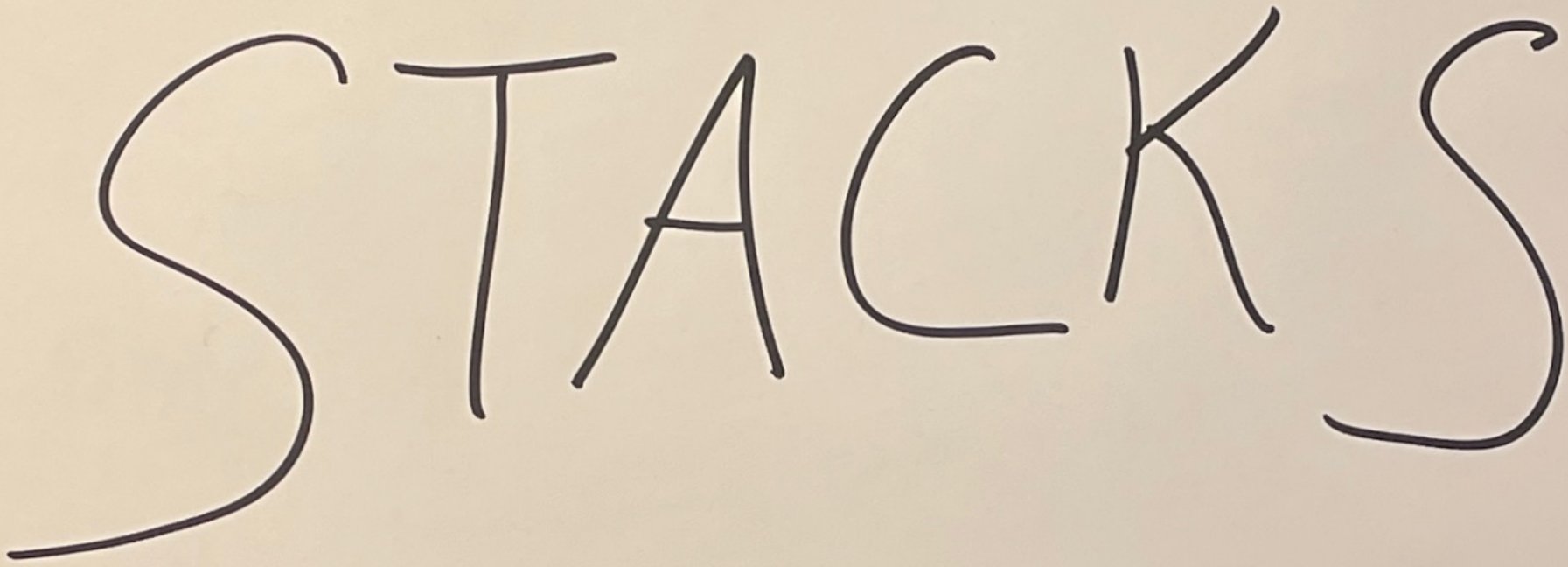

STACKS

Why stacks?

$$X = \operatorname{Spec} \mathbb{Z}[x_1, \ldots, x_n] / (f_1, \ldots, f_m)$$

Rings $\longrightarrow$ Sets

$A \longmapsto X(A) \overset{\text{def}}{=} \{ \bar{a} \in A^n \text{ s.t } \forall i,\ f_i(\bar{a}) = 0 \}$

IS

$$h_X(A) \overset{\text{def}}{=} \operatorname{Hom}(\operatorname{Spec} A, X)$$

Yoneda: $\mathrm{Sch} \longrightarrow \mathrm{Fun}(\mathrm{Sch}^{op}, \mathrm{Sets})$

$X \longmapsto h_X = \mathrm{Hom}(-, X)$

is fully faithful, i.e. $\mathrm{Hom}(X,Y) \xrightarrow{\sim} \mathrm{Hom}(h_X, h_Y)$

Fact: h_X is a sheaf.

(Zariski) Morphisms glue

(Étale) L/K Galois

$$X(K) \xrightarrow{\sim} X(L)^{\mathrm{Gal}(L/K)}$$

Many functors are not sheaves.

