Progress on Mazur's program B

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Explicit Methods in Number Theory
Oberwolfach

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Background - Galois Representations

$$G_{\mathbb{Q}} := \mathrm{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$$

$$E[n](\overline{\mathbb{Q}}) \cong (\mathbb{Z}/n\mathbb{Z})^2$$

$$\begin{aligned}
\rho_{E,n}\colon \quad G_{\mathbb{Q}} \to \mathrm{Aut}\, E[n] \quad &\cong \quad \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z}) \\
\rho_{E,\ell^\infty}\colon \quad G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_\ell) \quad &= \quad \varprojlim_n \mathrm{GL}_2\left(\mathbb{Z}/\ell^n\mathbb{Z}\right) \\
\rho_{E}\colon \quad G_{\mathbb{Q}} \to \mathrm{GL}_2(\widehat{\mathbb{Z}}) \quad &= \quad \varprojlim_n \mathrm{GL}_2\left(\mathbb{Z}/n\mathbb{Z}\right)
\end{aligned}$$

Background - Image of Galois

$$\rho_{E,n} \colon G_{\mathbb{Q}} \twoheadrightarrow H(n) \hookrightarrow \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

$$G_{\mathbb{Q}}\left\{\begin{array}{c}\overline{\mathbb{Q}}\\ \Big|\\ \left.\begin{array}{c}\overline{\mathbb{Q}}^{\ker\rho_{E,n}} = \mathbb{Q}(E[n])\\ \Big|\\ \mathbb{Q}\end{array}\right\}H(n)\end{array}\right.$$

Problem (Mazur's "program B")

Classify all possibilities for $H(n)$.

Example - torsion on an ellitpic curve

If E has a K-rational **torsion point** $P \in E(K)[n]$ (of exact order n) then:

$$H(n) \subset \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$$

since for $\sigma \in G_K$ and $Q \in E(\overline{K})[n]$ such that $E(\overline{K})[n] \cong \langle P, Q \rangle$,

$$\begin{aligned} \sigma(P) &= P \\ \sigma(Q) &= a_\sigma P + b_\sigma Q \end{aligned}$$

Example - Isogenies

If E has a K-rational, **cyclic isogeny** $\phi\colon E \to E'$ with $\ker\phi = \langle P \rangle$ then:

$$H(n) \subset \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

since for $\sigma \in G_K$ and $Q \in E(\overline{K})[n]$ such that $E(\overline{K})[n] \cong \langle P, Q \rangle$,

$$\begin{aligned} \sigma(P) &= a_\sigma P \\ \sigma(Q) &= b_\sigma P + c_\sigma Q \end{aligned}$$

Example - other maximal subgroups

Normalizer of a split Cartan:

$$N_{\mathrm{sp}} = \left\langle \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$$

$H(n) \subset N_{\mathrm{sp}}$ and $H(n) \not\subset C_{\mathrm{sp}}$ iff

- there exists an unordered pair $\{\phi_1, \phi_2\}$ of cyclic isogenies,
- neither of which is defined over K
- but which are both defined over some quadratic extension of K
- and which are Galois conjugate.

$$\rho_{E,n} \colon G_{\mathbb{Q}} \twoheadrightarrow H(n) \hookrightarrow \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})$$

$$G_{\mathbb{Q}} \left\{ \begin{array}{l} \begin{array}{c} \overline{\mathbb{Q}} \\ | \end{array} \\ \left. \begin{array}{c} \overline{\mathbb{Q}}^{\ker \rho_{E,n}} = \mathbb{Q}(E[n]) \\ | \\ \mathbb{Q} \end{array} \right\} H(n) \end{array} \right.$$

Problem (Mazur's "program B")

Classify all possibilities for $H(n)$.

Modular curves

Definition

- $X(N)(K) := \{(E/K, P, Q) : E[N] = \langle P, Q\rangle\} \cup \{\text{cusps}\}$
- $X(N)(K) \ \ni \ (E/K, P, Q) \Leftrightarrow \rho_{E,N}(G_K) = \{I\}$

Definition

$\Gamma(N) \subset H \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$ (finite index)

- $X_H := X(N)/\widetilde{H}$
- $X_H(K) \ni (E/K, \iota) \Leftrightarrow H(N) \subset H \mod N$

Stacky disclaimer

This is only true up to twist; there are some subtleties if

1. $j(E) \in \{0, 12^3\}$ (plus some minor group theoretic conditions), or
2. if $-I \in H$.

