

Moduli Spaces & Arithmetic Statistics

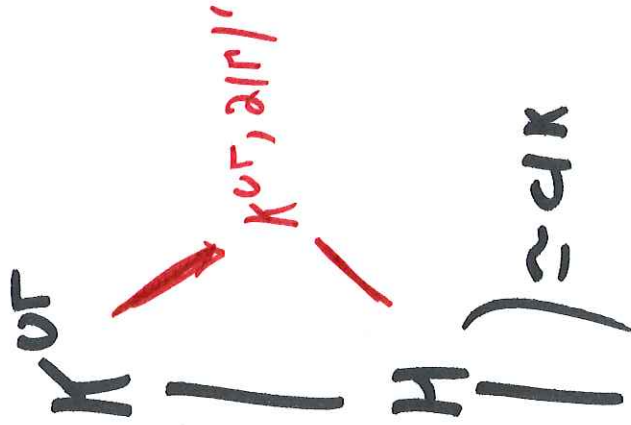
'A Predicted Distribution for Galois
Groups of Max's Unramified extns'

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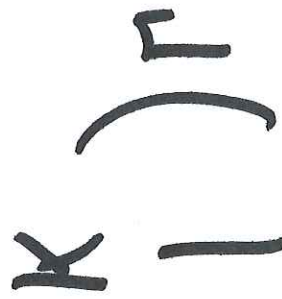
fix Γ (f. gr)

Max'l unram. extn of K



Q: How does K^{ur} vary?
How to "model"?

Hit class field



(var)

Cohen - Lenstra + C-Martinet:

How is $E(K)$ distributed

Q

$$\mathbb{Q} = \mathbb{Q}^{\text{ur}}$$

$$K = \mathbb{Q}(\sqrt{-1367})$$

$$K_H \neq K^{\text{ur}}$$

$$K \subseteq H_1 \subseteq H_2 \subseteq \dots \subseteq K_H \subseteq K^{\text{ur}}$$

$$H_i = \text{HCF}(H_{i-1}) \cup H_i$$

Class field towers

$$[\mathbb{Q}(\zeta_{877})^{\text{ur}} : \mathbb{Q}(\zeta_{877})] = \infty$$

$$K = \mathbb{Q}(\zeta_n) \quad \text{w/ } \in \text{ least } \& \text{ } p \mid n \text{ w/ } p-1$$

if $K = \mathbb{Q}(\sqrt{-D})$ &

$\text{rank}(\mathcal{O}_K)_2 \geq 5$

then K_H is infinite.

How to model Gal(K^{ur.abst}/K)?

$\Rightarrow G$
(Γ -group) $\Gamma \curvearrowright G$

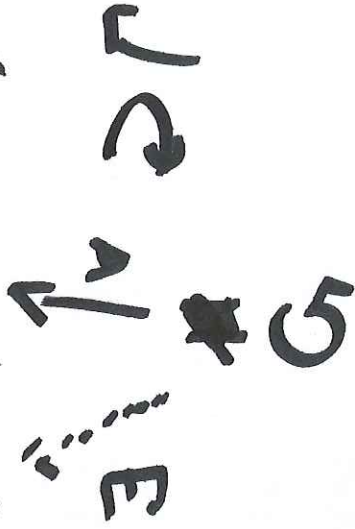
(admissible) $G = \langle \sigma_i \rangle : \sigma_i \in G, \text{ref}$

+ order prime to $|\Gamma|$

(Properly E) $\text{Aut } K/A$

(non-split)

$\Gamma \curvearrowright G \rightarrow G \rightarrow 1, A$



THM: \exists measure μ_T on set of iso classes
of such groups.

Conj: $\text{Gal}(K^{\text{ur}}, \mu_T^1/k)$ eq vidistributes

w.r. to μ_T .

THM: Conj is true for moments

over $\mathbb{F}_q(x)$, as $q \rightarrow \infty$.

Function field analogy

$$K/\mathbb{Q} \quad K/\mathbb{F}_q(t) \quad \leftrightarrow \quad K = \mathbb{F}_q(c)$$

Spec \mathbb{Q}

\mathbb{C}

Cl \mathbb{Q}

Pic \mathbb{C}

$|\Delta_K|$

g^{2g-1}

$$g = g(c)$$

different

ran. div. of $\sum_{i=1}^r \mathbb{P}^1$

See [Math. mit. edu/~poonen/papers/Curves.pdf](http://Math.mit.edu/~poonen/papers/Curves.pdf)

Why Bother?



$\text{Jac}_C := \text{Pic}$ is a variety

$$\mathfrak{J}(\mathbb{F}_q^n) = \text{Pic} \subset \mathbb{F}_q^n$$

$$\parallel \mathbb{F}_q^n$$
$$\mathfrak{J}(\overline{\mathbb{F}_q^n})$$

Weil Conjectures

$$\#X(\mathbb{F}_p) = \sum_{i=0}^{\infty} (-1)^i \text{Tr} \text{Frob}_C^i H_{2i}^1(X)$$

$$(\text{Pic}^0 C)_1 = \text{coker}(F - \mathbb{F})$$

FHM (Friedman + Washington)

$$P(\text{coker}(F - \mathbb{F}) \simeq L) = \frac{1}{\#A_{0,1}^L}$$

$$F \in \text{Gal}_{\mathbb{Q}}(\mathbb{Z}_\ell)$$

Geometric Analytic # Theorem

Ellenberg
AWS

$sf_0(n) :=$ # of square free monic polynomials in $\mathbb{F}_q[x]$
of degree n

$$= q^n - q^{n-1}$$

Note: $\frac{sf_0(n)}{q^n} = 1 - \frac{1}{q} = \sum_{\mathbb{A}_{\mathbb{F}_q}} (\alpha)^{-1}$

$$\text{Conf } X := (X^n - \Delta) / S^n$$

$$Y(k) = Y(\bar{k})^{G_K}$$

$$\text{Conf } A'(k) \xleftrightarrow{\text{diag } dn} \pi(X - \Delta)$$

$$\begin{aligned} \text{Goal: } \# \text{Conf } A'(\mathbb{F}_0) &= \sum_{i \in \mathbb{F}_0} \text{Tr } C^i H_{2c}^i(\text{Conf } i \alpha_i) \\ &= 0^{n-2c} \end{aligned}$$

$$\begin{aligned} \text{THM: } H^i(\text{Conf}^n(\mathbb{C}); \mathbb{Q}) &= \begin{cases} \mathbb{Q} & i=0 \\ 0 & i>1 \end{cases} \\ \pi_1(\text{Conf}^n(\mathbb{C})) &= \mathbb{B}_n \end{aligned}$$

Adj \hat{G}

$X \downarrow \uparrow R'$



$H_{U\tau G, *}$

\downarrow
 $Conf P'$

$$= \coprod H_{U\tau G, c}^n$$

\subset Multiset of G^{\dagger}
 \uparrow
 Conj. classes

(*) $\leftarrow \rightarrow$ f unram $\subset \infty$

Main Point

$$Q = F_0(t)$$

$$G_0 = H \times \Gamma$$

$$H_{\text{loc}}(F_0) \longleftrightarrow \text{Surj Gal}(\bar{Q}/Q) \rightarrow \rightarrow H \times \Gamma$$

"THM": We can compute components of

H_{loc} via Braid group action