Problem

1. *Given X, compute $X(\mathbb{Q})$ exactly.*
2. *Compute bounds on $\#X(\mathbb{Q})$.*

Conjecture (Uniformity)

There exists a constant $N(g)$ such that every smooth curve of genus g over $\mathbb{Q}$ has at most $N(g)$ rational points.

Theorem (Caporaso, Harris, Mazur)

Lang's conjecture $\Rightarrow$ uniformity.

Uniformity numerics

g	2	3	4	5	10	45	g
$B_g(\mathbb{Q})$	642	112	126	132	192	781	$16(g+1)$

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Remark

Elkies studied K3 surfaces of the form

$$y^2 = S(t, u, v)$$

with lots of rational lines, such that S restricted to such a line is a square.

Main Theorem (uniformity for curves of small rank)

Theorem (Katz–Rabinoff–ZB)

Let X be any curve of genus g and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $r < g - 2$. Then

$$\#X(\mathbb{Q}) \leq 84g^2 - 98g + 28$$

Tools

p-adic integration on annuli

comparison of different analytic continuations of p-adic integration

Non-Archimedean (Berkovich) structure of a curve [BPR]

Combinatorial restraints coming from the Tropical canonical bundle

Coleman's bound

Theorem (Coleman, 1985)

Let X be a curve of genus g and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $p > 2g$ is a prime of good reduction. Suppose $r < g$. Then

$$\#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2g - 2.$$

Remark

1. A modified statement holds for $p \leq 2g$ or for $K \neq \mathbb{Q}$.
2. This does not prove uniformity (since the first good p might be large).

Tools

p-adic integration and Riemann–Roch

Chabauty's method

(*p*-**adic integration**) There exists $V \subset H^0(X_{\mathbb{Q}_p}, \Omega^1_X)$ with $\dim_{\mathbb{Q}_p} V \geq g - r$ such that

$$\int_P^Q \omega = 0 \qquad \forall P, Q \in X(\mathbb{Q}), \omega \in V.$$

(**Coleman, via Newton Polygons**) Number of zeroes in a residue disc D_P is $\leq 1 + n_P$, where $n_P = \#(\operatorname{div}\omega \cap D_P)$

(**Riemann–Roch**) $\sum n_P = 2g - 2$.

(**Coleman's bound**) $\sum_{P \in X(\mathbb{F}_p)}(1 + n_P) = \#X(\mathbb{F}_p) + 2g - 2$.

Example (from McCallum–Poonen's survey paper)

Example

$$X\colon y^2 = x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1$$

1. Points P_t reducing mod 3 to $\widetilde{Q} = (0,1)$ are given by

$$\begin{aligned} x &= 3 \cdot t, \text{ where } t \in \mathbb{Z}_3 \\ y &= \sqrt{x^6 + 8x^5 + 22x^4 + 22x^3 + 5x^2 + 6x + 1} = 1 + x^2 + \cdots \end{aligned}$$

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Stoll's hyperelliptic uniformity theorem

Theorem (Stoll, 2013)

Let X be a hyperelliptic curve of genus g and let $r = \operatorname{rank}_{\mathbb{Z}} \operatorname{Jac}_X(\mathbb{Q})$. Suppose $r < g - 2$.

Then

$$\#X(\mathbb{Q}) \leq 8(r+4)(g-1) + \max\{1, 4r\} \cdot g$$

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Comments

Corollary ((Partially) effective Manin-Mumford)

There is an effective constant $N(g)$ such that if $g(X) = g$, then

$$\#\left(X \cap \mathrm{Jac}_{X,tors}\right)(\mathbb{Q}) \leq N(g)$$

Corollary

There is an effective constant $N'(g)$ such that if $g(X) = g > 3$ and $X/\mathbb{Q}$ has totally degenerate, trivalent reduction mod 2, then

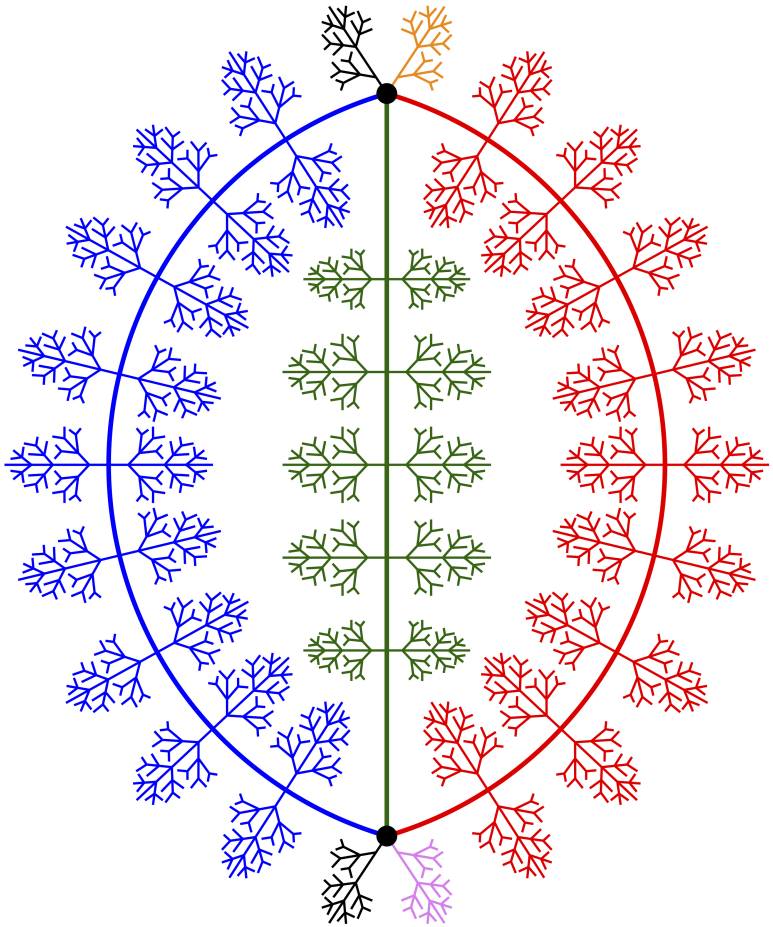
$$\#\left(X \cap \mathrm{Jac}_{X,tors}\right)(\mathbb{C}) \leq N'(g)$$