Rational Points on modular curves

Mazur's program B

Compute $X_H(\mathbb{Q})$ for all H.

Remark

- Sometimes $X_H \cong \mathbb{P}^1$ or elliptic with rank $X_H(\mathbb{Q}) > 0$.
- Some X_H have *sporadic* points.
- Can compute $g(X_H)$ group theoretically (via Riemann–Hurwitz).

Fact

$$g(X_H), \gamma(X_H) \to \infty \text{ as } \left[\mathrm{GL}_2(\widehat{\mathbb{Z}}) : H\right] \to \infty.$$

Mazur's Program B

As presented at Modular functions in one variable V in Bonn

Theorem 1 also fits into a general program:

B. Given a number field K and a subgroup H of $GL_2\hat{\mathbb{Z}} = \prod_p GL_2 \mathbb{Z}_p$ classify all elliptic curves $E_{/K}$ whose associated Galois representation on torsion points maps $\mathrm{Gal}(\overline{K}/K)$ into $H \subset GL_2\hat{\mathbb{Z}}$.

Mazur - Rational points on modular curves (1977)

Sample subgroup (Serre)

$$\begin{array}{ccccccc} \ker\phi_2 & \subset & H(8) & \subset & \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) & & \dim_{\mathbb{F}_2}\ker\phi_2 = 3 \\ & & \downarrow{\scriptstyle \phi_2} & & \downarrow & & \\ I + 2M_2(\mathbb{Z}/2\mathbb{Z}) & \subset & H(4) & = & \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) & & \dim_{\mathbb{F}_2}\ker\phi_1 = 4 \\ & & \downarrow{\scriptstyle \phi_1} & & \downarrow & & \\ & & H(2) & = & \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) & & \end{array}$$

$\chi\colon \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/8\mathbb{Z})^* \to \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/8\mathbb{Z})^* \cong \mathbb{F}_2^3.$

$\chi = \mathrm{sgn} \times \det$

$H(8) := \chi^{-1}(G),\ G \subset \mathbb{F}_2^3.$

Sample subgroup (Dokchitser^2)

$$\begin{array}{ccccc} \langle I + 2E_{1,1}, I + 2E_{2,2} \rangle & \subset & H(4) & \subset & \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) \\ & & \downarrow & & \downarrow \\ & & H(2) & = & \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) \end{array}$$

$\dim_{\mathbb{F}_2} \ker \phi_1 = 2$

$$H(2) = \left\langle \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\rangle \cong \mathbb{F}_3 \rtimes D_8.$$

$$\operatorname{im} \rho_{E,4} \subset H(4) \Leftrightarrow j(E) = -4t^3(t+8).$$

$$X_H \cong \mathbb{P}^1 \xrightarrow{j} X(1).$$

A typical subgroup

$$\begin{array}{ccccccc}
\ker\phi_4 & \subset & H(32) & \subset & \mathrm{GL}_2(\mathbb{Z}/32\mathbb{Z}) & & \dim_{\mathbb{F}_2}\ker\phi_4 = 4 \\
& & \downarrow{\scriptstyle\phi_4} & & \downarrow & & \\
\ker\phi_3 & \subset & H(16) & \subset & \mathrm{GL}_2(\mathbb{Z}/16\mathbb{Z}) & & \dim_{\mathbb{F}_2}\ker\phi_3 = 3 \\
& & \downarrow{\scriptstyle\phi_3} & & \downarrow & & \\
\ker\phi_2 & \subset & H(8) & \subset & \mathrm{GL}_2(\mathbb{Z}/8\mathbb{Z}) & & \dim_{\mathbb{F}_2}\ker\phi_2 = 2 \\
& & \downarrow{\scriptstyle\phi_2} & & \downarrow & & \\
\ker\phi_1 & \subset & H(4) & \subset & \mathrm{GL}_2(\mathbb{Z}/4\mathbb{Z}) & & \dim_{\mathbb{F}_2}\ker\phi_1 = 3 \\
& & \downarrow{\scriptstyle\phi_1} & & \downarrow & & \\
& & H(2) & = & \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) & &
\end{array}$$