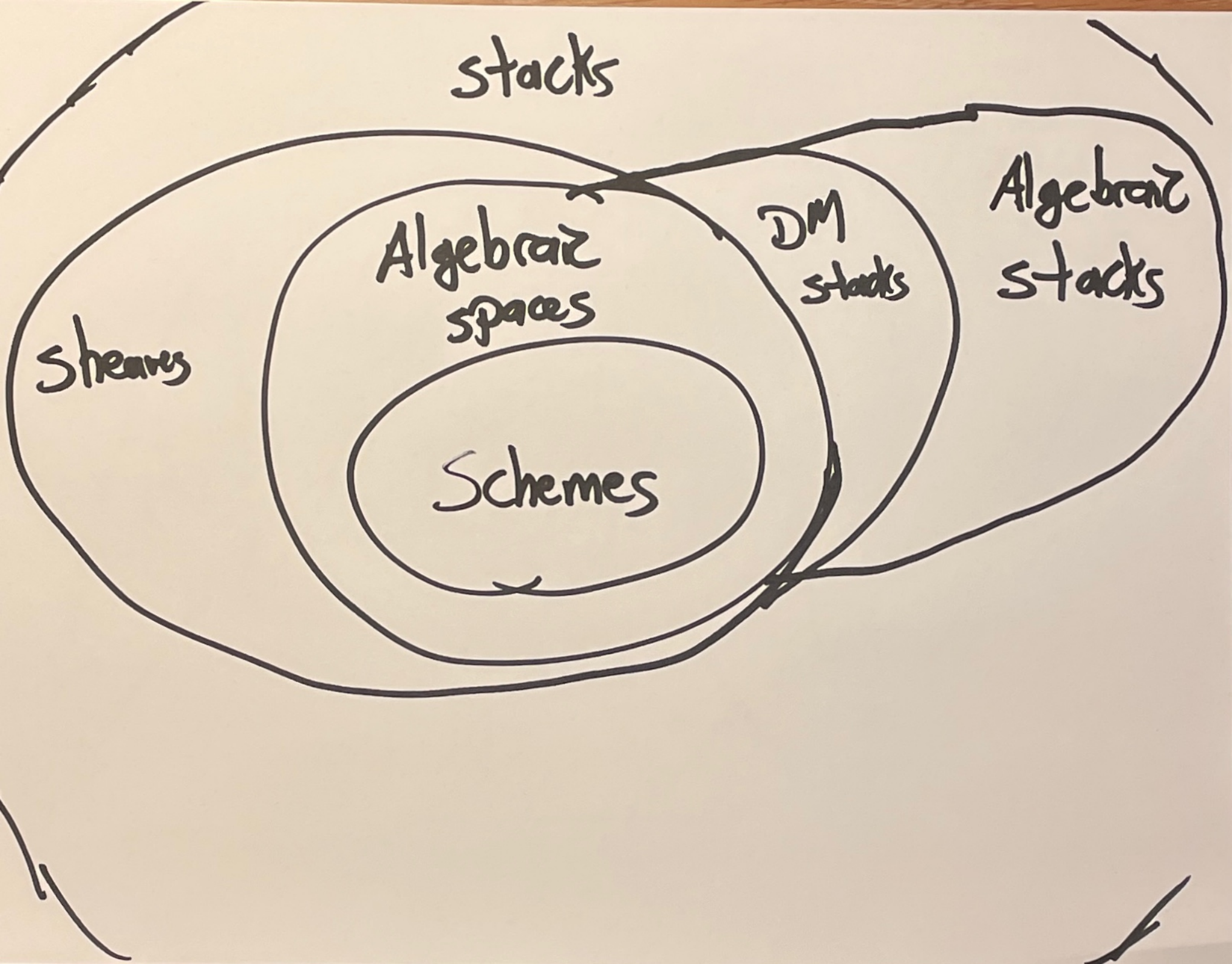
$$M_g : \mathrm{Sch} \longrightarrow \mathrm{Sets}$$

$$B \longmapsto M_g(B) \overset{\text{def}}{=} \left\{ \begin{array}{c} \text{smooth, proper genus } g \text{ curves} \\ /B \end{array} \right\} /\!\cong$$

