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Cohomological descent on the overconvergent site

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Abstract
We prove that cohomological descent holds for finitely presented crystals on the overconvergent site with respect to proper or fppf hypercovers.

Background

Cohomological descent is a robust computational and theoretical tool, central to p-adic cohomology and its applications. On the one hand, it facilitates explicit calculations (analogous to the computation of coherent cohomology in scheme theory via Čech cohomology); on the other, it allows one to deduce results about singular schemes (e.g., finiteness of the cohomology of overconvergent isocrystals on singular schemes [1]) from results about smooth schemes, and, in a pinch, sometimes allows one to bootstrap global definitions from local ones (for example, for a scheme X which fails to embed into a formal scheme smooth near X, one actually *defines* rigid cohomology via cohomological descent; see [2], comment after Proposition 8.2.17).

The main result of the series of papers [3-5] is that cohomological descent for the rigid cohomology of overconvergent isocrystals holds with respect to both flat and proper hypercovers. The barrage of choices in the definition of rigid cohomology is burdensome and makes their proofs of cohomological descent very difficult, totaling to over 200 pages. Even after the main cohomological descent theorems [3] Theorems 7.3.1 and 7.4.1 are proved, one still has to work a bit to get a spectral sequence [3], Theorem 11.7.1. Actually, even to state what one means by cohomological descent (without a site) is subtle.

The situation is now more favorable. More than 25 years after Berthelot's seminal papers [6-8], key foundational aspects have now been worked out. Le Stum's recent advance [9] is the construction of an 'overconvergent site' [9] which gives an alternative, equivalent definition of rigid cohomology as the cohomology of the structure sheaf of a ringed site $(X_{\mathrm{AN}^\dagger}, \mathcal{O}_X^\dagger)$ (and also of course an equivalence between the category of overconvergent isocrystals on X and the category of finitely presented $\mathcal{O}_X^\dagger$-modules). This formalism is the correct setting for many problems; for instance, [10] uses the overconvergent site to develop a theory of constructible ∇-modules on curves.

More applications are expected. And indeed, the main result of this paper is the application of the abstract machinery of [11], Exposé Vbis and VI to the overconvergent site to give a short proof of the following.

Theorem 1.1. *Cohomological descent for locally finitely presented modules on the overconvergent site holds with respect to*

(i) *fppf hypercovers of varieties and*
(ii) *proper hypercovers of varieties.*

Remark 1.2. (New ideas and comparison to prior work) Classical results in rigid cohomology have no access to site theoretic techniques and instead one works directly with a choice of embedding into a formal scheme. This adds difficulties (again we highlight the length of [3-5]) and restricts the applicability of results (e.g., generalization to stacks; see Remark 1.1 (3)).

On the other hand, Theorem 1.1 is not merely a formal consequence of the techniques of [11], Exposé Vbis and VI. Cohomological descent for abelian sheaves on the étale site with respect to smooth hypercovers is simply the Čech theory (as in Theorem 3.1 (i)). In the overconvergent setting, an étale surjection is not a covering; and hence, the Čech theory does not apply. Another technical difficulty is that one cannot check triviality of an overconvergent sheaf $\mathcal{F} \in \mathrm{AN}^{\dagger}X$ by restricting to points of X, so that the template of the proof of proper cohomological descent for étale cohomology therefore does not apply to overconvergent cohomology, and a new argument is needed.

Moreover, we emphasize that, while similar-looking results appear in the literature (see e.g. section Zariski covers and Lemma 5.14), these results are not powerful enough to directly imply cohomological descent for the overconvergent site. New tools and ideas – the use of techniques from [11] and for instance, the use of Raynaud-Gruson's theorem on 'flattening stratifications' [12], Théorèm 5.2.2, and le Stum's main theorems (e.g. Proposition 4.17) - greatly simplify and extend the generality of our proof, and as an additional indication of this, we remark that (in contrast to cohomological descent for étale or rigid cohomology) the proofs of the étale and proper cases are intertwined.

Finally, in light of the central role that one expects le Stum's work to play in the future development of rigid cohomology, we note that various ingredients of our proofs are useful lemmas which facilitate computations on the overconvergent site; see for instance Lemma 5.2.

Applications

We highlight a few direct applications of our main theorem.

1. (Spectral sequence). By le Stum's comparison theorems between rigid and overconvergent cohomology [9], Corollary 3.5.9, we obtain a spectral sequence (see Remark 3.3) computing rigid cohomology; this gives a shorter proof of Theorem 11.7.1 of [3]. While this corollary of our work and [3,4] are similar, the main results are independent and cannot be deduced from one another.
2. (Overconvergent de Rham-Witt cohomology). A recent result [13] proves directly that overconvergent de Rham-Witt cohomology agrees with classical rigid cohomology for smooth affine varieties, and a long argument with dagger spaces is needed to deduce agreement with general rigid cohomology. Use of the overconvergent site and Theorem 1.1 simplifies the globalization argument (this will appear in future work (Davis and Zureick-Brown: Overconvergent de

Rham-Witt cohomology for stacks, in preparation); the main theorem is that the étale cohomology of each overconvergent de Rham-Witt sheaf vanishes on affines, and the globalization argument is then a direct application of Theorem 1.1 and the spectral sequence 3.3).

3. (Rigid cohomology for stacks.) Motivated by applications to geometric Langlands, Kedlaya proposed the problem of generalizing rigid cohomology to stacks. There are three approaches – le Stum's site gives (via Definition 4.5) a direct approach realized in the author's thesis [14]); the overconvergent de Rham-Witt complex of [13] gives an alternative, explicit and direct construction. Theorem 1.1 gives a third approach and a direct comparison of the first two approaches; moreover, Theorem 1.1 also gives a direct proof that the rigid cohomology of a stack is finite dimensional, and allows one to make various constructions (e.g., to define a Gysin map). Rigid cohomology for stacks is most naturally defined via the overconvergent site (one can take the same definition for a stack as for a scheme); cohomological descent for the overconvergent cohomology of a stack follows from the case of schemes by standard arguments, and both of these applications (finiteness and the ability to bootstrap constructions from the case of schemes) follow directly from the spectral sequence.

Organization of the paper

This paper is organized as follows: In section Notation and conventions, we recall notation. In section Simplicial methods, we review the machinery of cohomological descent. In section The overconvergent site, we recall the construction of the overconvergent site of [9]. In section Cohomological descent for overconvergent crystals, we prove Theorem 1.1, first in the case of Zariski hypercovers, then in the case of fppf hypercovers, and finally for proper hypercovers.

Notation and conventions

Throughout, K will denote a field of characteristic 0 that is complete with respect to a non-trivial non-archimedean valuation with valuation ring $\mathcal{V}$, whose maximal ideal and residue field we denote by $\mathfrak{m}$ and k. We denote the category of schemes over k by $\mathbf{Sch}_k$. We define an algebraic variety over k to be a scheme such that there exists a *locally finite* cover by schemes of finite type over k (recall that a collection $\mathcal{S}$ of subsets of a topological space X is said to be locally finite if every point of X has a neighborhood which only intersects finitely many subsets $X \in \mathcal{S}$). Note that we do not require an algebraic variety to be reduced, quasi-compact, or separated.

Formal schemes: As in [9], 1.1 we define a formal $\mathcal{V}$-scheme to be a locally topologically finitely presented formal scheme P over $\mathcal{V}$, i.e., a formal scheme P with a locally finite covering by formal affine schemes Spf A, with A topologically of finite type (i.e., a quotient of the ring $\mathcal{V}\{T_1, \cdots, T_n\}$ of convergent power series by an ideal $I + \mathfrak{a}\mathcal{V}\{T_1, \cdots, T_n\}$, with I an ideal of $\mathcal{V}\{T_1, \cdots, T_n\}$ of finite type and $\mathfrak{a}$ an ideal of $\mathcal{V}$). This finiteness property is necessary to define the 'generic fiber' of a formal scheme (see [15], Section 1).

We refer to [16], 1.10 for basic properties of formal schemes. The first section of [17] is another good reference; a short alternative is [9], Section 1, which contains everything we will need.

K-**analytic spaces**: We refer to [18] (as well as the brief discussion in [9], 4.2) for definitions regarding K-analytic spaces. As in [9], 4.2, we define an analytic variety over K to be a locally Hausdorff topological space V together with a maximal affinoid atlas τ which is locally defined by *strictly* affinoid algebras (i.e., an algebra A is strict if it is a quotient of a Tate algebra $K\{T_1, \cdots, T_n\}$) and denote by $\mathcal{M}(A)$ the Gelfand spectrum of an affinoid algebra A. Moreover, recall that a K-analytic space is said to be good if every point has an open affinoid neighborhood.