Non-abelian entanglements

There exists a surjection $\theta\colon \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z})$.

$$\begin{tikzcd}
 & H(6) := \Gamma_\theta \arrow[dddl] \arrow[dddr] & \\
 & \cap & \\
 & \mathrm{GL}_2(\mathbb{Z}/6\mathbb{Z}) \arrow[dl] \arrow[dr] & \\
\mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) & & \mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})
\end{tikzcd}$$

$$\operatorname{im} \rho_{E,6} \subset H(6) \Rightarrow K(E[2]) \subset K(E[3])$$

Classification of Images - Mazur's Theorem

Theorem

Let E be an elliptic curve over $\mathbb{Q}$. Then for $\ell > 11$, $E(\mathbb{Q})[\ell] = \{0\}$.

In other words, for $\ell > 11$ the mod ℓ image is not contained in a subgroup conjugate to

$$\begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}.$$

Classification of Images - Mazur; Bilu, Parent, Rebolledo

Theorem (Mazur)

Let E be an elliptic curve over $\mathbb{Q}$ without CM. Then for $\ell > 37$ the mod ℓ image is not contained in a subgroup conjugate to

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

Theorem (Bilu, Parent, Rebolledo)

Let E be an elliptic curve over $\mathbb{Q}$ without CM. Then for $\ell > 13$ the mod ℓ image is not contained in a subgroup conjugate to

$$\left\langle \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle.$$

Main conjecture

Conjecture (Serre)

Let E be an elliptic curve over $\mathbb{Q}$ without CM. Then for $\ell > 37$, $\rho_{E,\ell}$ is surjective.

Serre's Open Image Theorem

Theorem (Serre, 1972)

Let E be an elliptic curve over K without CM. The image of ρ_E

$$\rho_E(G_K) \subset \mathrm{GL}_2(\widehat{\mathbb{Z}})$$

is open.

Note:

$$\mathrm{GL}_2(\widehat{\mathbb{Z}}) \cong \prod_p \mathrm{GL}_2(\mathbb{Z}_p)$$

"Vertical" image conjecture

Conjecture

There exists a constant N such that for every $E/\mathbb{Q}$ without CM

$$\left[\mathrm{GL}_2(\widehat{\mathbb{Z}}) : \rho_E(G_{\mathbb{Q}})\right] \leq N.$$

Remark

This follows from the "$\ell > 37$" conjecture.

Problem

Assume the "$\ell > 37$" conjecture and compute N.

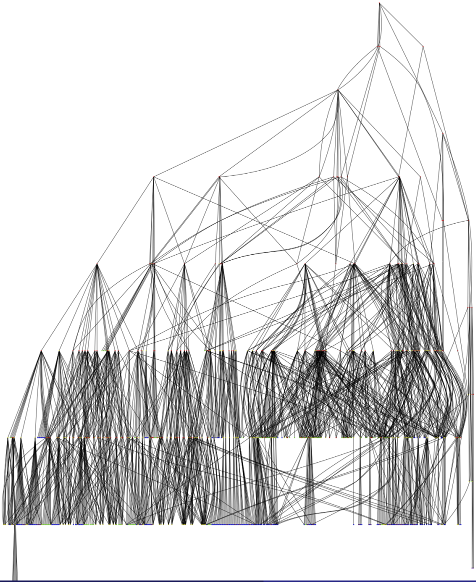
Main Theorem

Rouse, ZB (2-adic)

The index of $\rho_{E,2^\infty}(G_{\mathbb{Q}})$ divides 64 or 96; all such indices occur.

1. All indices dividing 96 occur infinitely often; 64 occurs only twice.
2. The 2-adic image is determined by the mod 32 image
3. 1208 different images can occur for non-CM elliptic curves
4. There are 8 "sporadic" subgroups.

