# Notes for BAGS: Seminar on Surfaces.

## Contents

# 1 Moduli of Surfaces and the semi-stable reduction theorem

Speaker: Fred.

Today the goal is to give the framework of the moduli space of surfaces.

## Setup for the Moduli of surfaces.

We usually mean of *general type* up to birational equivalence (instead of *up to isomorphism* as in the case of curves).

**Idea**: Take a minimal model. This is smooth and unique (since we are in dimension 2). Unfortunately the canonical divisor $\omega_X$ is nef but not ample.

**Idea**: Instead take the canonical model.

- This is still unique in the birational equivalence class.
- $\omega_X$ is ample (by definition).
- It is no longer smooth, but we at least know what type of singularities it can have - it can have at worst rational double points.

So the moduli functor we should think about is 'canonically polarised surfaces with rational double points'; i.e. we study the functor

$$F(B) := \{ \text{ proper flat morphisms } \mathscr{X} \to B \text{ such that the geometric fibers have}$$
$$\text{at worst rational double points and } \omega_X \text{ is ample } \}/\sim$$

where $\sim$ means isomorphisms respecting $\omega_X$ up to pulling back line bundles from the base.

**Definition 1.1.** The analogue of $g$ for curves is the *Hilbert polynomial*

$$h(n) = \chi(\omega_X^{\otimes n})$$

and we study $M_h$. ◇

For the purposes of Riemann-Roch we want to know $K^2$ and $\chi = \chi(\mathcal{O}_X)$, which is the same data as $h(n)$.

**Proposition 1.2.** *$M_h$ is*

*(1)* ***Bounded****: can pluricanonically embed all the surfaces into a single $\mathbb{P}^N$ (not obvious).*

*(2)* ***Locally closed:*** *in particular we get a locally closed subset* $\mathscr{U}$ *of the Hilbert scheme of pluricanonically embedded surfaces.*

*(3)* ***Separated:*** *the valuative criterion holds. This implies the* lemma *that the PGL action on* $U$ *is proper and has finite stabilizers.*

*Proof.* For references see

- Viehweg, *Moduli of Manifolds.*
- Matsusaka's 'big theorem', *Matsusaka-Mumford.*

□

**Remark 1.3.** Now the idea is that you use GIT to check stability. This was done by Gieseker in 1977. This means that you can take the GIT quotient, and thus you get a *coarse moduli space* for $M_h$. Furthermore you will get a quasi-projective scheme. ◇

**Warning 1.4.** In general the GIT part won't hold for compactifications. ⌟

**Remark 1.5.** Even the stack is not smooth; a smooth surface can be obstructed. Thus the obstruction to smoothness is not that we have rational double points.

Indeed, the obstruction group for a surface is just $H^2(\theta_S)$ ($\theta_S$ is the tangent bundle). This can be non-zero. This example is somewhat involved and can be found in Sernesi, Deformations. (The point will be that you can understand the deformations of the blow up.) ◇

## Compactification.

**Idea**: We use the semi-stable reduction theore to check the valuative criterion of properness.

The following theorem is in KKMS-D.

**Theorem 1.6** (Semi-stable reduction)**.** *Let* $\mathscr{X} \to B$ *be a surface with* $B$ *a non-singular curve,* $O \in B$*, and* $\mathscr{X}$ *smooth over* $B - \{O\}$*. Then*

$$\begin{array}{ccccc} \mathscr{X}'' & & & & \\ & \overset{f}{\searrow} & & & \\ & & \mathscr{X}' & \longrightarrow & \mathscr{X} \\ & & \downarrow & & \downarrow \\ & & B' & \xrightarrow{g} & B \end{array}$$

*where* $g$ *is finite,* $g^{-1}(0) - \{0'\}$*,* $f$ *is proper and birational,* $\mathscr{X}''$ *is smooth, and the new special fiber is reduced, with simple normal crossings.*

In the case of curves, we can get special fibers like (see pictures in handwritten notes). You can get a curve intersecting a $\mathbb{P}^1$, and blowing down gives you a stable model. Blowing down $-1$ curves gives you the minimal regular proper model, and blowing down $-2$-curves gives you the relative canonical model of the total space. Similarly, given the minimal model program in dimension three, we can try the same.