Topoi: We follow the conventions of [19] (exposited in [9], 4.1) regarding sites, topologies, topoi, and localization. When there is no confusion, we will identify an object X of a category with its associated presheaf $h_X \colon Y \mapsto \operatorname{Hom}(Y, X)$. For an object X of category C, we denote by $C_{/X}$ the comma category - objects of $C_{/X}$ are morphisms $Y \to X$, and morphisms are commutative diagrams - and by $\widetilde{C_{/X}}$ (resp. $\widehat{C_{/X}}$) the associated topos (resp. category of presheaves on $C_{/X}$). Often (in this paper), a morphism $(f^{-1}, f_*) \colon (T, \mathcal{O}_T) \to (T', \mathcal{O}_{T'})$ of ringed topoi will satisfy $f^{-1}\mathcal{O}_{T'} = \mathcal{O}_T$, so that there is no distinction between the functors f^{-1} and f^*; in this case, we will write f^* for both. Finally, we note that the category $\operatorname{Mod}_{\mathrm{fp}}\mathcal{O}_T$ of $\mathcal{O}_T$-modules which locally admit a finite presentation $\bigoplus_{i=1}^n \mathcal{O}_T \to \bigoplus_{i=1}^m \mathcal{O}_T \to M$, is generally *larger* than $\operatorname{Coh}\mathcal{O}_T$, since in general, the sheaf of rings $\mathcal{O}_T$ is not itself coherent.

Simplicial methods

Here we recall the setup and main results of cohomological descent. The standard reference is [11], Exposé Vbis and VI; some alternatives are Deligne's paper [20] and Conrad's notes [21]; the latter has a lengthy introduction with a lot of motivation and gives more detailed proofs of some theorems of [11] and [20].

A morphism $p_0 \colon X \to Y$ of presheaves on a site C induces a morphism $\widetilde{C_{/Y}} \to \widetilde{C_{/X}}$ of topoi. Setting X_n to be the $(n+1)$-fold fiber product of p_0 and $p_{ij}^k \colon X_i \to X_j$ to be the kth map induced by composing projections, p_0 also induces a morphism of topoi

$$\left(p^{-1}, p_*\right) : \widetilde{C_{/Y}} \to \widetilde{C_{/X_\bullet}},$$

where $\widetilde{C_{/X_\bullet}}$ is (equivalent to) the category of simplicial sheaves, i.e., collections of pairs $\mathcal{F}_\bullet := \left(\left\{ \mathcal{F}_n \in \widetilde{C_{/X_n}} \right\}, \left\{ \iota_{ij}^k \colon \left(p_{ij}^k\right)^{-1} \mathcal{F}_j \to \mathcal{F}_i \right\} \right)$ (where the ι_{ij}^k's are maps of sheaves satisfying the usual cocycle compatibilities). The functors (p^{-1}, p_*) are

$$p_*\mathcal{F}_\bullet := \operatorname{eq}\left(p_{(0)*}\mathcal{F}_0 \rightrightarrows p_{1*}\mathcal{F}_1 \right), \ (p^{-1}\mathcal{F})_n := (p_{ij}^k)^{-1}\mathcal{F}$$

with ι_{ij}^k the canonical isomorphisms; there is a clear natural map $\mathcal{F} \to p_*p^*\mathcal{F}$. Since $\widetilde{C_{/X_\bullet}}$ is actually the topos of the simplicial site $C_{/X_\bullet}$, to $\mathcal{F}_\bullet \in \widetilde{C_{/X_\bullet}}$, we can associate the derived pushforward $\mathbb{R}p_*\mathcal{F}_\bullet$ as well as the ith relative cohomology $\mathbb{R}^i p_*\mathcal{F}_\bullet$ and absolute cohomology $H^i(C_{X_\bullet}, \mathcal{F}_\bullet)$.

We say that $p_0 \colon X \to Y$ is of cohomological descent (resp. universally of cohomological descent) with respect to $\mathcal{F} \in \widetilde{C_{/Y}}$ if $\mathcal{F} \cong \mathbb{R}p_*p^*\mathcal{F}$ (resp. if for every $f \colon Y' \to Y$, the map $X \times_Y Y' \to Y'$ is of cohomological descent with respect to $f^*\mathcal{F}$). Similarly, we say that p_0 is (universally) of cohomological descent with respect to a subcategory $\mathcal{C} \subset \operatorname{Ab} C_{/Y}$ if p_0 is (universally) of cohomological descent with respect to every $\mathcal{F} \in \mathcal{C}$.

Theorem 3.1.

1. *Suppose that* $p_0 \colon X \to Y$ *in* $\widehat{C}$ *sheafifies to a covering in* $\widetilde{C}$. *Then* p_0 *is universally of cohomological descent. Moreover, for* $F \in \mathrm{Ab}\,\widetilde{Y}$, *the map* $F \to p_* p^* F$ *is a quasi-isomorphism.*
2. *Any morphism in* $\widehat{C}$ *which has a section locally (in* $\widetilde{C}$*) is universally of cohomological descent.*
3. *Consider a Cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, "\pi_0'"] \arrow[d, "f_0'"'] & X \arrow[d, "f_0"] \\ S' \arrow[r, "\pi_0"'] & S \end{tikzcd}$$

in $\widehat{C}$ *and let* $\mathcal{F} \in \widetilde{C_{/S}}$ *be a sheaf of abelian groups. Suppose* π_0 *is universally of cohomological descent with respect to* $\mathcal{F}$. *Then* f_0 *is universally of cohomological descent with respect to* $\mathcal{F}$ *if and only if* f_0' *is universally of cohomological descent with respect to* $\pi_0^* \mathcal{F}$.

4. *Let* $f \colon X \to Y$ *and* $g \colon Y \to Z$ *be maps in* C *and let* $\mathcal{F} \in \widetilde{Z}$ *be a sheaf of abelian groups. Suppose that the composition* $X \to Z$ *is universally of cohomological descent with respect to* $\mathcal{F}$. *Then* $Y \to Z$ *is as well.*
5. *Let* $f \colon X \to Y$ *and* $g \colon Y \to Z$ *be maps in* C *and let* $\mathcal{F} \in \widetilde{Z}$ *be a sheaf of abelian groups. If* g *is universally of cohomological descent with respect to* $\mathcal{F}$ *and* f *is universally of cohomological descent with respect to* $g^*\mathcal{F}$, *then the composition* $g \circ f$ *is universally of cohomological descent with respect to* $\mathcal{F}$.
6. *Let* $f_i \colon X_i \to Y_i$ *be maps in* C *indexed by some arbitrary set* I. *For each* $i \in I$ *let* $\mathcal{F}_i \in \widetilde{Y_i}$ *be a sheaf of abelian groups. Suppose that for each* i, f_i *is of cohomological descent relative to* $\mathcal{F}_i$. *Then* $\coprod f_i \colon \coprod X_i \to \coprod Y_i$ *is of cohomological descent relative to* $\coprod \mathcal{F}_i$ *(where disjoint unions are taken in* $\widehat{C}$*).*

Proof. Statement (1) is ([22], Lemma 1.4.24), (2) follows from (1) since any morphism with a section is a covering in the canonical topology. The proofs of (3) to (5) are identical to the proof of Theorem [21], Theorem 7.5, and (6) follows from the fact that, setting $p_0 = \coprod f_i$, the induced morphism of simplicial topoi

$$(p^{-1}, p_*) \colon \widetilde{\left(\coprod X_i\right)_\bullet} \to \widetilde{\coprod Y_i}$$

is also a morphism of topoi fibered over I, so that in particular, the natural map

$$\coprod \mathcal{F}_i \to \mathbb{R}p_* p^* \coprod \mathcal{F}_i$$

is an isomorphism if and only if, for all $i \in I$, the map $\mathcal{F}_i \to \mathbb{R}f_{i\bullet *} f_{i\bullet}^* \mathcal{F}_i$ is an isomorphism. □

Corollary 3.2. *Let*

$$\begin{tikzcd} \coprod X_i \arrow[r] \arrow[d] & X \arrow[d] \\ \coprod Y_i \arrow[r, "\coprod v_i"'] & Y \end{tikzcd}$$

be a commutative diagram in $\widehat{C}$ and let $\mathcal{F} \in \widetilde{C_{/Y}}$. Suppose that $\mathcal{F}$ is universally of cohomological descent with respect to $\coprod Y_i \to Y$ and that for each i, $v_i^*\mathcal{F}$ is universally of cohomological descent with respect to $X_i \to Y_i$. Then $\mathcal{F}$ is universally of cohomological descent with respect to $X \to Y$.

