SERRE SPECTRAL SEQUENCES AND APPLICATIONS

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Abstract. Spectral sequences are a key theoretical and computational tool in algebraic topology. This paper is a tutorial essay, intended for students and researchers as a quick introduction, providing a few nice examples, applications, and a short list of references for further inquiry.

Contents

1. Introduction

Algebraic topology focuses on the properties of spaces and maps between spaces which are invariant under 'wiggling'. Some central tools are algebraic gadgets such as homotopy, homology, and cohomology; these give a way to distinguish spaces and occasionally completely

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characterize spaces. Serre Spectral Sequences are a powerful theoretical and computational tool with numerous applications to Algebraic Topology. Indeed, a main application is the computation of various homotopy groups of spheres, and we will demonstrate this by the calculation of $\pi_4(S^3)$.

The exposition will first introduce the notion of a spectral sequence. As an inspiring example we will calculate $\pi_4(S^3)$. We next shift gears and explain an application of spectral sequences to the author's first love, number theory. Finally, we conclude with a few pages of abstract nonsense (see [Lan02, Part Four, Introduction] for a precise definition of abstract nonsense), illuminating the notion of a spectral sequence as we get our hands dirty with the pure homological algebra at its core and discover its essence.

We use as our main reference for any definitions or theorems Allen Hatcher's *Algebraic Topology* [Hat02]. The more topological pieces of this exposition follows Hatcher's upcoming book *Serre Spectral Sequences in Algebraic Topology* [Hat04]. The more abstract parts follow McCleary's book *A User's Guide to Spectral Sequences* [McC01], with insights from Chow's article *You Could Have Invented Spectral Sequences* [Cho06].

Remark 1.1. Basic definitions from algebraic topology will be occasionally given and occasionally omitted, the idea being that certain notions are really prerequisite for this exposition to have any meaning. For example, it would be silly to define notions such as Fibration or Cohomology because if you don't already know and care about these notions then briefly defining them and proving enough properties to make this exposition precise would be pointless, because you won't learn anything or care about Serre Spectral Sequences. Conversely, it is easy to forget precise statements of theorems like Universal Coefficients, so where appropriate I will state them, and in any case I will always try to give concise citations.

2. What is a Spectral Sequence?

The notion of a spectral sequence generalizes many typical algebraic situations that pop up in topology. For instance, instead of a pair $A \subset X$, consider a filtration

$$X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_i \subset \ldots \subset X. \tag{2.1}$$

Then instead of the long exact sequence of the pair (X, A), which relates the cohomology rings $H^i(X)$, $H^i(A)$, and $H^i(X, A)$, we get an algebraic relation between the cohomology of everything appearing in (2.1). Similarly one can generalize the Mayer-Vietoris sequence of a decomposition $X = A \cup B$ to a covering by any number of sets.

So again, what is a spectral sequence? Poetically, we begin with some initial data (like a filtration of a topological space X) and construct an infinite book. We can give each page the structure of a graded complex. Then the $(p+1)^{\text{th}}$ page is formed from the homology of the p^{th} page. Under suitable conditions each entry of a page eventually stabilizes, and we can pass to a limit, forming page ∞. And again under suitable conditions, the entries on page ∞ tells us a lot about our initial data (for instance, information about the cohomology ring $H^*(X)$).

3. Theorems from Algebraic Topology

Here we recall without proof standard theorems from algebraic topology that will be used later in this exposition.

Theorem 3.1. *(Universal Coefficients for Cohomology) Let C be a chain complex of free abelian groups with homology groups $H_n(C)$, and let G be a group. Then the cohomology groups of the chain complex $Hom(C_n, G)$ fit into the split exact sequences*

$$0 \to Ext(H_{n-1}(C), G) \to H^n(C; G) \to Hom(H_n(C), G) \to 0.$$

In practice, the Ext term either vanishes or is easily computable. See [Hat02, Chapter 3, Section 1, Theorem 3.2] for proofs and to further understand the Ext term.

Theorem 3.2. *(Hurewicz) Let X be a simply connected space, and let $i > 1$ be the first integer such that $\pi_i(X) \neq 0$. Then we have an isomorphism*

$$\pi_i(X) \cong H_i(X)$$

and a surjection

$$\pi_{i+1}(X) \to H_{i+1}(X) \to 0.$$

See [Hat02, Chapter 4, Section 2, Theorem 4.32] for a proof.

Theorem 3.3. *(Long Exact Sequence of a Fibration) Let $F \to E \to B$ be a fibration (see* [Hat02, Chapter 4, Section 3] *for a definition). Then there is a long exact sequence*

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(B) \to \pi_{n-1}(F) \to \cdots \to \pi_0(E) \to 0.$$

See [Hat02, Chapter 4, Section 2, Theorem 4.41] for a proof.

4. Main Application to Algebraic Topology: Cohomology Via Fibrations

We begin with a (somewhat) concrete example. Consider a fibration

$$F \to X \to B.$$

We will start by defining 'page 2'; page 0 and 1 are interesting and will later be illuminating, but the journey from page 0 to 2 is best saved for the abstract nonsense of the appendix. Our goal will be to deduce *something* about $H^*(X)$ based on information about $H^*(B)$ and $H^*(F)$.