Subgroups of $GL_2(\mathbb{Z}_2)$



Cremona Database, 2-adic images

Index, # **of isogeny classes**

1 , 727995

2 , 7281

3 , 175042

4 , 1769

6 , 57500

8 , 577

12 , 29900

16 , 235

24 , 5482

32 , 20

48 , 1544

64 , 0 (two examples)

96 , 241 (first example - $X_0(15)$)

CM , 1613

Cremona Database

Index, $\#$ **of isogeny classes**

64 , 0

$j = -3 \cdot 2^{18} \cdot 5 \cdot 13^3 \cdot 41^3 \cdot 107^3 \cdot 17^{-16}$

$j = -2^{21} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13^3 \cdot 23^3 \cdot 41^3 \cdot 179^3 \cdot 409^3 \cdot 79^{-16}$

Rational points on $X_{ns}^{+}(16)$ (Heegner, Baran)

Applications

Theorem (R. Jones, Rouse, ZB)

1. **Arithmetic dynamics**: *let* $P \in E(\mathbb{Q})$.
2. *How often is the order of* $\widetilde{P} \in E(\mathbb{F}_p)$ *odd?*
3. *Answer depends on* $\rho_{E,2^\infty}(G_{\mathbb{Q}})$.
4. *Examples:* $11/21$ *(generic),* $121/168$ *(maximal),* $1/28$ *(minimal)*

Theorem (Various authors)

Computation of $S_{\mathbb{Q}}(d)$ *for particular* d.

Theorem (Daniels, Lozano-Robledo, Najman, Sutherland)

Classification of $E(\mathbb{Q}(3^\infty))_{\mathrm{tors}}$

Theorem

Gonzalez–Jimenez, Lozano–Robledo Classify $E/\mathbb{Q}$ *with* $\rho_{E,N}(G_{\mathbb{Q}})$ *abelian.*

More applications

Theorem (Sporadic points)

Najman's example $X_1(21)^{(3)}(\mathbb{Q})$*; "easy production" of other examples.*

Theorem (Jack Thorne)

Elliptic curves over $\mathbb{Q}_\infty$ *are modular.*
(One step is to show $X_0(15)(\mathbb{Q}_\infty) = X_0(15)(\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$*.)*

Recent theorems

Zywina (mod ℓ)

Classifies $\rho_{E,\ell}(G_{\mathbb{Q}})$ (modulo some conjectures).

Zywina (indices occuring infinitely often; modulo conjectures)

The **index** of $\rho_{E,N}(G_{\mathbb{Q}})$ divides $220, 336, 360, 504, 864, 1152, 1200, 1296$ or 1536.

Sutherland–Zywina

Parametrizations in all **prime power** levels, $g = 0$ and $g = 1, r > 0$ cases.

Brau–N. Jones, N. Jones–McMurdy (in progress)

Equations for X_H for entanglement groups H.

In progress

Morrow; Camacho–Li–Morrow–Petok–ZB (composite level)

Classifies $\rho_{E,\ell_1^n \cdot \ell_2^m}(G_{\mathbb{Q}})$ (partially).

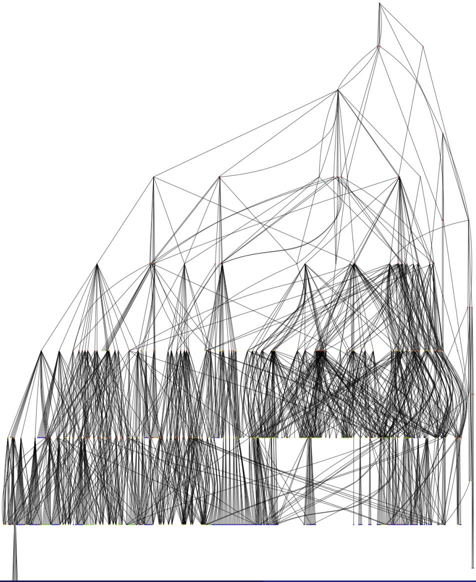
Rouse–ZB for other prime powers (in progress)

Partial progress; e.g. for $N = 3^n$.

Proof template

1. Compute all arithmetically minimal $H \subset \mathrm{GL}_2(\mathbb{Z}_2)$
2. Compute equations for each X_H
3. Find (with proof) all rational points on each X_H.