**Remark 1.7.** This process gives us semi log canonical singularities. If you don't know what this is define it in this way. ◇

We have been proving that a functor is proper but haven't defined that functor yet.

**Problem 1.8.** Semi-log canonical implies Cohen-McCauley, and so we have a canonical sheaf $\omega$, but it may not be invertible. ◇

To solve this we introduce more terminology.

**Definition 1.9.** Set $\omega_X^{[N]} := \left(\omega_X^{\otimes N}\right)^{**}$. Then $\omega_X^{[N]}$ is invertible for some $N$ and we define the *index* of a surface $X$ to be the smallest such $N$. ◇

Finally we can make the definition:

**Definition 1.10.** We say that a surface is a *stable surface of index* $N$ if $X$ has semi-log canonical singularities and $\omega_X^{[N]}$ is ample (and of index $N$). We define $M_h^N$ to be the moduli functor of stable surfaces of index $N$ with Hilbert polynomial $h$. ◇

Strictly speaking this is the wrong definition.

**Warning 1.11.** The total space need not be $\mathbb{Q}$-Gorenstein. The solution is to just require this explicitly. ⌟

**Question 1.12.** Do reflexive powers commute with base change? If not then you get two different definitions of this functor. ◇

**Proposition 1.13.** *$M_h^N$ is*

- ***Bounded*** *(Kollar, ...Singular..., 1980's).*
- ***Locally closed*** *(Hassett-Kovacs) ($\mathscr{U} \subset$ the Hilbert scheme of pluri-canonical embedded stable surfaces of index $N$).*
- ***Separated*** *(at least the uniqueness of the relative canonical model). As before we get a lemma that the PGL action on $\mathscr{U}$ is proper and has finite stabilizers.*

Now we can't use GIT, so we diverge from what we did before. Lets at least form the quotient as an algebraic space. This will then be the coarse moduli space for the stack. The condition of properness and finite stabilizers tells us this by the Artin/Mumford/Popp theorem.

**Remark 1.14.** You also need to check that points have no infinitesmal automorphisms to get that the stack is Deligne-Mumford. The scheme cover is the versal deformation space (this is just a different way to construct this stack). ◇

**Question 1.15.** Is $M_h^N$ compact? Is it a scheme? ◇

**Theorem 1.16.** *$M_h^N$ is complete (and hence proper).*

*Proof.* Here's an outline:

- Semi-stable reduction.
- Take the relative canonical model of the total space.
- Prove that this gives you semi-log canonical singularities.
- Control the index of the new special fiber.

□

Most of this is due to Kollar and Shepard-Baron, but without the index part. Recently, we have the

**Theorem 1.17** (Alexeev)**.** *Given a fixed $K^2$ and $\chi$, there exists $N$ such that for all semi-log canonical surfaces with such $K^2$ and $\chi$, index$(X)|N$.*

We could talk about this paper. Now we have a proper algebraic space. Finally, apply Kollar's Ampleness Criterion (Kollar, Projectivity) to get an ample line bundle on $M_h^N$, making it into a projective algebraic space, which is then a projective scheme.

**Remark 1.18.** $M_h^N$

- is not smooth;
- is not irreducible (at least analytically locally at the versal deformation space);
- the irreducible components are not known (i.e. what are the stable surfaces you are adding)

◇

So there are lots of things to do in this seminar.

# 2 Stable reductions for Surfaces

Today's speaker is Maksym Fedorchuk.

What we will do today is give a few definitions, a few proofs, and a few examples. One example will explain why we need $\mathbb{Q}$-Gorenstein in the definition of $M_h$.

We begin with the moduli space of canonically polarized surfaces (i.e. the canonical class is ample and it has at most rational double points as singularities) $M_h$. We defined the moduli functor as

$M_h : \mathrm{Sch} \to \Big\{ \mathscr{X} \xrightarrow{\pi} B :$ geometric fibers are surfaces with at most rational double points and $\omega_{\mathscr{X}/B}$ is

Everything is $\mathbb{Q}$-Gorenstein so we have a well-defined canonical divisor.

**Remark 2.1.** (1) $M_h$ has a coarse moduli space which is a quasi-projective scheme (Gieselar).

(2) $M_h$ is separated but not proper.