Set

$$E_2^{p,q} := \begin{cases} H^p(B, H^q(F; G)) & p, q \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

In the appendix we explain how to construct maps $E_2^{p,q} \xrightarrow{\partial} E_2^{p+2,q-1}$ giving $E_2^\bullet$ the structure of a graded complex:

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
E_2^{0,3} \arrow[drr] & E_2^{1,3} \arrow[drr] & E_2^{2,3} \arrow[drr] & E_2^{3,3} & E_2^{4,3} & \cdots \\
E_2^{0,2} \arrow[drr] & E_2^{1,2} \arrow[drr] & E_2^{2,2} \arrow[drr] & E_2^{3,2} & E_2^{4,2} & \cdots \\
E_2^{0,1} \arrow[drr] & E_2^{1,1} \arrow[drr] & E_2^{2,1} \arrow[drr] & E_2^{3,1} & E_2^{4,1} & \cdots \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & E_2^{3,0} & E_2^{4,0} & \cdots
\end{tikzcd}
$$

We now take homology, setting

$$E_3^{p,q} := \frac{\ker\left(E_2^{p,q} \xrightarrow{\partial} E_2^{p+2,q-1}\right)}{\operatorname{im}\left(E_2^{p-2,q+1} \xrightarrow{\partial} E_2^{p,q}\right)}$$

to get page three. Similarly, we get maps $E_3^{p,q} \xrightarrow{\partial} E_3^{p+3,q-2}$ and a graded complex:

$$
\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
E_3^{0,3} \arrow[ddrrr] & E_3^{1,3} \arrow[ddrrr] & E_3^{2,3} & E_3^{3,3} & E_3^{4,3} & \cdots \\
E_3^{0,2} \arrow[ddrrr] & E_3^{1,2} \arrow[ddrrr] & E_3^{2,2} & E_3^{3,2} & E_3^{4,2} & \cdots \\
E_3^{0,1} \arrow[ddrrr] & E_3^{1,1} \arrow[ddrrr] & E_3^{2,1} & E_3^{3,1} & E_3^{4,1} & \cdots \\
E_3^{0,0} & E_3^{1,0} & E_3^{2,0} & E_3^{3,0} & E_3^{4,0} & \cdots \\
0 & 0 & 0 & 0 & 0 & \cdots
\end{tikzcd}
$$

We take homology to get page 4.

More generally we get maps $E_r^{p,q} \xrightarrow{\partial} E_r^{p+r,q-r+1}$ on the r^{th} page and take homology to get the $(r+1)^{\text{th}}$ page.

Observe that each ∂ eventually becomes trivial; for a fixed (p,q) and large r the map $E_r^{p,q} \xrightarrow{\partial} E_r^{p+r,q-r+1}$ will have trivial co-domain, and the map $E_r^{p-r,q+r-1} \xrightarrow{\partial} E_r^{p,q}$ with have trivial domain. Thus for a fixed (p,q) and large r (depending on p and q) we have

$$E_r^{p,q} = E_{r+1}^{p,q} = E_{r+2}^{p,q} = \dots$$

and so for such an r we set $E^{\infty}_{p,q} := E_r^{p,q}$.

The punchline is that with sufficient assumptions on our fibration, the groups $E_\infty^{p,q}$, which we constructed from the data $H^p(B, H^q(F))$, are related to the groups $H^{p+q}(X)$.

More precisely, let us specialize to the case of a fibration of CW-complexes $F \to X \xrightarrow{\pi} B$ such that B is simply-connected. The p skeletons

$$B^0 \subset B^1 \subset \dots \subset B^p \subset \dots \subset B$$

induce a filtration on X; set $X^p := \pi^{-1}(B^p)$. Then

$$X^0 \subset X^1 \subset \dots \subset X^p \subset \dots \subset X$$

is a filtration. The inclusions $X_p \hookrightarrow X$ induce maps $H^n(X) \to H^n(X_p)$ and so we set

$$F_n^p := \ker H^n(X) \to H^n(X_p).$$

This gives a filtration of groups

$$0 \subset F_n^n \subset \dots \subset F_p^n \subset \dots \subset F_0^n = H^n(X).$$

Then the theorem of Serre says

$$E_\infty^{p,n-p} \cong F_p^n / F_{p+1}^n.$$

See [Hat04] for a more lengthy discussion and more general theorems.

5. Additional Data

5.1. **Naturality.** Given maps of fibrations, i.e. commutative diagrams

$$\begin{tikzcd} F \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & B \arrow[d, "\phi"] \\ F' \arrow[r] & X' \arrow[r] & B' \end{tikzcd}$$

such that the fiber over $b \in B$ maps to the fiber over $\phi(b) \in B'$, we get maps

$$E_r^{p,q} \to E_r^{'\,p,q}$$

(including the case $r = \infty$) which respect all differentials and isomorphisms.