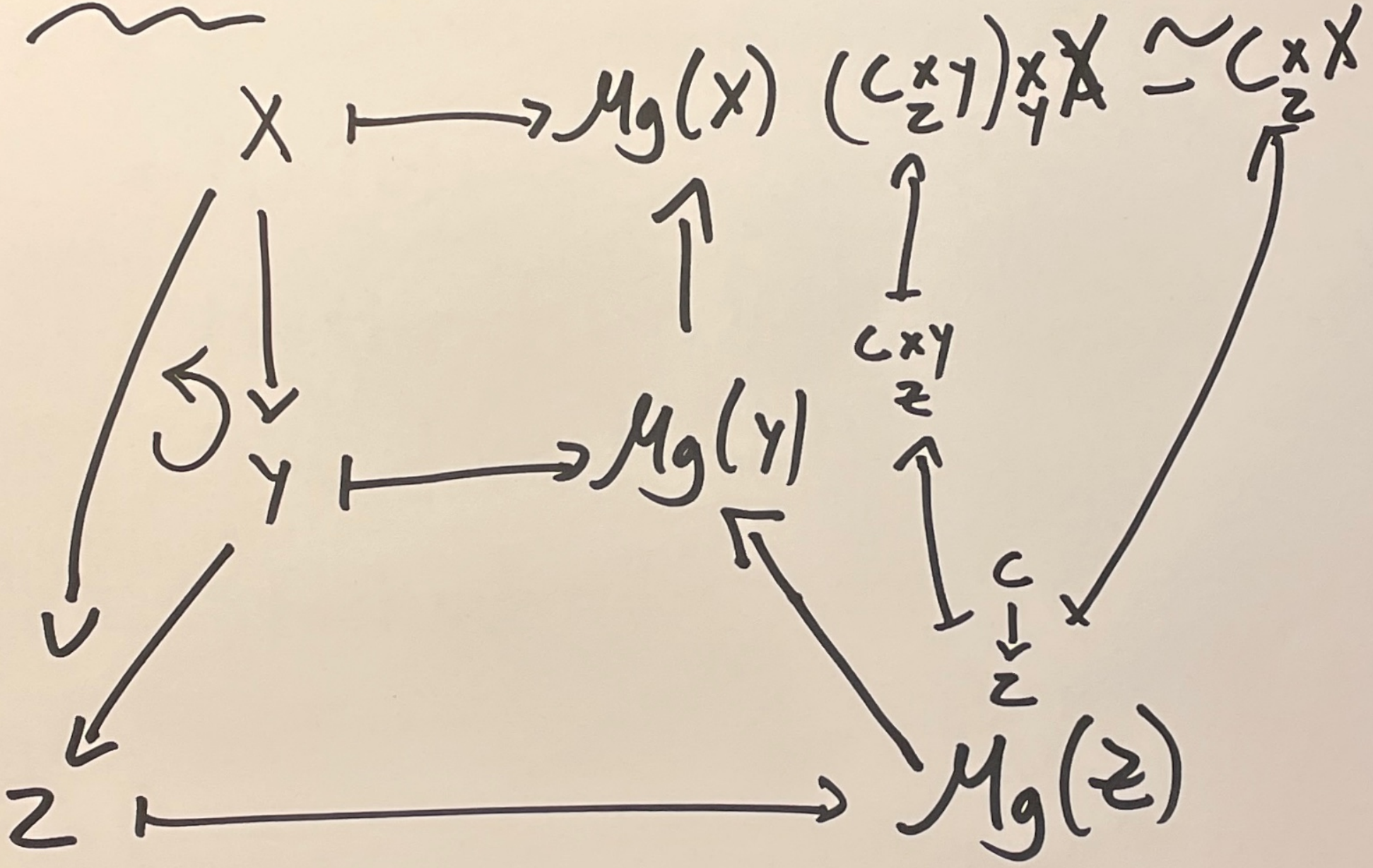
$M_g(\mathbb{C}) \longrightarrow M_g(\mathbb{C}(\sqrt{a}))$ not inj

$H: y^2 - f(x)$

$H_a: ay^2 - f(x)$


stacks
Algebraic stacks
DM stacks
Algebraic spaces
Schemes
Sheaves

Idea: slightly weaken sheaf axioms



Why bother?