Subgroups of $GL_2(\mathbb{Z}_2)$



Finding Equations – Basic idea

1. The canoncial map $C \hookrightarrow \mathbb{P}^{g-1}$ is given by $P \mapsto [\omega_1(P) : \cdots : \omega_g(P)]$.
2. For a general curve, this is an embedding, and the relations are quadratic.
3. For a modular curve,

$$M_k(H) \cong H^0(X_H, \Omega^1(\Delta)^{\otimes k/2})$$

given by

$$f(z) \mapsto f(z)\, dz^{\otimes k/2}.$$

Equations – Example: $X_1(17) \subset \mathbb{P}^4$

$$
\begin{aligned}
q - 11q^5 + 10q^7 + O(q^8) \\
q^2 - 7q^5 + 6q^7 + O(q^8) \\
q^3 - 4q^5 + 2q^7 + O(q^8) \\
q^4 - 2q^5 + O(q^8) \\
q^6 - 3q^7 + O(q^8)
\end{aligned}
$$

$$
\begin{aligned}
xu + 2xv - yz + yu - 3yv + z^2 - 4zu + 2u^2 + v^2 &= 0 \\
xu + xv - yz + yu - 2yv + z^2 - 3zu + 2uv &= 0 \\
2xz - 3xu + xv - 2y^2 + 3yz + 7yu - 4yv - 5z^2 - 3zu + 4zv &= 0
\end{aligned}
$$

Equations – general

1. $H' \subset H$ of index 2, $X_{H'} \to X_H$ degree 2.
2. Given equations for X_H, compute equations for $X_{H'}$.
3. Compute a new modular form on H', compute (quadratic) relations between this and modular forms on H.
4. **Main technique** – if $X_{H'}$ has "new cusps", then write down Eisenstein series which vanish at "one new cusp, not others".

Rational points rundown, $\ell = 2$

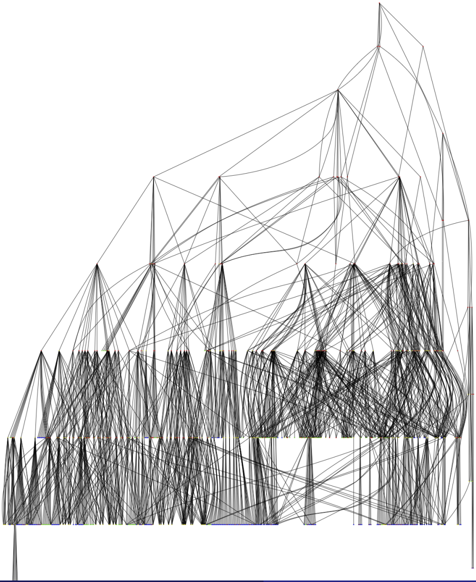
318 curves (excluding pointless conics)

Genus	0	1	2	3	5	7
Number	175	52	56	18	20	4
Rank of Jacobian						
0		25	46	–	–	??
1		27	3	9	10	??
2			7	–	–	??
3				9	–	??
4					–	??
5					10	??

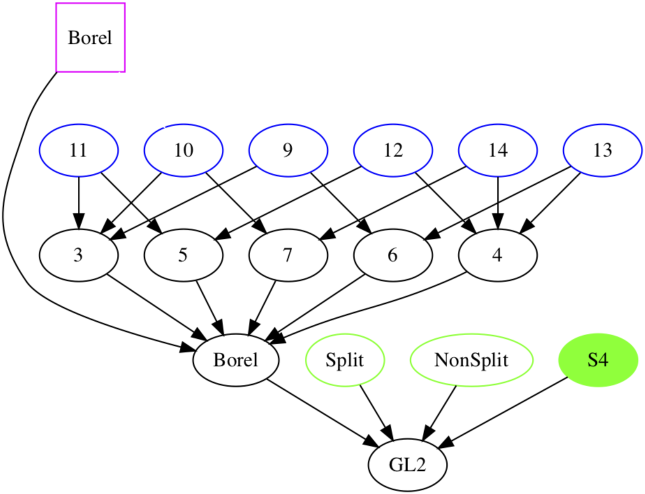
More 2-adic facts

1. There are 8 "sporadic" subgroups
   1. Only one genus 2 curve has a sporadic point
   2. Six genus 3 curves each have a single sporadic point
   3. The genus 1, 5, and 7 curves have no sporadic points
2. Many accidental isomorphisms of $X_H \cong X_{H'}$.
3. There is one H such that $g(X_H) = 1$ and $X_H \in X_H(\mathbb{Q})$.