5.2. **Derivations.** Often we have extra structure, which the spectral sequence respects. For example, when X is a topological space $H^*(X)$ has a ring structure induced by the cup product, and the differentials ∂ respect this ring structure. More precisely, ∂ is a derivation, and the Leibintz rule is satisfied:

$$\partial(\alpha \cdot \beta) = \partial(\alpha) \cdot \beta + (-1)^{p+q} \alpha \partial(\beta).$$

More precisely, there are bilinear products $E_r^{p,q} \times E_r^{s,t} \to E_r^{p+s,q+t}$ which are just the standard cup product when $r = 2$. As we will soon see, this extra structure can allow us to compute information about the spectral sequence that we otherwise could not. See [Hat04, Chapter 1, Section 2] for proofs and further discussion.

6. Eilenberg-MacLane Spaces

Given a group G and an integer n, one can use CW-complexes to elegantly construct spaces $K(G, n)$ with the nice property that

$$\pi_i\left(K(G, n)\right) = \begin{cases} G & \text{if } i = 0, n \\ 0 & \text{otherwise} \end{cases}.$$

The construction is simple; choose a presentation for G (generators and relations), and create a CW-complex with one zero-cell, one n-cell for each generator, one (n+1)-cell for each relation, then (n+2)-cells to kill of any homotopy's added by the (n+1)-cells, and so on.

This construction tends to have cells in every dimension, giving a possibility of non-trivial homology and cohomology groups of every dimension. Indeed, they are often non-trivial! While we know from the Hurewicz theorem (3.2) and Universal Coefficients for cohomology (3.1) that

$$G = \pi_n(K(G, n)) \cong H_n(K(G, n)),$$

$$H^n(K(G, n)) \cong \mathrm{Hom}(H_n(K(G, n)), \mathbb{Z}) \cong \mathrm{Hom}\left(G, \mathbb{Z}\right),$$

this is pretty much all we know in general, and calculating higher homotopy groups is one task we will now demonstrate using a spectral sequence.

See [Hat02, 4.2] for more facts on Eilenberg-MacLane Spaces.

6.1. $K(\mathbb{Z}, 2)$**.** Using the fact that S^1 is a $K(\mathbb{Z}, 1)$, we calculate $H^*(K(\mathbb{Z}, 2))$.

Consider the pathspace fibration [Hat02, 4.3] $F \to P \to B$, where B is a $K(\mathbb{Z}, 2)$. P is contractible, so from the long exact sequence of a fibration (3.3) we get for $i > 0$ that

$$\pi_i(K(\mathbb{Z}, 1)) \cong \pi_{i+1}(K(\mathbb{Z}, 2))$$

and conclude that F is a $K(\mathbb{Z},1)$:

$$\pi_i(F) = \begin{cases} \mathbb{Z} & \text{if } i = 0,1 \\ 0 & \text{otherwise} \end{cases}.$$

Setting

$$E_2^{p,q} := H^p(K(\mathbb{Z},2); H^q(S^1)) = \begin{cases} H^p(S^1,\mathbb{Z}) & \text{if } q = 0,1 \\ 0 & \text{otherwise} \end{cases},$$

we get the following for page one:

$$\begin{array}{c|ccccc}
2 & 0 & 0 & 0 & 0 & 0 \\
1 & H^0(K(\mathbb{Z},2)) & H^1(K(\mathbb{Z},2)) & H^2(K(\mathbb{Z},2)) & H^3(K(\mathbb{Z},2)) & H^4(K(\mathbb{Z},2)) \\
0 & H^0(K(\mathbb{Z},2)) & H^1(K(\mathbb{Z},2)) & H^2(K(\mathbb{Z},2)) & H^3(K(\mathbb{Z},2)) & H^4(K(\mathbb{Z},2)) \\
\hline
 & 0 & 1 & 2 & 3 & 4
\end{array}.$$

The differentials on page 2 are

$$H^r(K(\mathbb{Z},2)) \to H^{r+2}(K(\mathbb{Z},2)).$$

Furthermore, these are the only differentials that have a chance of killing anything (because all other differentials travel at least two rows and thus start or end with a 0), and so page 3 will be the same as page ∞. Consider also that the middle space is contractible, so that each $E_\infty^{p,q}$ must be 0. We finally conclude that each differential is an isomorphism, and thus

$$H^r(K(\mathbb{Z},2)) \cong \begin{cases} \mathbb{Z} & r \text{ even} \\ 0 & r \text{ odd} \end{cases}.$$

6.2. $K(\mathbb{Z},3)$. Similarly, we have a pathspace fibration $F \to P \to B$, where B is a $K(\mathbb{Z},3)$, and again P is contractible and (from the long exact sequence of a fibration (3.3)) F is a $K(\mathbb{Z},2)$ (and so homotopy equivalent to $\mathbb{C}P^\infty$). The Hurewicz theorem (3.2) tells us that $H^3(K(\mathbb{Z},3)) = \mathbb{Z}$. Furthermore, below (7) we will see that we can find a model of $K(\mathbb{Z},3)$ with no 4-cells, so $H^4(K(\mathbb{Z},3))$ and $H_4(K(\mathbb{Z},3)) = 0$. From Universal Coefficients (3.1) we see that

$$H^5(K(\mathbb{Z},3)) = \mathrm{Ext}(H_4(K(\mathbb{Z},3),\mathbb{Z}) \oplus \mathrm{Hom}(H_5(K(\mathbb{Z},3)),\mathbb{Z}) = 0.$$