Proof. This follows immediately from Theorem 3.1 (4), (5), and (6). □

Remark 3.3. (Spectral sequence) If $\mathcal{F} \cong \mathbb{R}p_*p^{-1}\mathcal{F}$, then composing derived functors gives

$$\mathbb{R}\Gamma(\mathcal{F}) \cong \mathbb{R}\Gamma \circ \mathbb{R}p_*p^{-1}\mathcal{F} = \mathbb{R}(\Gamma \circ p_*)p^{-1}\mathcal{F} = \mathbb{R}\Gamma p^{-1}\mathcal{F};$$

in particular, there is a spectral sequence

$$H^j(C_{/X_i}, \mathcal{F}_i) \Rightarrow H^{i+j}(C_{/Y}, \mathcal{F}).$$

Corollary 3.4. *Let $p_0 \colon X \to Y$ be an and suppose that $\mathcal{F} \in \widetilde{Y}$ is an abelian sheaf such that*

(1) for $i \geq 0$ and $j > 0$, $\mathbb{R}^j p_{i*}p_i^*\mathcal{F} = 0$, and
(2) $\mathcal{F} \to p_*p^*\mathcal{F}$ is a quasi-isomorphism.

Then $\mathcal{F} \to \mathbb{R}p_*p^*\mathcal{F}$ is a quasi-isomorphism.

Proof. This follows from the spectral sequence. □

Remark 3.5. (Hypercoverings). We omit discussion of more general **P**-hypercovers except to note that by [21], Theorem 7.10, our results immediately extend to étale hypercovers and proper hypercovers.

The overconvergent site

In this section, we recall the setup and results of [9].

Definition 4.1. ([9], 1.2). Define an overconvergent variety over $\mathcal{V}$ to be a pair $(X \subset P, V \xrightarrow{\lambda} P_K)$, where $X \subset P$ is a locally closed immersion of an algebraic variety X over k into the special fiber P_k of a formal scheme P (recall our convention that all formal schemes are topologically finitely presented over $\operatorname{Spf}\mathcal{V}$), and $V \xrightarrow{\lambda} P_K$ is a morphism of analytic varieties, where P_K denotes the generic fiber of P, which is an analytic space. When there is no confusion, we will write (X, V) for $(X \subset P, V \xrightarrow{\lambda} P_K)$ and (X, P) for $(X \subset P, P_K \xrightarrow{\mathrm{id}} P_K)$. Define a formal morphism $(X', V') \to (X, V)$ of overconvergent varieties to be a commutative diagram

$$\begin{array}{ccccccc} X' & \hookrightarrow & P' & \leftarrow & P'_K & \leftarrow & V' \\ \downarrow f & & \downarrow v & & \downarrow v_K & & \downarrow u \\ X & \hookrightarrow & P & \leftarrow & P_K & \leftarrow & V \end{array}$$

where f is a morphism of algebraic varieties, v is a morphism of formal schemes, and u is a morphism of analytic varieties.

Finally, define $\mathrm{AN}(\mathcal{V})$ to be the category whose objects are overconvergent varieties and morphisms are formal morphisms. We endow $\mathrm{AN}(\mathcal{V})$ with the analytic topology, defined to be the topology generated by families $\{(X_i, V_i) \to (X, V)\}$ such that for each i, the maps $X_i \to X$ and $P_i \to P$ are the identity maps, V_i is an open subset of V, and $V = \bigcup V_i$ is an open covering (recall that an open subset of an analytic space is admissible in the G-topology and thus also an analytic space – this can be checked locally in the G-topology, and for an affinoid this is clear because there is a basis for the topology of open affinoid subdomains).

Definition 4.2 ([9], Section 1.1)**.** The specialization map $P_K \to P_k$ induces by composition a map $V \to P_k$ and we define the tube $]X[_V$ of X in V to be the preimage of X under this map. The tube $]X[_{P_K}$ admits the structure of an analytic space and the inclusion $i_X\colon\,]X[_{P_K} \hookrightarrow P_K$ is a locally closed inclusion of analytic spaces (and generally not open, in contrast to the rigid case). The tube $]X[_V$ is then the fiber product $]X[_{P_K} \times_{P_K} V$ (as analytic spaces) and in particular is also an analytic space. A formal morphism $(f, u)\colon (X', V') \to (X, V)$ induces a morphism $]f[_u :]X'[_{V'} \to]X[_V$ of tubes. (Since $]f[_u$ is induced by u, when there is no confusion, we will sometimes denote it by u).

The fundamental topological object in rigid cohomology is the tube $]X[_V$, in that most notions are defined only up to neighborhoods of $]X[_V$. To make this precise, modify $\mathrm{AN}(\mathcal{V})$.

Definition 4.3 ([9], Definition 1.3.3)**.** Define a formal morphism

$$(f, u)\colon (X', V') \to (X, V)$$

to be a strict neighborhood if f and $]f[_u$ are isomorphisms and u induces an isomorphism from V' to a neighborhood W of $]X[_V$ in V.

Definition 4.4. We define the category $\mathrm{AN}^\dagger(\mathcal{V})$ of overconvergent varieties to be the localization of $\mathrm{AN}(\mathcal{V})$ by strict neighborhoods (which is possible by [9], Proposition 1.3.6): the objects of $\mathrm{AN}^\dagger(\mathcal{V})$ are the same as those of $\mathrm{AN}(\mathcal{V})$ and a morphism $(X', V') \to (X, V)$ in $\mathrm{AN}^\dagger(\mathcal{V})$ is a pair of formal morphisms

$$(X', V') \leftarrow (X', W) \to (X, V),$$

where $(X', W) \to (X', V')$ is a strict neighborhood.

The functor $\mathrm{AN}(\mathcal{V}) \to \mathrm{AN}^\dagger(\mathcal{V})$ induces the image topology on $\mathrm{AN}^\dagger(\mathcal{V})$ (i.e., the largest topology on $\mathrm{AN}^\dagger(\mathcal{V})$ such that the map from $\mathrm{AN}(\mathcal{V})$ is continuous). By [9], Proposition 1.4.1, the image topology on $\mathrm{AN}^\dagger(\mathcal{V})$ is generated by the pretopology of collections $\{(X, V_i) \to (X, V)\}$ with $\bigcup V_i$ an open covering of a neighborhood of $]X[_V$ in V and $]X[_V = \bigcup]X[_{V_i}$.

Definition 4.5. For any presheaf $T \in \widehat{\mathrm{AN}^\dagger(\mathcal{V})}$, we define $\mathrm{AN}^\dagger(T)$ to be the localized category $\mathrm{AN}^\dagger(\mathcal{V})_{/T}$ whose objects are morphisms $h_{(X,V)} \to T$ (where $h_{(X,V)}$ is the presheaf associated to (X, V)) and morphisms are morphisms $(X', V') \to (X, V)$ which induce a commutative diagram

$$\begin{tikzcd} h_{(X',V')} \arrow[rr] \arrow[dr] & & h_{(X,V)} \arrow[dl] \\ & T & \end{tikzcd} .$$

We may endow $\mathrm{AN}^\dagger(T)$ with the induced topology (i.e., the smallest topology making continuous the projection functor $\mathrm{AN}^\dagger(T) \to \mathrm{AN}^\dagger(\mathcal{V})$; see [9], Definition 1.4.7); concretely, the covering condition is the same as in 4.4. When $T = h_{(X,V)}$ we denote $\mathrm{AN}^\dagger(T)$ by $\mathrm{AN}^\dagger(X, V)$. Since the projection $\mathrm{AN}^\dagger T \to \mathrm{AN}^\dagger \mathcal{V}$ is a fibered category, the projection is also cocontinuous with respect to the induced topology. Finally, an algebraic space X over k defines a presheaf $(X', V') \mapsto \mathrm{Hom}(X', X)$, and we denote the resulting site by $\mathrm{AN}^\dagger(X)$.

There will be no confusion in writing (X, V) for an object of $\mathrm{AN}^\dagger(T)$.

We use subscripts to denote topoi and continue the above naming conventions – i.e., we denote the category of sheaves of sets on $\mathrm{AN}^\dagger(T)$ (resp. $\mathrm{AN}^\dagger(X, V)$, $\mathrm{AN}^\dagger(X)$) by $T_{\mathrm{AN}^\dagger}$ (resp. $(X, V)_{\mathrm{AN}^\dagger}, X_{\mathrm{AN}^\dagger}$). Any morphism $f\colon T' \to T$ of presheaves on $\mathrm{AN}^\dagger(\mathcal{V})$ induces a morphism $f_{\mathrm{AN}^\dagger}\colon T'_{\mathrm{AN}^\dagger} \to T_{\mathrm{AN}^\dagger}$ of topoi. For a morphism $(f, u)\colon (X', V') \to (X, V)$ of overconvergent varieties, we denote the induced morphism of topoi by $(u^*_{\mathrm{AN}^\dagger}, u_{\mathrm{AN}^\dagger *})$.

For an analytic space V we denote by $\mathrm{Open}\,V$ the category of open subsets of V and by V_{an} the associated topos of sheaves of sets on $\mathrm{Open}\,V$. Recall that for an analytic variety (X, V), the topology on the tube $]X[_V$ is induced by the inclusion $i_X\colon]X[_V \hookrightarrow V$.