- Moduli
- Better quotients
- "Fractional" points
- Stack of all "answers"

Quotients

$X = \operatorname{Spec} A$

G finite group

$G \curvearrowright X$ free action

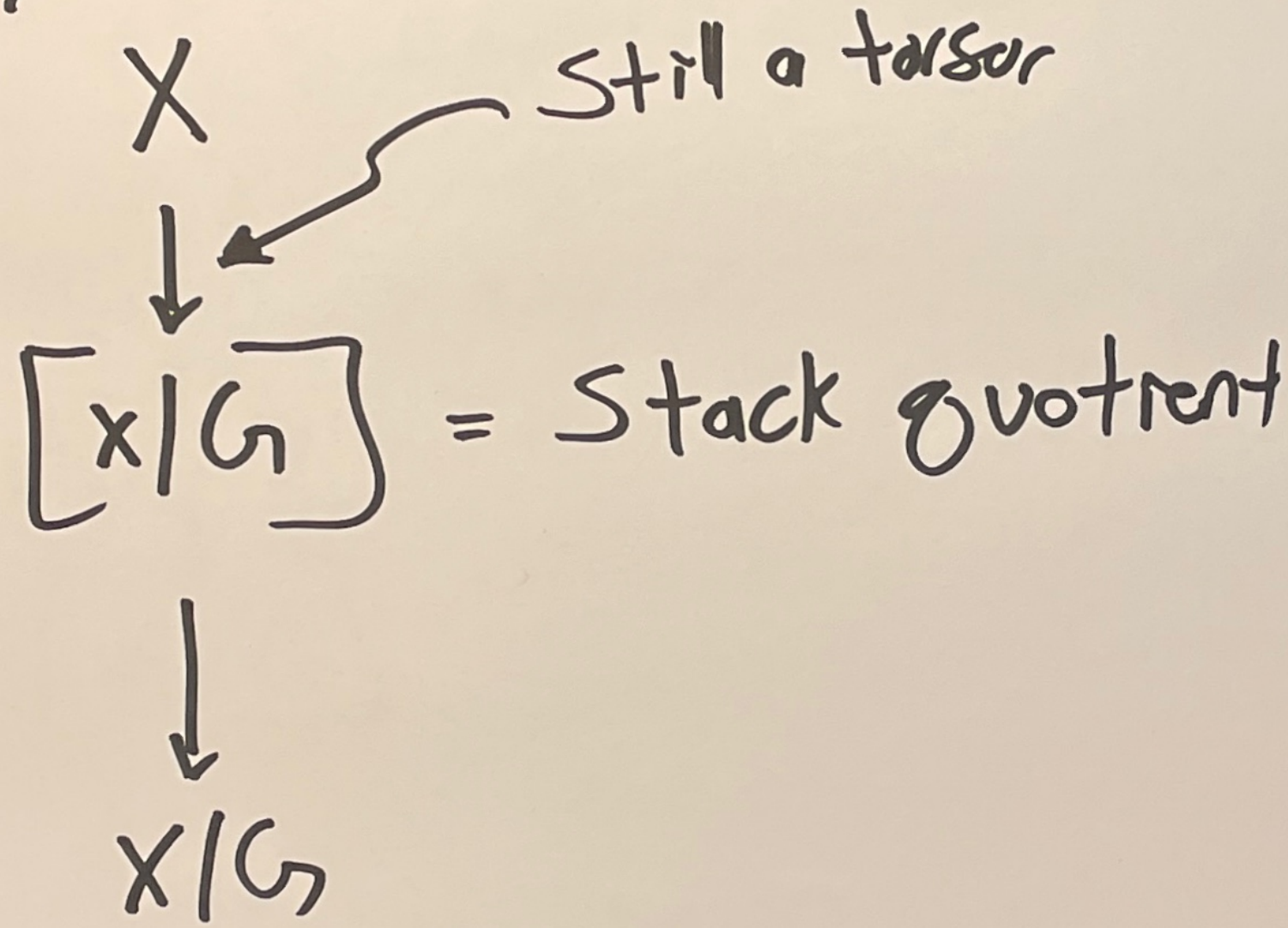
X

$\downarrow$

$X/G = \operatorname{Spec} A^G$

is a torsor

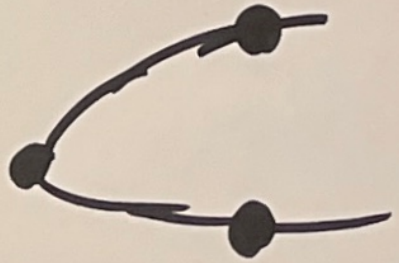
Non-free action



Example

$G = \mathbb{Z}/2\mathbb{Z}$

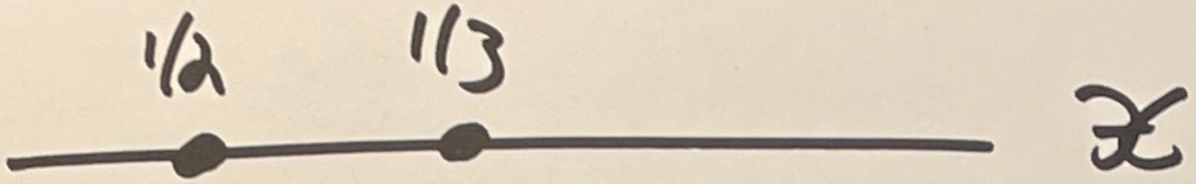
$\mathbb{A}^1 \quad z \mapsto -z$



$[\mathbb{A}^1/G]$

$\mathbb{A}^1/G = \operatorname{Spec} k[z^2] \simeq \mathbb{A}^1$

1. Stacky Curves



Problem: Local-to-global principles

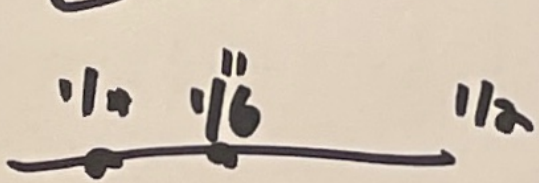
Arithmetic of stacky curves

2. Canonical rings

$$\mathcal{H} \xrightarrow{f} \mathbb{C} \qquad (f(z)dz)|_\gamma = f(z)dz, \quad \gamma \in \Gamma$$

$\curvearrowright$