The second corollary is a big improvement

1. It requires working over a non-discretely valued field.
2. The bound only depends on the reduction type.
3. Integration over wide opens (c.f. Coleman) instead of discs and annuli.

Berkovich picture

X^{an}



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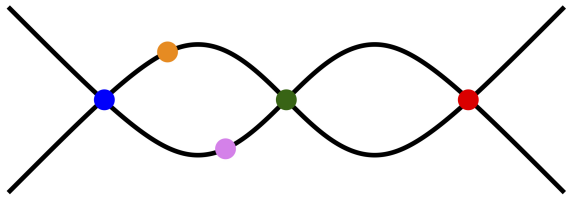
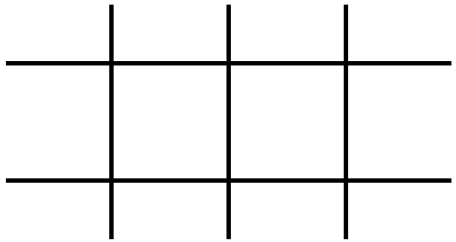
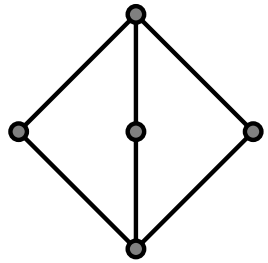


$\mathcal{X}_k$

Baker–Payne–Rabinoff and the slope formula

(**Dual graph Γ of $X_{\mathbb{F}_p}$**)





(**Contraction Theorem**) $\tau\colon X^{\mathrm{an}} \to \Gamma$.

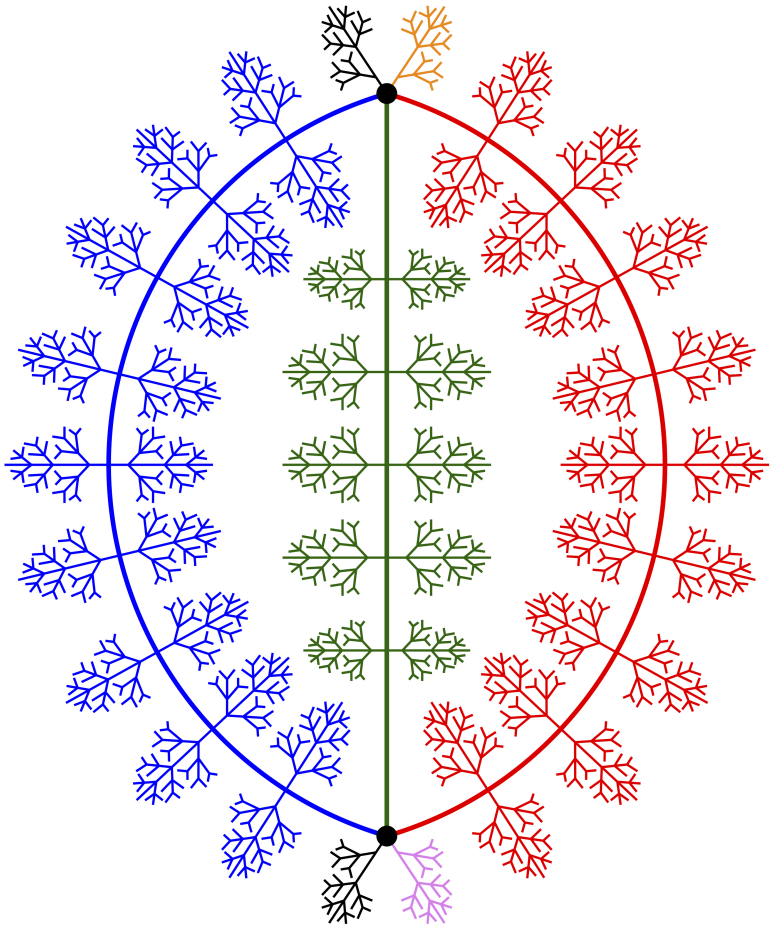
(**Combinatorial harmonic analysis/potential theory**)

f	a meromorphic function on X^{an}
$F := (-\log\lvert f\rvert)\big\vert_\Gamma$	associated tropical, piecewise linear function
$\operatorname{div} F$	combinatorial record of the slopes of F

(**Slope formula**) $\tau_* \operatorname{div} f = \operatorname{div} F$

Berkovich picture

X^{an}



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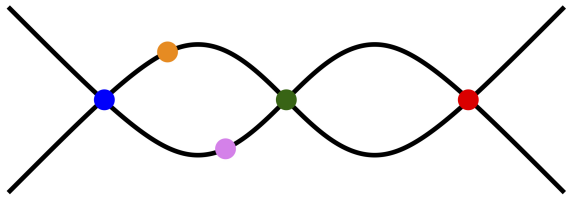

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