Definition 4.6 ([9], Corollary 2.1.3)**.** Let (X, V) be an overconvergent variety. Then there is a morphism of sites

$$\varphi_{X,V}\colon \mathrm{AN}^\dagger(X, V) \to \mathrm{Open}\,]X[_V .$$

The notation as usual is in the 'direction' of the induced morphism of topoi and in particular backward; it is associated to the functor $\mathrm{Open}\,]X[_V \to \mathrm{AN}^\dagger(X, V)$ given by $U = W \cap\,]X[_V \mapsto (X, W)$ (and is independent of the choice of W up to strict neighborhoods). This induces a morphism of topoi

$$(\varphi^{-1}_{X,V}, \varphi_{X,V*})\colon (X, V)_{\mathrm{AN}^\dagger} \to (]X[_V)_{\mathrm{an}}.$$

Definition 4.7 ([9], 2.1.7)**.** Let $(X, V) \in \mathrm{AN}^\dagger(T)$ be an overconvergent variety over T and let $F \in T_{\mathrm{AN}^\dagger}$ be a sheaf on $\mathrm{AN}^\dagger(T)$. We define the realization $F_{X,V}$ of F on $]X[_V$ to be $\varphi_{(X,V)*}(F|_{(X,V)_{\mathrm{AN}^\dagger}})$, where $F|_{(X,V)_{\mathrm{AN}^\dagger}}$ is the restriction of F to $\mathrm{AN}^\dagger(X, V)$.

We can describe the category $T_{\mathrm{AN}^\dagger}$ in terms of realizations in a manner similar to sheaves on the crystalline or lisse-étale sites.

Proposition 4.8 ([9], Proposition 2.1.8)**.** *Let T be a presheaf on $AN^\dagger(\mathcal{V})$. Then the category $T_{AN^\dagger}$ is equivalent to the following category:*

(1) *An object is a collection of sheaves $F_{X,V}$ on $]X[_V$ indexed by $(X,V) \in AN^\dagger(T)$ and, for each $(f,u)\colon (X',V') \to (X,V)$, a morphism $\phi_{f,u} :]f[_u^{-1} F_{X,V} \to F_{X',V'}$, such that as (f,u) varies, the maps $\phi_{f,u}$ satisfy the usual compatibility condition.*

(2) *A morphism is a collection of morphisms $F_{X,V} \to G_{X,V}$ compatible with the morphisms $\phi_{f,u}$.*

To obtain a richer theory, we endow our topoi with sheaves of rings and study the resulting theory of modules.

Definition 4.9 ([9], Definition 2.3.4)**.** Define the sheaf of overconvergent functions on $\mathrm{AN}^\dagger(\mathcal{V})$ to be the presheaf of rings

$$\mathcal{O}_{\mathcal{V}}^\dagger\colon (X,V) \mapsto \Gamma(]X[_V, i_X^{-1}\mathcal{O}_V)$$

where i_X is the inclusion of $]X[_V$ into V; this is a sheaf by [9], Corollary 2.3.3. For $T \in \widehat{\mathrm{AN}^\dagger(\mathcal{V})}$ a presheaf on $\mathrm{AN}^\dagger(\mathcal{V})$, define $\mathcal{O}_T^\dagger$ to be the restriction of $\mathcal{O}_{\mathcal{V}}^\dagger$ to $\mathrm{AN}^\dagger(T)$. Following our naming conventions above, we denote by $\mathcal{O}_{(X,V)}^\dagger$ the restriction of $\mathcal{O}_{\mathcal{V}}^\dagger$ to $\mathrm{AN}(X,V)$.

Remark 4.10. By [9],Proposition 2.3.5, (i), the morphism of topoi of Definition 4.6 can be promoted to a morphism of ringed sites

$$(\varphi_{X,V}^*, \varphi_{X,V*})\colon (\mathrm{AN}^\dagger(X,V), \mathcal{O}_{(X,V)}^\dagger) \to (]X[_V, i_X^{-1}\mathcal{O}_V).$$

In particular, for $(X,V) \in \mathrm{AN}^\dagger T$ and $M \in \mathcal{O}_T^\dagger$, the realization $M_{X,V}$ is an $i_X^{-1}\mathcal{O}_V$-module. For any morphism $(f,u)\colon (X',V') \to (X,V)$ in $\mathrm{AN}^\dagger(T)$, one has a map

$$(]f[_u^\dagger,]f[_{u*})\colon (]X'[_{V'}, i_{X',V'}^{-1}\mathcal{O}_{V'}) \to (]X[_V, i_{X,V}^{-1}\mathcal{O}_V).$$

of ringed sites, and functoriality gives transition maps

$$\phi_{f,u}^\dagger\colon]f[_u^\dagger M_{X,V} \to M_{X',V'}$$

which satisfy the usual cocycle compatibilities.

Definition 4.11 ([9], Definition 2.3.7)**.** Define the category of overconvergent crystals on T, denoted $\mathrm{Cris}^\dagger T$, to be the full subcategory of $\mathrm{Mod}\mathcal{O}_T^\dagger$ such that the transition maps $\phi_{f,u}^\dagger$ are isomorphisms.

Example 4.12. The sheaf $\mathcal{O}_T^\dagger$ is a crystal, and in fact $\mathrm{QCoh}\mathcal{O}_T^\dagger \subset \mathrm{Cris}^\dagger T$. More generally, the pair $\varphi_{X,V}^*$ and $\varphi_{X,V*}$ induces (see [9], Proposition 2.3.8) an equivalence of categories

$$\mathrm{Cris}^\dagger(X,V) \to \mathrm{Mod}i_X^{-1}\mathcal{O}_V.$$

One minor subtlety is the choice of an overconvergent variety as a base.

Definition 4.13. Let $(C,O) \in \mathrm{AN}^\dagger(\mathcal{V})$ be an overconvergent variety and let $T \to C$ be a morphism from a presheaf on $\mathbf{Sch}_k$ to C. Then T defines a presheaf on $\mathrm{AN}^\dagger(C,O)$ which sends $(X,V) \to (C,O)$ to $\mathrm{Hom}_C(X,T)$, which we denote by T/O. We denote the associated site by $\mathrm{AN}^\dagger(T/O)$, and when $(C,O) = (S_k,S)$ for some formal $\mathcal{V}$-scheme S, we write instead $\mathrm{AN}^\dagger(T/S)$.

The minor subtlety is that there is no morphism $T \to h_{(C,O)}$ of presheaves on $\mathrm{AN}^\dagger(\mathcal{V})$. A key construction is the following.

Definition 4.14 ([9], Paragraph after Corollary 1.4.15)**.** Let $(X, V) \to (C, O) \in \mathrm{AN}^\dagger(\mathcal{V})$ be a morphism of overconvergent varieties. We denote by X_V/O the image presheaf of the morphism $(X, V) \to X/O$, considered as a morphism of presheaves. Explicitly, a morphism $(X', V') \to X/O$ lifts to a morphism $(X', V') \to X_V/O$ if and only if there exists a morphism $(X', V') \to (X, V)$ over X/O, and in particular different lifts $(X', V') \to (X, V)$ give rise to the same morphism $(X', V') \to X_V/O$. When $(C, O) = (\mathrm{Spec}\ k, \mathcal{M}(K))$, we may write X_V instead $X_V/\mathcal{M}(K)$.

Many theorems will require the following extra assumption of [9], Definition 1.5.10. Recall that a morphism of formal schemes $P' \to P$ is said to be proper at a subscheme $X \subset P'_k$ if, for every component Y of $\overline{X}$, the map $Y \to P_k$ is proper (see [9], Definition 1.1.5).

Definition 4.15. Let $(C, O) \in \mathrm{AN}^\dagger(\mathcal{V})$ be an overconvergent variety and let $f : X \to C$ be a morphism of k-schemes. We say that a formal morphism $(f, u) : (X, V) \to (C, O)$, written as

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "f"] & P \arrow[d, "v"] & V \arrow[l] \arrow[d, "u"] \\ C \arrow[r, hook] & Q & O \arrow[l] \end{tikzcd} ,$$

is a geometric realization of f if v is proper at X, v is smooth in a neighborhood of X, and V is a neighborhood of $]X[_{P_K \times_{Q_K} O}$ in $P_K \times_{Q_K} O$. We say that f is realizable if there exists a geometric realization of f.

We need a final refinement to $\mathrm{AN}^\dagger(\mathcal{V})$.

Definition 4.16. We say that an overconvergent variety (X, V) is good if there is a good neighborhood V' of $]X[_V$ in V (i.e., every point of $]X[_V$ has an affinoid neighborhood in V). We say that a formal scheme S is good if the overconvergent variety (S_k, S_K) is good. We define the good overconvergent site $\mathrm{AN}^\dagger_{\mathrm{g}}(T)$ to be the full subcategory of $\mathrm{AN}^\dagger(T)$ consisting of good overconvergent varieties. Given a presheaf $T \in \mathrm{AN}^\dagger(\mathcal{V})$, we denote by T_{g} the restriction of T to $\mathrm{AN}^\dagger_{\mathrm{g}}(\mathcal{V})$.