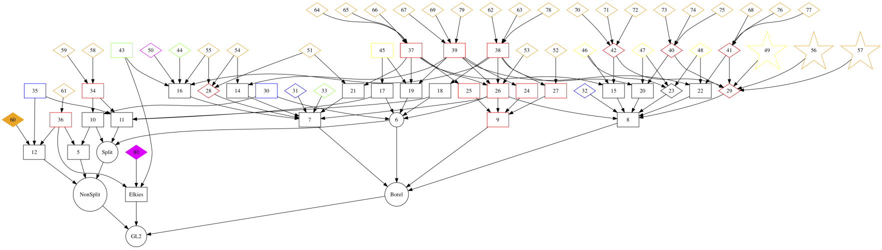
Subgroups of $GL_2(\mathbb{Z}_2)$

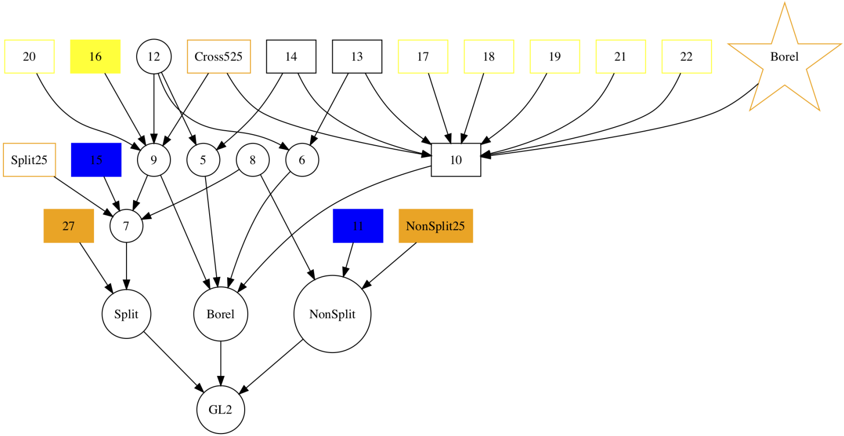


Subgroups of $\mathrm{GL}_2(\mathbb{Z}_{13})$

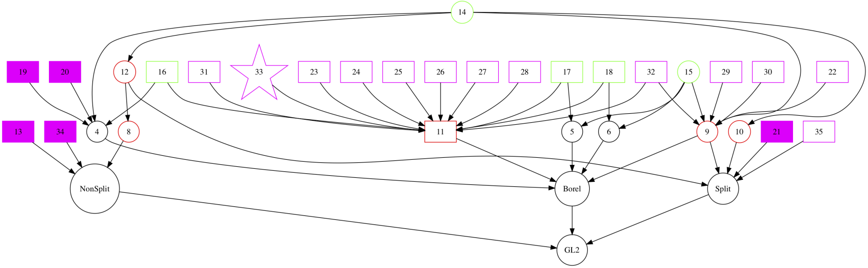


Subgroups of $GL_2(\mathbb{Z}_3)$

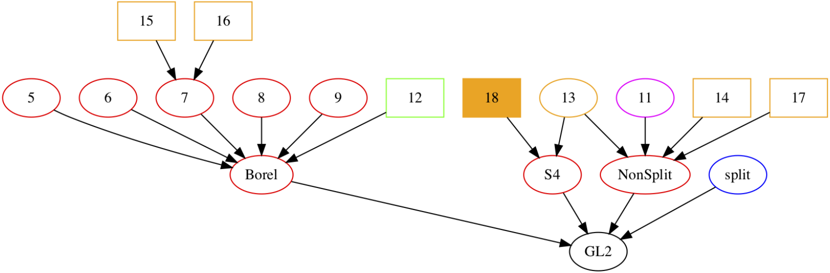



20
16
12
Cross525
14
13
17
18
19
21
22
Borel
Split25
15
9
5
8
6
10
27
7
11
NonSplit25
Split
Borel
NonSplit
GL2

Subgroups of $GL_2(\mathbb{Z}_7)$



Subgroups of $GL_2(\mathbb{Z}_{11})$



Rational Points rundown: $\ell = 3$

3	$g = 0$	Handled by Sutherland–Zywina
	$g = 1$	all rank zero
	$g = 4$	map to $g = 1$
	$g = 2$	Chabauty works
	$g = 4$	no 3-adic points
	$g = 3$	Picard curves; map to rank 0 AV
	$g = 4$	Admits étale triple cover
	$g = 6$	Admits étale triple cover
	$g = 12$	gonality ≤ 9, plane model, degree 121
	$g = 43$	New ideas needed