$\Gamma \overset{\text{eg}}{=} SL_2(\mathbb{Z})$ $\qquad \left[\mathcal{H}/SL_2\right] \to \mathcal{H}/SL_2 \simeq \mathbb{P}^1$



Problems: study $\bigoplus_k H^\bullet(\mathcal{X}, \omega_{\mathcal{X}}^{\otimes k})$ on stacks

3. Heights and rational points

$\mathbb{P}^N(\mathbb{Q}) \ni P = [x_0 : \cdots : x_N]$ w/ gcd = 1

$$ht(P) = \max\{|x_0|, \ldots, |x_N|\}$$

$X \hookrightarrow \mathbb{P}^N$

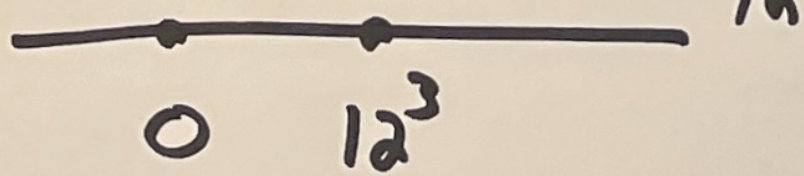
$$N_X(B) = \#\{P \in X(\mathbb{Q}) \text{ w/ } ht(P) \leq B\}$$

Conj $\sim c B^a (\log B)^b$

Problem: Study $N_X(B)$ for stacks

4. Cohomology of Stacky Curves

5. Chow rings + intersection theory

$\mathcal{X} = \mathcal{M}_{1,1} =$ 

$j \downarrow$

$\mathbb{A}^1$

$\operatorname{Pic} \mathcal{X} \simeq \mathbb{Z}/12\mathbb{Z}$

6. Global quotient stacks

7. Functorial resolution of singularities

8. Expository projects