Setting $E_2^{p,q} := H^p(K(\mathbb{Z},3), H^q(\mathbb{C}P^\infty))$, we get the following for page two:

$$\begin{array}{c|ccccccc}
5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & H^5(K(\mathbb{Z},3)) & H^6(K(\mathbb{Z},3)) \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & H^5(K(\mathbb{Z},3)) & H^6(K(\mathbb{Z},3)) \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & H^5(K(\mathbb{Z},3)) & H^6(K(\mathbb{Z},3)) \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}.$$

Now we use the spectral sequence to calculate $H^5(K(\mathbb{Z},3))$ and $H^6(K(\mathbb{Z},3))$. Since every other row is 0, all of the differentials on page 2 are trivial and page 3 is the same as page 2. To fully analyze page three we need to use the multiplicative structure (5.2). Let x be a generator of $H^3(K(\mathbb{Z},3))$, and consider the map

$$H^0(K(\mathbb{Z},3); H^2(\mathbb{C}P^\infty)) \xrightarrow{\partial} H^3(K(\mathbb{Z},3); H^0(\mathbb{C}P^\infty)).$$

We need to understand the image of $z = z \cdot 1$. Upon inspection we see that the only differential which can affect $E_\infty^{0,2}$ or $E_\infty^{3,0}$ is the differential on page 3. Furthermore, we know that $H^n(*) = 0$ for all n, and so $E_\infty^{p,q} = 0$ whenever either p or q is non-zero. Thus $E_\infty^{0,2}$ and $E_\infty^{3,0}$ are both zero. This can only happen if $\partial(z)$ is a generator (and so we choose the sign of x so that $\partial(z) = x$).

Similarly, we can use the Leibintz property of the differentials to calculate

$$H^0(K(\mathbb{Z},3), H^4(\mathbb{C}P^\infty)) \xrightarrow{\partial} H^3(K(\mathbb{Z},3), H^2(\mathbb{C}P^\infty)).$$

Indeed, the generator of the left term is z^2, and $\partial(z^2) = \partial(z)z + z\partial(z) = x \cdot z + z \cdot x$. Since z has even degree, the cup product of x and z is commutative, and we get $z \cdot x + z \cdot x = 2z \cdot x$. In particular the differential is multiplication by 2. We deduce that $E_4^{0,4}$ is zero (since the differential is injective. Furthermore, the differential

$$\mathbb{Z}/2\mathbb{Z} \cong E_3^{3,2}/\mathrm{Im}\partial \xrightarrow{\partial} H^6(K(\mathbb{Z},3))$$

must be an isomorphism; indeed everything on the $p+q=5$ and $p+q=6$ diagonals must eventually die, and this is the last differential with a chance to kill either. We conclude that

$$H^6(K(\mathbb{Z},3)) \cong \mathbb{Z}/2\mathbb{Z}.$$

Similarly, $E_3^{5,0} = H^5(K(\mathbb{Z},3))$ must die too. The only differential with a chance to kill it is

$$0 = E_5^{0,4} \xrightarrow{\partial} H^5(K(\mathbb{Z},3)),$$

and so it must be zero.

6.3. **General Strategy.** Let $1 \to A \to B \to C \to 1$ be an exact sequence of groups, and assume the further technical condition that A is in the center of B (then the action of $/pi_1(K(C,1)) = C$ on $H_*(K(A,1);G)$ is trivial. Then there is a fibration

$$K(A,1) \to K(B,1) \to K(C,1).$$

See [Hat04, Chapter 1, Section 1, Exercise 3]). This together with pathspace fibrations give a general first attempt at computing homology and cohomology of $K(\mathbb{G}, n)$'s.

7. Homotopy groups of spheres - $\pi_4(S^3)$

This is one of the more interesting applications of spectral sequences. A good reference for more general facts about homotopy groups of spheres is [Rav86]. Here we will use a spectral sequence to calculate $\pi_4(S^3)$.

S^3 has a CW-structure consisting of one 0-cell and one 3-cell. We can kill off all higher homotopy groups by attaching higher dimensional cells, forming a $K(\mathbb{Z},3)$, and giving a fibration $S^3 \hookrightarrow K(\mathbb{Z},3)$

$$\begin{tikzcd} F \arrow[r] & S^3 \arrow[d] \\ & K(\mathbb{Z},3) \end{tikzcd}.$$

From the long exact sequence of a fibration (3.3), we deduce that

$$\pi_i(F) = \begin{cases} \pi_i(S^3) & \text{if } i \geq 4 \\ 0 & i \leq 3 \end{cases}.$$

The Hurewicz theorem (3.2), says that the first non-trivial homology and homotopy groups are isomorphic, so we conclude that

$$\pi_4(S^3) \cong \pi_4(F) \cong H_4(F).$$

We want to use a spectral sequence to compute $H^5(F)$ and $H^4(F)$. We can then use universal coefficients to compute $H_4(F)$. Indeed, using the additional data that $H_i(F)$ is a finitely generated abelian group [Hat04, Chapter 1, Section 1], we write $H_4(F) \cong \mathbb{Z}^r \oplus T$ (where T is the finite and torsion). Universal Coefficients for cohomology (3.1) gives the exact sequence

$$1 \to \mathrm{Ext}(H_{i-1}(F), \mathbb{Z}) \to H^i(F) \to \mathrm{Hom}(H_i(F), \mathbb{Z}) \to 1.$$