Note that localization commutes with passage to good variants of our sites (e.g., there is an isomorphism $\mathrm{AN}^\dagger_{\mathrm{g}}(\mathcal{V})_{/T_{\mathrm{g}}} \cong \mathrm{AN}^\dagger_{\mathrm{g}}(T)$). When making further definitions we will often omit the generalization to $\mathrm{AN}^\dagger_{\mathrm{g}}$ when it is clear.

The following proposition will allow us to deduce facts about $\mathrm{Mod}_{\mathrm{fp}}\mathcal{O}^\dagger_{X_g}$ from results about (X, V) and X_V.

Proposition 4.17. *Let $(C, O) \in \mathrm{AN}^\dagger_{\mathrm{g}}(\mathcal{V})$ be a good overconvergent variety and let $(X, V) \to (C, O)$ be a geometric realization of a morphism $X \to C$ of schemes. Then the following are true:*

(i) *The map* $(X, V)_{\mathrm{g}} \to (X/O)_{\mathrm{g}}$ *is a covering in* $\mathrm{AN}^{\dagger}_{\mathrm{g}}(\mathcal{V})$.
(ii) *There is an equivalence of topoi* $(X_V/O)_{\mathrm{AN}^{\dagger}_{g}} \cong (X/O)_{\mathrm{AN}^{\dagger}_{g}}$.
(iii) *The natural pullback map* $\mathrm{Cris}^{\dagger}_{\mathrm{g}}X/O \to \mathrm{Cris}^{\dagger}_{\mathrm{g}}X_V/O$ *is an equivalence of categories.*
(iv) *Suppose that* (X, V) *is good. Then the natural map* $\mathrm{Cris}^{\dagger}X_V/O \to \mathrm{Cris}^{\dagger}_{\mathrm{g}}X_V/O$ *is an equivalence of categories. In particular, the natural map* $\mathrm{Mod}_{\mathrm{fp}}\mathcal{O}^{\dagger}_{X_{\mathrm{g}}} \to \mathrm{Mod}_{\mathrm{fp}}\mathcal{O}^{\dagger}_{(X_V)_{\mathrm{g}}} \cong \mathrm{Mod}_{\mathrm{fp}}\mathcal{O}^{\dagger}_{X_V}$ *is an equivalence of categories.*

Proof. The first two claims are [9], 1.5.14, 1.5.15, the third follows from the second, and the last is clear. □

Technical lemmas

We state here a few technical lemmas that will be useful in the proof of Theorem 1.1.

Lemma 4.19. *Let* $(Y, W) \to (X, V)$ *be a morphism of overconvergent varieties. Let* $Y' = Y \times_X X'$ *and* $W' = W \times_V X'$. *Then* $(Y', W') \cong (Y, W) \times_{(X,V)} (Y', V')$.

Proof. This is the comment after [9], Proposition 1.3.10. □

Lemma 4.20. *Let* $p\colon (X', V') \to (X, V)$ *be a morphism of overconvergent varieties such that the induced map on tubes is an inclusion of a closed subset. Then* p_* *is exact.*

Proof. It suffices to check that, for any Cartesian diagram

$$\begin{tikzcd} (Y', W') \arrow[r] \arrow[d, "p'"] & (X', V') \arrow[d] \\ (Y, W) \arrow[r] & (X, V) \end{tikzcd}$$

the map induced on tubes by p' is exact; the lemma follows since for any base change of p, the induced map on tubes is also an inclusion of a closed subset and such maps are exact. □

Cohomological descent for overconvergent crystals

In this section, we prove Theorem 1.1. The proof naturally breaks into cases: Zariski covers, modifications, finitely presented flat covers, and proper surjections. The full proof fails without the goodness assumption, but many special cases (e.g., cohomological descent with respect to Zariski hypercovers) hold without the goodness assumption.

Throughout, when discussing cohomological descent, we consider k-varieties, analytic varieties, etc., as presheaves on the overconvergent site.

Zariski covers

We begin with the case of a Zariski cover. One can restate the main result of [9], section 3.6 as the statement that a Zariski covering is universally of cohomological descent with respect to crystals. Throughout this subsection, we omit distinction between $\mathrm{AN}^{\dagger}$ and $\mathrm{AN}^{\dagger}_{g}$, but remark here that each result is true for either site.

Some care is needed to interpret le Stum's results in the language of cohomological descent; to that end, we first prove a few lemmas that will be useful in later proofs as well.

Lemma 5.2. *Let $p_0 = (f_0, u_0)\colon (X_0, V_0) \to (X_{-1}, V_{-1})$ be a morphism of overconvergent varieties such that*

(i) the induced map $]f_0[\colon\,]X_0[_{V_0} \to\,]X_{-1}[_{V_{-1}}$ is an isomorphism, and
(ii) the natural map $]f_0[^{-1} i_{X_{-1}}^{-1}\mathcal{O}_{V_{-1}} \to i_{X_0}^{-1}\mathcal{O}_{V_0}$ is an isomorphism.

Then p_0 is universally of cohomological descent with respect to crystals.

Remark 5.3. Note that condition (ii) of Lemma 5.2 is satisfied if u_0 is a finite quasi-immersion and thus in particular is satisfied if u_0 is an isomorphism or if V_0 is a neighborhood of $]X_{-1}[_{V_{-1}}$. (Note that these are non-trivial conditions, since f_0 may not be an isomorphism.) Moreover, condition (i) holds if f_0 is surjective.

We also note that condition (ii) is necessary; in general, a morphism

$$(X,]X[_V) \to (X, V)$$

is not universally of cohomological descent for crystals, since the Čech complex is not exact. For example, when $X = \mathbb{A}^1$ and $V = \mathbb{P}^1_K$, condition (ii) fails, and indeed the Čech complex

$$0 \to K\{t\}^\dagger \to K\{t\} \xrightarrow{0} K\{t\} \to \ldots$$

is not exact.

Proof. We check the hypotheses of Corollary 3.4. Denote by $p_i\colon (X_i, V_i) \to (X_{-1}, V_{-1})$ the $i+1$-fold fiber product of the map $p_0\colon (X_0, V_0) \to (X_{-1}, V_{-1})$. Noting that formation of tubes commutes with base change (and in particular that the map on tubes induced by p_i is an isomorphism), it follows from Lemma 4.20 that $p_{i,*}$ is exact.

For each $i \geq 0$ and $j > 0$, each projection $p_i^j\colon (X_i, V_i) \to (X_{i-1}, V_{i-1})$ also induces an isomorphism $]X_i[_{V_i} \cong\,]X_{i-1}[_{V_{i-1}}$ on tubes. Moreover, for a fixed i, the maps p_i^j are all equal. Finally, note that by condition (ii), the natural maps $\mathcal{F} \to p_{i,*}^j p_i^{j,*}\mathcal{F}$ are all isomorphisms (contrast with Remark 5.3). It follows that the Čech complex $\mathcal{F} \to p_* p^* \mathcal{F}$ is exact (since the maps alternate between an isomorphism and the zero map). The lemma follows. □

Lemma 5.4. *Let $\{(X_i, V_i) \to (X, V)\}$ be a collection of morphisms of overconvergent varieties such that each $V_i \to V$ is an open immersion and $\{]X_i[_{V_i}\}$ is an open covering of $]X[_V$. Then the map*

$$u_0\colon \coprod (X_i, V_i) \to (X, V)$$

is universally of cohomological descent with respect to crystals.

Proof. Let W_i' be an open subset of V such that $W_i' \cap\,]X[_V =\,]X_i[_{V_i}$ (which exists since$]X_i[_{V_i}$ is an open subset of $]X[_V$). Let W_i be the preimage of W_i' under the map $V_i \to V$. By Corollary 3.1 (4), it suffices to prove that the map $u_0'\colon \coprod (X_i, W_i) \to (X, V)$ is universally of cohomological descent with respect to crystals.

Let v_i denote the morphism $(X_i, W_i) \to (X, V)$. Then the morphism u_0' factors as

$$\coprod (X_i, W_i) \xrightarrow{\coprod v_i} \coprod (X, W_i) \xrightarrow{w_0} (X, V).$$

The map w_0 is a covering in $\mathrm{AN}^\dagger(X, V)$ and thus universally of cohomological descent by Theorem 3.1 (1). Since, by the construction of W_i, the induced map $]X_i[_{W_i} \to]X[_{W_i}$ is an isomorphism, it follows from Lemma 5.2 that each map v_i is universally of cohomological descent with respect to crystals. By Theorem 3.1 (6), $\coprod v_i$ is universally of cohomological descent with respect to crystals; by 3.1 (5), the composition is also universally of cohomological descent with respect to crystals and the lemma follows. ☐

Definition 5.5. We say that a collection $\{X_i\}$ of subspaces of a topological space X is a locally finite covering if $X = \cup X_i$ and if each point x of X admits an open neighborhood U_x on which $\{X_i \cap U_x\}$ admits a finite refinement which covers U_x.

