Moduli Spaces + Arithmetic Stastics

"A Predicted Distribution for Galois Groups of Max'l unramified extns"

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fix Γ (f. gp)

K^{ur} — Max'l unram. extn of K

Q: How does K^{ur} ram?. How to "model"?

$K^{ur, 2|\Gamma|'}$

H — Hilb class field

$\Big| \Big) \simeq \mathrm{Cl}\,K$

(ram) K — Cohen-Lenstra + C-Martinet: How is $\mathrm{Cl}(K)$ distributed

$\Big| \Big) \Gamma$

$\mathbb{Q}$

$$\mathbb{Q} = \mathbb{Q}^{ur}$$

$$K = \mathbb{Q}(\sqrt{-1367})$$

$$K_H \neq K^{ur}$$

$$K \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq K_H \leq K^{ur}$$

$$K_H = \bigcup H_i$$

$$H_i = HCF(H_{i-1})$$

Class field towers

$$[\mathbb{Q}(\zeta_{877})^{ur} : \mathbb{Q}(\zeta_{877})] = \infty$$

$K = \mathbb{Q}(\zeta_n)$ w/ @ least 8 $p|m$ w $\ell | p-1$

if $K = \mathbb{Q}(\sqrt{-D})$ &

$\operatorname{rank}(\mathrm{Cl}\,K)_2 \geq 5$

then K_H is infinite.

How to model $\mathrm{Gal}\left(K^{ur, a|\Gamma|'}/K\right)$? $=: G$

(Γ-group) $\Gamma \circlearrowright G$

(admissible) $G = \overline{\langle g^{-1}\gamma(g) : g \in G, \gamma \in \Gamma \rangle}$

$+$ order prime to $|\Gamma|$

(Property E) $\forall p \nmid a|\Gamma|$,

(non-split)

$$\forall \begin{tikzcd} 1 \arrow[r] & \mathbb{Z}/p \arrow[r] & \hat{G}' \arrow[r] & G' \arrow[r] & 1, \\ & & & G \arrow[u] \arrow[ul, dashed, "\exists"] & \circlearrowleft \Gamma \end{tikzcd}$$

THM: $\exists$ measure μ_Γ on set of isoclasses of such groups.

Conj: $\mathrm{Gal}(K^{ur, \partial|\Gamma|'}/K)$ equidistributes w.r.to μ_Γ.

THM: Conj is true for moments over $\mathbb{F}_q(t)$, as $q \to \infty$.

Function field analogy

$K/\mathbb{Q}$	$K/\mathbb{F}_q(t)$ $\longleftrightarrow$	$K = \mathbb{F}_q(C)$
$\operatorname{Spec} \mathcal{O}_K$	C	
$\operatorname{Cl} \mathcal{O}_K$	$\operatorname{Pic}^0 C$	
$\lvert\Delta_K\rvert$	q^{g-1}	$g = g(C)$
different	ram. div. of $\begin{matrix} C \\ \downarrow \\ \mathbb{P}^1 \end{matrix}$	

See Math.mit.edu/~poonen/papers/curves.pdf

Why Bother ?

$\mathrm{Jac}_C := \mathrm{Pic}^\circ C$ is a <u>variety</u>

$$J(\mathbb{F}_q^n) = \mathrm{Pic}^\circ C_{\mathbb{F}_q^n}$$

$$\| $$

$$J(\overline{\mathbb{F}}_q^n)^{\mathrm{Frob}^n}$$

Weil Conj's

$$\#X(\mathbb{F}_p) = \sum_{i=0}^{\infty} (-1)^i \operatorname{Tr} \operatorname{Frob} \circlearrowright H^i_{et}(X)$$

$$(\operatorname{Pic}^0 C)_\ell = \operatorname{coker}(F - I)$$

$\boxed{\text{THM}}$ (Friedman + Washington)

$$P(\operatorname{coker}(\bar{F} - I) \cong L) = \frac{1}{\# \operatorname{Aut} L}$$

$$\bar{F} \in GL_{2g}(\mathbb{Z}_\ell)$$

Geometric Analytic # Theory

Ellenberg
AWS

$sf_q(n) :=$ # of square free monic polynomials in $\mathbb{F}_q[x]$ of degree n

$$= q^n - q^{n-1}$$

Note: $$\frac{sf_q(n)}{q^n} = 1 - \frac{1}{q} = \zeta_{\mathbb{A}^1_{\mathbb{F}_q}}(2)^{-1}$$

$$\mathrm{Conf}_n X := (X^n - \Delta)/S_n$$

$$\left(\mathrm{Conf}_n \mathbb{A}^1\right)(k) \longleftrightarrow sf_k(n)$$

$$(\alpha_1, \dots, \alpha_n) \longmapsto \prod (x - \alpha_i)$$

$$Y(k) = Y(\bar{k})^{G_k}$$

Goal: $$\# \mathrm{Conf}_n \mathbb{A}^1(\mathbb{F}_q) = \sum (-1)^i \mathrm{Tr}\, \mathrm{Frob} \curvearrowright H^i_{et,c}(\mathrm{Conf}; \mathbb{Q}_\ell)$$
$$= q^n - q^{n-1}$$

THM: $$H^i(\mathrm{Conf}^n(\mathbb{C}); \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0, 1 \\ 0 & i > 1 \end{cases}$$

$$\pi_1(\mathrm{Conf}^n(\mathbb{C})) = B_n$$

$\mathrm{Aut}\, f \overset{\iota}{\simeq} G$

$$\begin{array}{c} X \\ \downarrow f \\ \mathbb{P}^1 \end{array} \longmapsto \Delta_f \qquad \mathrm{Hur}^n_{G,*} \longrightarrow \mathrm{Conf}^n \mathbb{P}^1$$

$$= \coprod_{c} \mathrm{Hur}^n_{G,c}$$

c multiset of G^*

conj. classes

(*) $\leftrightarrow$ f unram @ ∞

Main Point

$\mathbb{Q} = \mathbb{F}_q(t)$

$G = H \rtimes \Gamma$

$\mathrm{Hur}^n_{G,c}(\mathbb{F}_q) \longleftrightarrow \mathrm{Surj}\ \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \twoheadrightarrow H \rtimes \Gamma$

"THM": We can compute components of $\mathrm{Hur}^n_{G,c}$ via Braid gp action