Letting $i = 4$ and recalling that $H_3(F) = 0$, we get that $\mathrm{Hom}(H_4(F), \mathbb{Z}) = \mathbb{Z}^r$ is equal to the free part of $H^4(F)$, and thus the ranks of $H_4(F)$ and $H^4(F)$ are equal. Letting $i = 5$ we get $H^5(F) \cong \mathrm{Ext}(H_4(F), \mathbb{Z}) \oplus \mathrm{Hom}(H_5(F), \mathbb{Z})$. But $\mathrm{Ext}(H_4(F), \mathbb{Z}) = T$. Thus we can deduce the free part of $H_4(F)$ from $H^4(F)$ and the torsion part from $H^5(F)$.

To proceed we take one further fibration; by [Hat02, 4.3] we also have a fibration

$$\Omega(K(\mathbb{Z},3)) \to F \to S^3.$$

From the long exact sequence of a pathspace fibration (6.1), we wee that $\Omega(K(\mathbb{Z},3))$ is a $K(\mathbb{Z},2)$, which has the homotopy type of (and is thus homotopy equivalent to) $\mathbb{C}P^\infty$.

We can calculate (directly from the cell complexes!) that

$$H^*(S^3) \cong \mathbb{Z}[x]/(x^2), \text{degree } x = 3.$$

and

$$H^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[z], \text{degree } \alpha = 2.$$

Setting $E_2^{p,q} := H^p(S^3, H^q(\mathbb{C}P^2))$, we conclude that page two is

$$\begin{array}{c|cccccc}
5 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & 0 & 0 & \mathbb{Z} & 0 & 0 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5
\end{array}.$$

Rewriting page two with generators (for example, from (5.2) the generator of $H^3(S^3, H^2(\mathbb{C}P^2))$ is just the cup product $z \cdot x$)

$$\begin{array}{c|ccccccc}
6 & z^3 & 0 & 0 & z^3x & 0 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & z^2 & 0 & 0 & z^2x & 0 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & z & 0 & 0 & zx & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & x & 0 & 0 & 0 \\
\hline
 & 0 & 1 & 2 & 3 & 4 & 5 & 6
\end{array}$$

the formal argument comes through. Nothing happens on page two, so page three is identical. We know by Hurewicz (3.2)) that $H_2(F)$ and $H_3(F)$ are 0, and by the universal coefficients theorem for cohomology (3.1) $H^2(F)$ and $H^3(F) = 0$ too. Thus $E_\infty^{2,0}$ and $E_\infty^{0,3}$ must both be 0, and the differential on page three from the (2,0) spot to the (0,3) spot must be an isomorphism; $\partial(z) = x$.

Using that ∂ is a derivation, we reason that $\partial(z^2) = 2zx$. Thus the differential from (0,4) to (3,2) is injective, and thus $E_\infty^{0,4} = 0$. We conclude that $H^4(F) = 0$. Similarly, $\partial(z^p) = pz \cdot x$, and so for i even the $p + q = i$ diagonal is empty. We conclude that $H^i(F) = 0$ for i even.

Next consider the filtration

$$H^5(F) \supset F^5_0 \supset F^5_1 \supset F^5_2 \supset F^5_3 \supset F^5_4 \supset F^5_5 \supset F^5_6 = 0.$$

Observe that $E_\infty^{p,5-p} = 0$ for $i = 0 \ldots 2$, $H^5(F) = F^5_0 = \cdots F^5_3$ and $F^5_4 = F^5_5 = 0$. Thus $F^5_3/F^5_4 = F^5_3 = \mathbb{Z}/2\mathbb{Z}$. We conclude that $E_\infty^{3,2} = \mathbb{Z}/2\mathbb{Z}$ and $E_\infty^{p,5-p} = 0$ if $p \neq 3$. With similar reasoning along the $p + q = i$ (i odd) diagonal we conclude that

$$H^i(F) = \begin{cases} \mathbb{Z}/\left((i-1)/2\right)\mathbb{Z} & i \text{ odd} \\ 0 & i \text{ even} \end{cases}$$

We have that $H^4(F) = 0$, and so the free part of $H_4(F)$ is zero. Also $H^5(F) = \mathbb{Z}/2\mathbb{Z}$, and so the torsion part of $H_4(F)$ is $\mathbb{Z}/2\mathbb{Z}$. We conclude that $H_4(F) = \pi_4(F) = \mathbb{Z}/2\mathbb{Z}$.

8. Grothendieck Spectral Sequence

Let $\mathscr{A}, \mathscr{B}$, and $\mathscr{C}$ be abelian categories, and let $\mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{C}$ be two additive left exact functors such that the composition is also left exact. Suppose our abelian categories have nice enough properties for right derived functors R^iF, R^iG and $R^i(G \circ F)$ to exist, and suppose also that if I is injective in $\mathscr{A}$, then $F(I)$ is G-acyclic. (See [Lan02, XX, section 9] for these definitions, but don't worry much about them. They are just the most general technical conditions necessary for the following question to even make sense.) Once can ask the following

Question: Is there any relation between $R^{p+q}(G \circ F)$ and $R^qG \circ R^pF$?