Lemma 5.6. *Let $\{(f_i, u_i)\colon (X_i, V_i) \to (X, V)\}$ be a collection of morphisms of overconvergent varieties such that*

(a) the maps $]u_i[\,:\,]X_i[_{V_i} \to]X[_V$ are closed inclusions of topological spaces,
(b) $\{]X_i[_{V_i}\}$ is a locally finite covering of $]X[_V$, and
(c) for each i, the natural map $]f_i[^{-1} i_X^{-1}\mathcal{O}_V \to i_{X_i}^{-1}\mathcal{O}_{V_i}$ is an isomorphism.

Then the map

$$p\colon \coprod (X_i, V_i) \to (X, V)$$

is universally of cohomological descent with respect to crystals.

Proof. Let $F \in \mathrm{Cris}^\dagger(X, V)$ be a crystal. By Corollary 3.4, it suffices to prove that (i) $\mathbb{R}^q p_{\bullet,*} p_\bullet^* F = 0$ for $q > 0$, and (ii) the Čech complex $F \to p_{\bullet,*} p_\bullet^* F$ is exact. Let p_j be the j-fold fiber product of p. By the spectral sequence (Remark 3.3), it suffices to prove that $\mathbb{R}^q p_{j*} p_j^* F = 0$ for $j \geq 0$ and $q > 0$; since for each j, p_j is a disjoint union of maps which induce closed inclusions on tubes, this follows from Lemma 4.20.

For (ii), it suffices to check that, for every map $\pi\colon (X', V') \to (X, V)$, the realization with respect to (X', V') of the Čech complex of $\pi^{-1}F$ with respect to

$$p'\colon \coprod (X'_i, V'_i) \to (X', V')$$

(where $X'_i = X_i \times_X X'$ and $V'_i = V_i \times_V V'$) is exact, which (noting that our hypotheses are stable under base change) since the tubes form a locally finite closed covering, follows from condition (c) and the proof of [9], Proposition 3.1.4. ☐

Corollary 5.7. *Let $(X \hookrightarrow P \leftarrow V)$ be an overconvergent variety and let $\{P_i\}_{i \in I}$ be a collection of Zariski open formal subschemes of P. Let $(X_i, V) = (X \times_P P_i \hookrightarrow P \leftarrow V)$ and let $(X_i, V_i) = (X \times_P P_i \hookrightarrow P_i \leftarrow V \times_{P_K} (P_i)_K)$. Suppose that $\{X_i\}$ forms a locally finite Zariski open cover of X. Then the following are true:*

(1) The map $\coprod (X_i, V) \to (X, V)$ is universally of cohomological descent with respect to crystals.
(2) Suppose (X, V) is good and that the tubes $\{]X_i[_{V_i}\}$ cover a neighborhood of $]X[_V$ in V.
Then $\coprod (X_i, V_i) \to (X, V)$ is universally of cohomological descent with respect to finitely presented crystals.

Remark 5.8. By Remark 5.3, the extra hypothesis on the tubes in claim (2) is necessary.

Proof. Since specialization is anti-continuous, the tubes form a locally finite closed covering and claim (1) thus follows from Lemma 5.6. For claim (2), since (X, V) is good, we may assume that V is affinoid. The claim then follows by Tate's Acylicity Theorem [23], 8.2, Corollary 5. □

We say that a morphism of schemes $X \to Y$ over k is universally of cohomological descent (resp., with respect to a sheaf $\mathcal{F} \in \mathrm{Ab}(Y_{\mathrm{AN}^\dagger})$) if the associated morphism $X_{\mathrm{AN}^\dagger} \to Y_{\mathrm{AN}^\dagger}$ is universally of cohomological descent (resp., with respect to $\mathcal{F}$).

Theorem 5.9. *Let (C, O) be an overconvergent variety and let $X \to C$ be a morphism of algebraic varieties. Let $\{U_i\}_{i\in I}$ be a locally finite covering of X by open subschemes (resp., a covering of X by closed subschemes) and denote by $\alpha_0 \colon U = \coprod_{i\in I} U_i \to X$ the induced morphism of schemes. Denote by $\alpha \colon U_\bullet \to X$ the 0-coskeleton of α_0. Then the morphism of topoi $U_\bullet/O_{AN^\dagger} \to X/O_{AN^\dagger}$ is universally of cohomological descent with respect to $\mathcal{F}$.*

Proof. The proof for $\alpha_{\mathrm{AN}^\dagger_{\mathrm{g}}}$ is identical to the proof for $\alpha_{\mathrm{AN}^\dagger}$. We note that the map $\coprod(X', V) \to X$, where the coproduct is taken over $\mathrm{AN}^\dagger(X/O)$, is a covering in the canonical topology on $\mathrm{AN}^\dagger(X/O)$ and thus universally of cohomological descent. Setting $U_i' = X' \times_X U_i$, the diagram (of sheaves on $\mathrm{AN}^\dagger O$)

$$\begin{tikzcd}
\coprod_{\mathrm{AN}^\dagger(X/O)} \coprod_i (U_i', V) \arrow[r] \arrow[d] & \coprod_i U_i \arrow[d] \\
\coprod_{\mathrm{AN}^\dagger(X/O)} (X', V) \arrow[r] & X
\end{tikzcd}$$

commutes. By Lemma 5.6 (resp., Lemma 5.4) the maps $\coprod(U_i', V) \to (X', V)$ are universally of cohomological descent with respect to crystals; the theorem thus follows from Corollary 3.2. □

Remark 5.10. Let $\{X_i\}$ be a collection of schemes. Then the presheaf on $\mathrm{AN}^\dagger\mathcal{V}$ represented by the disjoint union $\coprod X_i$ (as schemes) is *not* equal to the disjoint union (as presheaves) of the presheaves represented by each X_i. Nonetheless, Theorem 3.1 (6) also holds for the map in $\mathrm{AN}^\dagger\mathcal{V}$ represented by a disjoint union $\coprod Y_i \to \coprod X_i$ of morphisms of schemes (taken as a disjoint union of schemes instead of as presheaves on $\mathrm{AN}^\dagger\mathcal{V}$); indeed, the sheafification of $\coprod X_i$ is the same in each case, and in general for a site C and a presheaf $F \in \widehat{C}$ with sheafification F^a, there is a natural equivalence

$$\widetilde{C_{/F}} \cong \widetilde{C}_{/F^a}$$

of topoi.

Corollary 5.11. *Let (C, O) be an overconvergent variety and let $X \to Y$ be a morphism of algebraic varieties over C. Let $\{Y_i\}$ be a locally finite open cover of Y and denote by X_i the fiber product $X \times_Y Y_i$. Then $X/O_{AN^\dagger} \to Y/O_{AN^\dagger}$ is universally of cohomological descent with respect to crystals if and only if for each i, the map $X_i/O_{AN^\dagger} \to Y_i/O_{AN^\dagger}$ is universally of cohomological descent with respect to crystals.*

Proof. This follows from Theorems 5.9 and 3.1 (3) and (6) applied to the diagram of sheaves on $\mathrm{AN}^\dagger(Y/O)$ induced by the Cartesian diagram of schemes

$$\begin{tikzcd} \coprod X_i \arrow[r] \arrow[d] & X \arrow[d] \\ \coprod Y_i \arrow[r] & Y \end{tikzcd}$$

□

The following direct corollary to Theorem 5.9 allows us to reduce to the integral case.

Corollary 5.12. *Let Y be an algebraic variety. Let $\{Y_i'\}$ be the set of irreducible components of Y and let $Y_i := (Y_i')_{red}$ be the reduction of Y_i'. Then the morphism $\coprod Y_i \to Y$ is universally of cohomological descent with respect to crystals.*

Modifications

In order to apply Raynaud-Gruson's theorem on 'flattening stratifications', we now address cohomological descent for modifications. The following lemma is an adaptation of [5], Lemma 3.4.5 to the overconvergent site, with a minor variation in that we work with non-archimedean analytic spaces. Note also that some care (e.g., the use of Lemma 5.2) is necessary to apply his argument to the overconvergent site.

Lemma 5.14. *Let Y be a scheme and let Z be a closed subscheme whose sheaf of ideals I is generated by two elements f and g. Then the blow up $X \to Y$ of Y with respect to I is universally of cohomological descent with respect to crystals.*

Proof. By Corollary 5.11, we may assume that Y is affine and thus admits an embedding $Y \hookrightarrow P := \widehat{\mathbb{P}^n_{\mathcal{V}}}$. Let $\{P_i := \widehat{\mathbb{A}^n}\}_{i=0}^n$ (where we invert the ith coordinate on $\mathbb{P}^n$) and let $Y_i = Y \times_P P_i$. Let $\overline{Y_i}$ be the closure of Y_i in P_i.