Rational Points rundown: $\ell = 5$

5	$g = 0$ (10 level 5, 3 level 25)	All level 5 curves are genus 0
	$g = 4$ (4 level 25)	No 5-adic points
	$g = 8, 22$	known (e.g., $X_{\text{ns}}(25)$)
	$g = 2$ (2 level 25)	Rank 2, A_5 mod 2 image
	$g = 4$ (3 level 25)	All isomorphic.
		Each has 5 rational points
		Each admits an order 5 aut
		Simple Jacobian
	$g = 14, 36$ (levels 25 and 125)	No models (or ideas, yet)

Rational Points rundown: $\ell = 7$

7	$g = 1, 3$	[Z, 4.4] handles these, $X_H(\mathbb{Q})$ is finite.
	$g = 19, 26$, level 49	Maps to one of the 6 above
	$g = 1$, level 49	[SZ] handles this one (rank 0)
	$g = 3, 19, 26$, level 49, 343	Map to curve on previous line
	$g = 12$, level 49	Handled by Greenberg–Rubin–Silverberg–Stoll
	$g = 94$	Known ($X_{\text{ns}}(49)$)
	$g = 9, 12, 69$	No models (or ideas, yet)

Rational Points rundown: $\ell = 11$

11	all maximal are genus one	
	only positive rank is $X_{ns}(11)$	
	All but one are ruled out by Zywina	some have sporadic points;
		[Z, Theorem 1.6]
	$g = 5$, level 11	[Z, Lemma 4.5]
	$g = 5776$, level 121	"Challenge..."

Rational Points rundown: $\ell = 13$

Zywina handles all level 13 except for the cursed curve

13	$g = 2, 3$, level 13 (8 total)	
	$g = 8$, level 169	$X_0(13^2)$, handled by Kenku
	$X_{ns}(13)$	Cursed. Genus 3, rank 3. No torsion. Some points Probably has maximal mod 2 image Solved! [BDMTV]
	$X_{S_4}(13)$	Also cursed.

Rational Points: summary of remaining work.

3	$g = 12, 43$
5	$g = 2, 4, 14, 36$
7	$g = 9, 12, 69$
11	a single genus 5776 curve remains
13	$X_{S_4}(13)$

Explicit methods: highlight reel

- Local methods
- Chabauty
- Elliptic Chabauty
- Mordell–Weil sieve
- étale descent
- Pryms
- **Equationless descent via group theory.**
- **New techniques for computing** $\operatorname{Aut} C$.

Pryms (via Nils Bruin)

$$\begin{tikzcd} D \arrow[r, "\iota - \mathrm{id} - (\iota(P) - P)"] \arrow[d, "\mathrm{et}"'] \arrow[loop below, "\iota"] & \ker_0(J_D \to J_C) =: \mathrm{Prym}(D \to C) \\ C & \end{tikzcd}$$

$$C(\mathbb{Q}) = \bigcup_{\delta \in \{\pm 1, \pm 2\}} \operatorname{im} D_\delta(\mathbb{Q})$$

Pryms

$$\begin{tikzcd} D \arrow[r, "\iota - \mathrm{id} - (\iota(P) - P)"] \arrow[d, "\mathrm{et}"'] \arrow[loop below, "\iota"] & \ker_0(J_D \to J_C) =: \mathrm{Prym}(D \to C) \\ C \end{tikzcd}$$

Example (Genus $C = 3 \Rightarrow$ Genus $D = 5$)

- $C\colon Q(x,y,z) = 0$
- $Q = Q_1 Q_3 - Q_2^2$.

$$\begin{aligned} D_\delta\colon\ Q_1(x,y,z) &= \delta u^2 \\ Q_2(x,y,z) &= \delta uv \\ Q_3(x,y,z) &= \delta v^2 \end{aligned}$$

- $\mathrm{Prym}(D_\delta \to C) \cong \mathrm{Jac}_{H_\delta}$,
- $H_\delta\colon y^2 = -\delta \det(M_1 + 2xM_2 + x^2 M_3)$.

Thanks

Thank you!