The answer to this loaded question is a resounding "YES!", and again comes from a spectral sequence. Indeed, in [Lan02, XX, Theorem 9.6] we find the following:

Theorem 8.1. *For each A in $\mathscr{A}$, there is a spectral sequence $\{E_r(A)\}$ such that*

$$E_2^{p,q}(A) = R^pG(R^qF(A))$$

such that $E_r^{p,q}$ 'converges' (in a similar sense as above, involving filtrations and extensions) to $R^{p+q}(G \circ F)(A)$.

Most of the spectral sequences we know and love arise as special cases of the Grothendieck Spectral Sequence, often as derived functors of $M \otimes -$ or $\mathrm{Hom}(-, C)$ (the spectral sequence described in (4) arises from Hom on a suitable category of cellular complexes). A fun exercise for the interested reader would be to derive the spectral sequences which appear 'in nature' as instances of the Grothendieck spectral sequence. See also [McC01, Chapter 12, Section 4] for more on this.

As demonstrated in the examples above, we can squeeze a lot of information out of the spectral sequence from purely formal considerations. The following lemma can be proved purely homologically (i.e. for a Grothendieck Spectral Sequence in an arbitrary abelian category), and nicely compactifies a few purely formal arguments one uses repeatedly. Setting $L^{p+q} := R^pG(R^qF(A))$ we get the following

Lemma 8.2. *There is an exact sequence*

$$0 \to E_2^{1,0} \to L^1 \to E_2^{0,1} \xrightarrow{\partial} E_2^{2,0} \to \ker\left(L^2 \to E_2^{0,2}\right) \to E_2^{1,1} \to E_2^{3,0}.$$

The proof of this is purely formal and requires only minimal manipulations of exact sequences. In the next section we apply this to group cohomology and observe as a corollary the inflation-restriction exact sequence.

Remark 8.3. Unfortunately, not all spectral sequences come from this construction; as we will see in the appendix the cohomology of the total complex of a double complex is the limit of a spectral sequence starting on page 0.

9. Group Cohomology and the Hochschild-Serre Spectral Sequence

The Grothendieck Spectral Sequence gives us a clue as to how one might apply spectral sequences to other areas, Algebraic Geometry or Number Theory. An amazing (to the author) fact is that spectral sequences can play a significant role in understanding and finding integer solutions to polynomial equations. Indeed, in [PSS05], spectral sequences are used as a key step in their classification of coprime integer solutions to the equation $x^2 + y^3 = z^7$.

Let G be a group and let M be a G-module (an abelian group equipped with an action of G). Consider the **fixed-point functor**

$$M^G := \{m \in M | gm = m \text{ for all } g \in G\}.$$

An easy computation shows that this is left-exact, i.e. if

$$1 \to M' \to M \to M'' \to 1$$

is exact, then

$$1 \to M'^G \to M^G \to M''^G$$

is also exact. It is a standard fact that the category of G-modules has 'enough injectives' and satisfies the conditions necessary for right-derived functors of $-^G$ to exist. We call the right derived functor $H^n(G, A) := R^i(-^G)(A)$ and call this 'group cohomology'. The canonical reference for these facts is [Ser02].

Consider a short exact sequence of groups

$$1 \to H \to G \to G/H \to 1.$$

One hopes that there is a relation between the fixed point functors $-^H$, $-^G$, and $-^{G/H}$, and indeed, here we have a composition of functors waiting to be exploited!

Let M be a G-module. Then M is also an H-module, and H acts trivially on M^H; thus M^H is a G/H-module. Furthermore, $(M^H)^{G/H} = M^G$. This is the crucial observation, because now the composition of the functors

$$\{G - \text{modules}\} \xrightarrow{-^H} \{G/H - \text{modules}\} \xrightarrow{-^{G/H}} \mathscr{A}$$

is

$$\{G - \text{modules}\} \xrightarrow{-^G} \mathscr{A}$$

If G and H are finite abelian groups, then $\mathscr{A}$ is the category of abelian groups. In full generality the best one can do is $\mathscr{A} = \{$ pointed sets $\}$, for if G is non-abelian, while M^G is still an abelian group, the cohomology no longer has any group structure, only a canonical element. ♠♠♠ David: [Changed] There are many non-trivial results in number theory, co-prime integer solutions to $x^2 + y^3 = z^7$ for instance, whose resolution entices one to care about the category of pointed sets.

Now we turn the crank of the Grothendieck Spectral Sequence, setting

$$E_2^{p,q} := H^p(G/H, H^q(H, A)),$$

and conclude that it 'converges' to $H^{p+q}(G, A)$.

Remark 9.1. The lemma from section (8.2) amounts to the exactness of

$$\begin{aligned} 0 \longrightarrow H^1(G/H, A^H) &\longrightarrow H^1(G, H) \longrightarrow H^1(H, A)^{G/H} \\ &\longrightarrow H^2(G/H, A^H) \longrightarrow \ker\left(H^2(G, A) \to H^2(H, A)\right) \longrightarrow H^1(G/H, H^1(H, A)) \\ &\longrightarrow H^3(G/H, A^H). \end{aligned}$$

This is a more general version of the *inflation-restriction exact sequence*, which plays a role in group cohomology similar to the role of the long exact sequence of a fibration in topology.