Let f_i (resp. g_i) denote the restriction of f (resp. g) to Y_i and denote by Z_i the subscheme defined by f_i and g_i. We claim that f and g lift to sections f' and g' of P. Indeed, the immersion $Y \hookrightarrow P$ factors through a closed immersion $Y \to \mathbb{A}^n_{\mathcal{V}}$; since this is a closed immersion the sections lift to $\mathbb{A}^n$, and we can lift to $\mathbb{P}^n$ by homogenizing.

Denote by $\overline{f_i}$ and $\overline{g_i}$ restrictions of the lifts f' and g' to $\Gamma(\mathcal{O}_{\overline{Y_i}})$ and define $\overline{Z_i}$ be the subscheme of $\overline{Y_i}$ defined by $\overline{f_i}$ and $\overline{g_i}$. Let $\overline{X_i}$ be the blow up of $\overline{Y_i}$ along $\overline{Z_i}$. Then $\overline{Z_i} \times_{\overline{Y_i}} Y_i = Z_i$ and $\overline{X_i} \times_{\overline{Y_i}} Y_i = X_i$.

Let $U_{i,1}$ be the tube $]\overline{Z_i}[_{(P_i)_K}$ of $\overline{Z_i}$ in $(P_i)_K$. Fix a rational number λ in $(0, 1)$, let $\hat{f_i}$ and $\hat{g_i}$ be lifts of $\overline{f_i}$ and $\overline{g_i}$ to $gamma(\mathcal{O}_{P_i})$, and define

$$U_{i,2} = \{x \in]\overline{X_i}[_{(P_i)_K} : |\hat{f_i}| > \lambda \text{or} |\hat{g_i}| > \lambda\};$$

by construction $U_{i,1} \cup U_{i,2}$ is a cover of $]\overline{X_i}[_{(P_i)_K}$.

The scheme $\overline{X_i}$ is a subscheme of $\mathbb{P}^1_{P_i}$ (indeed, if s and t are coordinates for $\mathbb{P}^1$, then $\overline{X_i}$ is defined by the equation $\overline{f_i}t - \overline{g_i}s$). Set $V_{i,1} := U_{i,1} \times_{(P_i)_K} (\mathbb{P}^1_{P_i})_K \cong \mathbb{P}^1_{U_{i,1}}$. The map $(X_i, V_{i,1}) \to (Y_i, U_{i,1})$ of overconvergent varieties factors as $(X_i, V_{i,1}) \to (Y_i, V_{i,1}) \to (Y_i, U_{i,1})$. The second map has a section and is thus universally of cohomological descent by Theorem 3.1 (2), and the first map is universally of cohomological descent with respect to crystals by Lemma 5.2 (for the first map, note that since $X_i \to Y_i$ is surjective, the map on tubes is an isomorphism); we conclude that $(X_i, V_{i,1}) \to (Y_i, U_{i,1})$ universally of cohomological descent with respect to crystals by Theorem 3.1 (5).

Let R_i be the closed formal subscheme of $\mathbb{P}^1_{P_i}$ defined by the equation $\hat{f}_i t - \hat{g}_i s$. Then $(R_i)_K \to (P_i)_K$ is an isomorphism away from the vanishing locus of $\hat{f}_i$ and $\hat{g}_i$ in $(P_i)_K$. Denote by $V_{i,2}$ the pre-image of $U_{i,2}$ under the map $(R_i)_K \to (P_i)_K$. Then the map $(X_i, V_{i,2}) \to (Y_i, U_{i,2})$ of overconvergent varieties factors as $(X_i, V_{i,2}) \to (Y_i, V_{i,2}) \to (Y_i, U_{i,2})$; the second map is an isomorphism, and the first map is universally of cohomological descent with respect to crystals by Lemma 5.2 (again, since $X_i \to Y_i$ is surjective, the map on tubes is an isomorphism); we conclude that $(X_i, V_{i,2}) \to (Y_i, U_{i,2})$ is universally of cohomological descent with respect to crystals by Theorem 3.1 (5).

We get a diagram

$$\begin{tikzcd}
\coprod ((X_i, V_{i,1}) \coprod (X_i, V_{i,2})) \arrow[rrr] \arrow[d] & & & X . \arrow[d] \\
\coprod ((Y_i, U_{i,1}) \coprod (Y_i, U_{i,2})) \arrow[r] & \coprod (Y_i, (P_i)_K) \arrow[r] & (Y, P_K) \arrow[r] & Y
\end{tikzcd}$$

The middle horizontal map is universally of cohomological descent with respect to crystals by Lemma 5.7 and the left and right horizontal maps are universally of cohomological descent by Theorems 4.17 and 3.1 (1); thus the composition is universally of cohomological descent with respect to crystals by Theorem 3.1 (5). By the previous two paragraphs and Theorem 3.1 (6), the left vertical map is universally of cohomological descent with respect to crystals; the lemma thus follows from Theorem 3.1 (4). ☐

The next lemma lets us reduce the case of a general blow up to the situation of Lemma 5.14.

Lemma 5.15. *Let Y be a Noetherian integral scheme, let $I \subset \mathcal{O}_Y$ be a sheaf of ideals globally generated by $r \geq 2$ many elements and let $X \to Y$ be the blow up of Y along I. Then there exists a map $X' \to X$ such that the composition $X' \to Y$ factors as*

$$X' = X_{r'} \to X_{r'-1} \to \cdots \to X_i \to X_{i-1} \to \cdots \to X_0 = Y,$$

where each map $X_i \to X_{i-1}$ is a blow up centered at an ideal which is globally generated by two elements.

Proof. This is a special case of [5], Lemma 3.4.4. ☐

Recall that a morphism $p: X \to Y$ is a modification if it is proper and an isomorphism over a dense open subscheme of Y. The next proposition shows that modifications are universally of cohomological descent with respect to crystals.

Proposition 5.16. *Let $p: X \to Y$ be a modification. Then p is universally of cohomological descent with respect to crystals.*

Proof. By Chow's lemma [24], Theorem 5.6.1 and 3.1 (4), we may assume that p is projective. By Corollary 5.12 and Theorem 3.1 (4) and (5), we may assume that Y is integral, and then by [25], Section 8, Theorem 1.24, there exists an affine open cover $\{Y_i\}$ of Y such that for each i, $Y_i \times_Y X \to Y_i$ is a blow up of Y_i along a closed subscheme; by Corollary 5.11, we may thus assume that p is a blow up. By the structure lemma for blow ups (Lemma 5.15), we may reduce to the case of a codimension one blow up which is Lemma 5.14. □

Flat covers

In this section, we prove Theorem 1.1 (i) – that finitely presented crystals are universally cohomologically descendable with respect to fppf (faithfully flat locally finitely presented) morphisms of schemes.

Definition 5.18. A map $(f, u) \colon (X', V') \to (X, V)$ of overconvergent varieties is said to be finite (see [9], Definition 3.2.3) if, up to strict neighborhoods, u is finite (see [18], paragraph after Lemma 1.3.7) and $u^{-1}(]X[_V) =]X'[_{V'}$. Moreover, u is said to be universally flat if u is quasi-finite and, locally for Grothendieck topology on V' and V, u is of the form $\mathcal{M}(A') \to \mathcal{M}(A)$ with $A \to A'$ flat (see [18], Definition 3.2.5).

Proposition 5.19. *Let $(f, u) \colon (X', V') \to (X, V)$ be a finite map of overconvergent varieties and suppose that, after possibly shrinking V' and V, u is universally flat and surjective. Then (f, u) is universally of cohomological descent with respect to finitely presented overconvergent crystals.*

Proof. To ease notation we set $p := (f, u)$. Let $F \in \mathrm{Mod}_{\mathrm{fp}}(X, V)$. By Corollary 3.4, it suffices to prove that (i) $\mathbb{R}^q p_{\bullet,*}p_\bullet^* F = 0$ for $q > 0$ and (ii) the Čech complex $F \to p_{\bullet,*}p_\bullet^* F$ is exact. Let $p_i := (f_i, u_i) \colon (X_i, V_i) \to (X, V)$ be the i-fold fiber product of p; p_i also satisfies the hypotheses of this proposition. By the spectral sequence (Remark 3.3), it suffices to prove that $\mathbb{R}^q p_{i*}p_i^* F = 0$ for $i \geq 0$ and $q > 0$.