Remark 9.2. Another application of the spectral sequence of a fibration is to develop group cohomology using twisted $K(G, 1)$ spaces.

10. Appendix

Here I talk about some of the more abstract theory behind spectral sequences.

Let $C^\bullet$ be a chain complex

$$\cdots \xrightarrow{\partial} C_{d+1} \xrightarrow{\partial} C_d \xrightarrow{\partial} C_{d-1} \xrightarrow{\partial} \cdots$$

whose homology we want to compute. For instance, we care about the cohomology ring $H^*(X)$ of a topological space X. Often our chain complex has some extra structure which makes things easier to compute. Suppose, for example, that each C_d was in fact a graded ring

$$C_d = \bigoplus_{i=1}^{\infty} C_{d,i},$$

and that the boundary maps ∂ respect the grading:

$$\partial(C_{d,i}) \subset C_{d-1,i}$$

Here, understanding the grading and the homology of each graded part allows us to recover the homology of the entire complex. Indeed, the homology of the larger complex is just the sum of the homology of each graded part:

$$h(C^\bullet) = \bigoplus_{i=1}^{\infty} h(C_i^\bullet).$$

In practice, we can't really compute the homology of all the graded parts, so we need more 'extra structure'.

10.1. **Filtrations.** Consider the following more common situation: for each d we have a filtration

$$0 = C_{d,0} \subset C_{d,1} \subset \ldots \subset C_{d,n} \subset \ldots \subset C_d,$$

such that the boundary maps respect the grading:

$$\partial(C_{d,i}) \subset C_{d-1,i}.$$

We can set

$$E^0_{d,i} := C_{d,i}/C_{d,i-1},$$
$$C'_d := \bigoplus_{i=1}^{\infty} E^0_{d,i}.$$

The boundary maps on $C_{d,i}$ induce maps

$$C_{d,i}/C_{d,i-1} = E^0_{d,i} \xrightarrow{\partial} E^0_{d-1,i} = C_{d-1,i}/C_{d-1,i-1}$$

(since $C_{d,i-1}$ lands inside $C_{d-1,i-1}$). Starting with a filtration, we have stumbled upon a graded complex; we call this the **associated graded complex** of our filtration. Taking homology, we set

$$E^1_{d,i} := \frac{\ker\left(E^0_{d,i} \xrightarrow{\partial} E^0_{d-1,i}\right)}{\operatorname{im}\left(E^0_{d+1,i} \xrightarrow{\partial} E^0_{d,i}\right)}$$

and make the observation that $E^1_{d,i}$ is the relative homology group

$$H^d(C_p, C_{p-1}).$$

While the relative homology is related to the homology we originally set out to calculate, they are not equal.

Intuitively, the discrepancy occurs because the relative homology doesn't notice that sometimes the boundary maps move things down too many levels. For instance, it may happen that

$$\partial(C_{d,i}) \subset C_{d-1,i-1},$$

and the relative homology simply doesn't record this, while the homology does.

10.2. **Spectral Sequences.** We begin by constructing a map

$$E^1_{d,i} \xrightarrow{\partial} E^1_{d-1,i-1}.$$

which measures the 'discrepancy' of relative homology.

Consider an $x \in C_{d,i}$. If $x \in \ker\left(E^0_{d,i} \xrightarrow{\partial} E^0_{d-1,i}\right)$, then actually x is in the kernel of the map

$$C_{d,i} \xrightarrow{\partial} C_{d-1,i}/C_{d-1,i-1},$$

i.e. $\partial(x) \in C_{d-1,i-1}$. This induces the desired map $E^1_{d,i} \xrightarrow{\partial} E^1_{d-1,i-1}$, and further computations show that this in fact gives a graded chain complex

$$0 \to \bigoplus_i E^1_{d+1,i+1} \to \bigoplus_i E^1_{d,i} \to \bigoplus_i E^1_{d-1,i-1} \to \dots.$$

We can take homology and set

$$E^2_{d,i} := \frac{\ker\left(E^1_{d,i} \xrightarrow{\partial} E^1_{d-1,i-1}\right)}{\operatorname{im}\left(E^1_{d+1,i+1} \xrightarrow{\partial} E^1_{d,i}\right)}.$$

Think of this as a second order approximation to H^*. Also, to make sense of this we should define

$$E^m_{d,i} := 0$$

wherever d or i hasn't been defined yet.