Shrink V and V_i such that u_i is finite and such that $F_{X,V}$ is isomorphic to $i_X^{-1}G$ for some $G \in \mathrm{Coh}\mathcal{O}_V$ (which is possible by [9], Proposition 2.2.10). To prove (i), one can work with realizations as in [9], Proof of Proposition 3.2.4; it thus suffices to prove that $\mathbb{R}^q]u_i[_*]u_i[^* F_{X,V} = 0$ for $q > 0$. Then $\mathbb{R}^q]u_i[_*]u_i[^* F_{X,V} = i_X^{-1}\mathbb{R}^q u_{i*}u_i^* G$; by [18], Corollary 4.3.2 $\mathbb{R}^q u_{i*}u_i^* G = 0$ and (i) follows.

For (ii), since one can check exactness of a complex of abelian sheaves on the collection of all good realizations and since our hypotheses are stable under base change, it suffices to prove that the Čech complex of $F_{X,V}$ with respect to $]u[$ is exact. Since i_X^{-1} is exact, it suffices to prove that the Čech complex of G with respect to u is exact.

By [18], Proposition 4.1.2, G is a sheaf in the flat quasi-finite topology, so by Theorem 3.1 (1), $G \to u_{\bullet,*}u_\bullet^* G$ is exact in the flat quasi-finite topology; since G is coherent and (X, V) is good, this is exact in the usual topology.

□

Recall that a monogenic map of rings is a map of the form $A \to A[t]/f(t)$, where $f \in A[t]$ is a monic polynomial, and a map of affine formal schemes is said to be monogenic if the associated map on rings is monogenic.

Proof of Theorem 1.1 (i). By Theorem 5.9 and Corollary 3.2, we may assume that everything is affine.

Step 0: (Reduction to the finite and locally free case). Let $p \colon X \to Y$ be an fppf cover. By [26], Lemma (http://math.columbia.edu/algebraic_geometry/stacks-git/locate.php?tag=05WN), there exists a map $X' \to X$ such that the composition $X' \to Y$ is a composition of surjective finite locally free morphisms and Zariski coverings; by Theorem 5.9 and Corollary 3.2, we may assume that $X \to Y$ is finite and locally free.

Step 1: (Monogenic case). Suppose that $X \to Y$ is monogenic and choose a closed embedding $Y \hookrightarrow \mathbb{A}^n_{\mathcal{V}}$ (which exists since Y is affine and of finite type) and then an open immersion $\mathbb{A}^n_{\mathcal{V}} \subset P := \mathbb{P}^n_{\mathcal{V}}$. The polynomial defining $X \to Y$ lifts to a monic polynomial with coefficients, giving a monogenic (and thus finite and flat) map $X_0 \to \mathbb{A}^n_{\mathcal{V}}$, and then homogenizing this polynomial gives a map $\pi : P' \to P$ of schemes over Spec$\mathcal{V}$ and an embedding $X \hookrightarrow P'$ which is compatible with the embedding $Y \hookrightarrow P$. The map π may not be finite or flat (see Remark 5.20 below), but (noting that π is projective) by [12], Théorèm 5.2.2, there exists a modification $\widetilde{P} \to P$, centered away from X, such that the strict transform $\widetilde{P'} \to \widetilde{P}$ of $P' \to P$ is flat and (since it is generically finite, flat, and proper) finite.

Replacing $P' \to P$ with the formal completion of $\widetilde{P'} \to \widetilde{P}$, we thus have a finite flat map $P' \to P$ of formal schemes and an embedding $X \hookrightarrow P'$ which is compatible with the embedding $Y \hookrightarrow P$. Consider the diagram

$$\begin{tikzcd} (X, P'_K) \arrow[r] \arrow[d] & X \arrow[d] \\ (Y, P_K) \arrow[r] & Y \end{tikzcd} .$$

By Theorems 4.17 and 3.1 (1), $(Y, P_K) \to Y$ is universally of cohomological descent with respect to crystals, so by Corollary 3.2 it suffices to prove that $(X, P'_K) \to (Y, P_K)$ is universally of cohomological descent with respect to finitely presented crystals. Since $X = Y \times_P P'$, $(X, P'_K) \to (Y, P_K)$ satisfies the hypotheses of Proposition 5.19 and step 1 follows.

Step 2: (Base extension). Now let $k \subset k'$ be a finite field extension of the residue field. We claim that it suffices to check that $X_{k'} \to Y_{k'}$ is universally of cohomological descent with respect to finitely presented crystals. Indeed, let $k = k_0 \subset k_1 \subset \ldots \subset k_{n-1} \subset k_n = k'$ be a sequence of field extensions such that $k_i = k_{i-1}(\alpha_i)$ for some $\alpha_i \in k_i$ (note that one may not be able to choose $n = 1$ since $k \subset k'$ may not be separable). Consider the diagram

$$\begin{tikzcd} X_{k_n} \arrow[rrrr] \arrow[d] & & & & X \arrow[d] \\ Y_{k_n} \arrow[r] & Y_{k_{n-1}} \arrow[r] & \cdots \arrow[r] & Y_{k_1} \arrow[r] & Y_{k_0} \end{tikzcd} .$$

Each map $Y_{k_i} \to Y_{k_{i-1}}$ is monogenic and thus universally of cohomological descent with respect to finitely presented crystals, the claim thus follows from Theorem 3.1 (4), (5), and (6).

Step 3: (Reduction to the monogenic case). Let $\overline{k}$ be the algebraic closure of k and let $p_{\overline{k}} : X_{\overline{k}} \to Y_{\overline{k}}$ be the base change of p to $\overline{k}$. Let $x \in X_{\overline{k}}$ be a closed point and set $y = p_{\overline{k}}(x)$. Let $\overline{k}(x)$ and $\overline{k}(y)$ denote the residue fields of x and y; since $\overline{k}$ is algebraically closed, $\overline{k}(x) =$

$\overline{k}(y)$. In particular, $\overline{k}(y)$ is a separable extension of $\overline{k}(x)$, and thus, by the argument of [27], 2.3, Proposition 3, there exists a (generally non-cartesian) commutative diagram

$$\begin{tikzcd} X_x \arrow[r] \arrow[d] & X_{\overline{k}} \arrow[d] \\ Y_y \arrow[r] & Y_{\overline{k}} \end{tikzcd}$$

where X_x (resp. Y_y) is an affine open neighborhood of x (resp. y) and $X_x \to Y_y$ is monogenic. By quasi-compactness of $Y_{\overline{k}}$, there thus exists a (generally non-cartesian) commutative diagram

$$\begin{tikzcd} \coprod X_i \arrow[r] \arrow[d, "\coprod f_i"] & X_{\overline{k}} \arrow[d] \\ \coprod Y_i \arrow[r] & Y_{\overline{k}} \end{tikzcd}$$

such that $\{Y_i\}$ is a finite cover of $Y_{\overline{k}}$ by affine open subschemes of finite type over $\overline{k}$, X_i is an affine open subscheme of $X_{\overline{k}}$, and each map $f_i \colon X_i \to Y_i$ is monogenic. Since the covering is finite, there exists a finite field extension $k \subset k'$ and a (generally non-Cartesian) commutative diagram

$$\begin{tikzcd} \coprod X_i' \arrow[r] \arrow[d, "\coprod f_i"] & X_{k'} \arrow[d] \\ \coprod Y_i' \arrow[r] & Y_{k'} \end{tikzcd}$$

with the same properties. By step 2, it suffices to check that $X_{k'} \to Y_{k'}$ is universally of cohomological descent with respect to finitely presented crystals. By Corollary 3.2, it suffices to prove this for each i, the map $X_i' \to Y_i'$, which follows from step 1.

Remark 5.20. The modification in step 1 of the proof is necessary. Indeed, the monogenic map $X \to \mathbb{A}^2$ given by $t^2 + x_1x_2t + x_1 + x_2$ homogenizes to the map $X' \to \mathbb{P}^2$ given by $x_0^2t^2 + x_1x_2ts + (x_1 + x_2)x_0s^2$ which is not flat, since it is generically quasi-finite but not quasi-finite (since the fiber over $x_0 = x_1 = 0$ is $\mathbb{P}^1$).

Proper surjections

The proper case of the main theorem will now follow from Chow's lemma and the Raynaud-Gruson theorem on 'Flattening Blow Ups'.

Proof of Theorem 1.1 (ii). Let $p \colon X \to Y$ be a proper surjection. By Chow's lemma [24], Theorem 5.6.1 and Theorem 3.1 (4), we may assume that p is projective. By Corollary 5.11 we may assume that Y is affine and thus by [12], Théorèm 5.2.2, there exists a modification $Y' \to Y$ such that the strict transform $X' \to Y'$ is flat. By Theorem 1.1 (i) (resp. 5.16) $X' \to Y'$ (resp. $Y' \to Y$) is universally of cohomological descent with respect to finitely presented crystals. By 3.1 (5), the composition $X' \to Y$ is universally of cohomological descent, and the proper case of the main theorem follows from 3.1 (4).

Competing interests
The author declares that he has no competing interests.

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