We think of this as only a second order approximation because, as before, we haven't taken into consideration chains $x \in C_{d,i}$ which go down two levels, i.e such that

$$\partial(x) \in C_{d-1,i-2}.$$

Of course, we can iterate this process; the same arguments give a graded complex

$$0 \to \bigoplus_i E^2_{d+1,i+2} \to \bigoplus_i E^2_{d,i} \to \bigoplus_i E^2_{d-1,i-2} \to \dots$$

and more homology groups

$$E^3_{d,i} := \frac{\ker\left(E^2_{d,i} \xrightarrow{\partial} E^2_{d-1,i-2}\right)}{\operatorname{im}\left(E^2_{d+1,i+2} \xrightarrow{\partial} E^2_{d,i}\right)}.$$

More generally, we get

$$0 \to \bigoplus_i E^r_{d+1,i+r-1} \to \bigoplus_i E^r_{d,i} \to \bigoplus_i E^r_{d-1,i-r+1} \to \dots$$

and

$$E^{r+1}_{d,i} := \frac{\ker\left(E^r_{d,i} \xrightarrow{\partial} E^r_{d-1,i-r+1}\right)}{\mathrm{im}\left(E^r_{d+1,i+r-1} \xrightarrow{\partial} E^r_{d,i}\right)}.$$

We can take a limit (upon inspection we see that all maps eventually become trivial and each individual homology group stabilizes) and set

$$E^\infty_{d,i} := E^r_{d,i}$$

for r large enough that $E^m_{d,i}$ stabilizes.

10.3. **Comparison Theorems.** Now we would like to understand how the graded ring

$$H^*(C^\bullet)$$

and our approximation, the doubly graded complex

$$E(C^\bullet) := \bigoplus E^\infty_{d,i}$$

are related. What is the best we can say?

In our setup above, we have maps (inclusions even) of *chain complexes*

$$\begin{tikzcd}
\cdots \arrow[r] & C_{d+1,i-1} \arrow[r] \arrow[d] & C_{d,i-1} \arrow[r] \arrow[d] & C_{d-1,i-1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{d+1,i} \arrow[r] \arrow[d] & C_{d,i} \arrow[r] \arrow[d] & C_{d-1,i} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & C_{d+1} \arrow[r] & C_{d} \arrow[r] & C_{d-1} \arrow[r] & \cdots
\end{tikzcd}$$

or more compactly

$$C^\bullet_{i-1} \to C^\bullet_i \to C^\bullet.$$

This induces maps

$$H(C^\bullet_{i-1}) \to H(C^\bullet_i) \to H(C^\bullet),$$

and if we set

$$F_i(H^*) := \ker H(C^\bullet_i) \to H(C^\bullet)$$

we get another filtration

$$0 \subset F_1(H^*) \subset F_2(H^*) \subset \ldots \subset F_i(H^*) \subset \ldots \subset H^*.$$

Finally, set

$$E^\infty_d := \bigoplus_i E^\infty_{i,d-i}$$

and

$$E = \bigoplus E_d^\infty.$$

Then the best we can do is the following

Theorem 10.1. *We have an isomorphism of graded rings*

$$\bigoplus E_d^\infty \cong \bigoplus F_d(H^*)/F_{d-1}(H^*).$$

This gives a lot of partial information about H^*, but not full information. For example, the filtrations

$$0 \subset \mathbb{Z}/2\mathbb{Z} \subset \mathbb{Z}/4\mathbb{Z}$$

and

$$0 \subset \mathbb{Z}/2\mathbb{Z} \subset \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

look the same under the lens of this theorem. What we really have determined is H^* up to extensions.

10.4. **Example: Cohomology of a CW-Complex.** We return now to the main example of the text. Consider a fibration $F \to X \xrightarrow{f} B$, with B a CW-complex. Setting $X^p := f^{-1}(B^p)$, we get a filtration

$$X_0 \subset X_1 \subset X_2 \subset \ldots \subset X_i \subset \ldots \subset X$$

We set $C_n := H^n(X)$ and $C_{n,p} := H^n(X_p)$. The inclusions

$$X_{i-1} \hookrightarrow X_i \hookrightarrow X$$

induce maps

$$H^*(X) \to H^*(X_i) \to H^*(X_{i-1})$$

and thus, setting

$$F_i(X) := \ker\left(H^*(X) \to H^*(X_i)\right),$$

a filtration

$$0 \subset F_1(X) \subset F_2(X) \subset \ldots \subset F_i(X) \subset \ldots \subset H^*(X).$$

The relative homology groups $E_1^{d,i}$ are the relative groups

$$E_1^{d,i} = H^d(X_i, X_{i-1}),$$

and the homology of this gives us page 2 of the spectral sequence of section (4).

References

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[Hat04] ———, *Spectral Sequences in Algebraic Topology*, In Preparation, http://www.math.cornell.edu/ hatcher/SSAT/SSATpage.html, 2004.

[Cho06] Timothy Y. Chow, *You could have invented spectral sequences*, Notices Amer. Math. Soc. **53** (2006), no. 1, 15–19.

[McC01] John McCleary, *A user's guide to spectral sequences*, Second, Cambridge Studies in Advanced Mathematics, vol. 58, Cambridge University Press, Cambridge, 2001.

[PSS05] Bjorn Poonen, Edward F. Schaefer, and Michael Stoll, *Twists of $X(7)$ and Primitive Solutions to* $x^2+y^3=z^7$, preprint, http://www.citebase.org/cgi-bin/citations?id=oai:arXiv.org:math/0508174, 2005.

[Rav86] Douglas C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, vol. 121, Academic Press Inc., Orlando, FL, 1986.

[Ser02] Jean-Pierre Serre, *Galois cohomology*, English, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002. Translated from the French by Patrick Ion and revised by the author.

[Lan02] Serge Lang, *Algebra*, third, Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002.

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