Note for (2) that to show that $M_h$ is not proper we need more than just a family which degenerates. ◇

## Good compactification of $M_h$

- Let $X$ be a normal surface. $X$ is then Cohen-McCauley, so the dualizing sheaf $\omega_X$ exists. As $X$ is smooth in codimension one $\omega_X$ is invertible in codimension one, and there thus exists a well defined Weil divisor class $K_X$. This is the same as $\mathbb{Q}$-Gorenstein of index $n$ (we make this our definition).
- $K_X$ has *index* $n$ iff $nK_X$ is a Cartier Divisor iff $(\omega_X^{\otimes n})^{**}$ is invertible.

So we now consider the non-normal case (because often we will have for example non-reduced special fibers). Consider the desingularization $\tilde{X} \to X$. $X$ is $\mathbb{Q}$-Gorenstein. Thus

$$K_{\tilde{X}} - f^* K_X - \sum a_i E_i, a_i \in \mathbb{Q}$$

where the sum is over the exceptional divisors of $f$. If $a_i$ are all positive then $X$ has *terminal singularities*, which for surfaces means smooth. If all $a_i \geq 0$, then $X$ has canonical singularities. This is the case of rational double points. If $a_i > -1$ then we have log terminal singularities, and if $a_i \geq -1$ then we have log terminal and log canonical.

**Remark 2.2.** The philosophy of the minimal model program is that we should allow log canonical singularities. ◇

We have a similar definition for log pairs $(X, D)$ where $D$ is a $\mathbb{Q}$-Cartier divisor. We then have

$$K_{\tilde{X}} - f^*(K_X + D) = \sum a_i E_i - \sum D_i, a_i \in \mathbb{Q}$$

where now $D_i$ is a component of the strict transform $\tilde{D}$. Now if $a_i > -1$ then $(X, D)$ is a log canonical pair.

We now give a definition of a class of non-normal surfaces that we will consider.

**Definition 2.3.** Let $X$ be a surface. We say that $X$ has *semi-log canonical* singularities if

(1) $X$ is Cohen-MacCauley.

(2) The only singularities in codimension 1 are normal crossings (a double curve is the closure of the codimension 1 singular locus; we think of the double curve as where two components meet or a component meets itself with normal crossings).

(3) (1) + (2) tells us that there exists a canonical Weil divisor class $K_X$ associated to $\omega_X$. For this we need to understand that $K_X$ is $\mathbb{Q}$-Cartier.

(4) Let $\pi : \tilde{X} \to X$ be the normalization. Then $(X^0, \pi^{-1}(D))$, with $D$ a double curve, is a log canonical pair.

◇

We always want our surface to be $\mathbb{Q}$-Gorenstein to make sense of the canonical divisor and singularities.

We want to start with a family of flat family $X \to \Delta$ with smooth generic fiber of general type. We want to replace this family with one where the central fiber is in our above log canonical class.

- **Step 1**: Semi-stable reduction (KKSM):

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ & \overset{g}{\searrow} & \downarrow \\ & & \Delta \end{array}$$

  such that

  - $\tilde{X}$ is smooth.
  - $g^{-1}(0)$ is a reduced normal crossings divisor.

  What we want is for $X' \to X$ to be an isomorphism on the complement of $\Delta - \{0\}$ and want $K_{X'}$ to be ample.

(2) Find the relative canonical model of $\tilde{X}$. This is very well explained in the book of Kollar and Mori, in the following theorem.

**Theorem 2.4** (Kollar, Mori 7.10)**.** *Let $\tilde{X} \to \Delta$ be a semistable family. Then there exists a birational model*

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & X \\ & \overset{g^c}{\searrow} & \downarrow g \\ & & \Delta \end{array}$$

*such that $g^c$ is log canonical and relatively ample (i.e. $K_{\tilde{X}}$ is relatively ample with respect to $g^c$).*

The following lemma is in the same book.

**Lemma 2.5.** *If the general fiber of $X^c$ is log canonical, then $\tilde{X}^c$ is too.*

Thus the total space will be canonical if the generic fiber is.

**Theorem 2.6** (Shepard-Baron, Kollar 5.1)**.** *Let $f : X \to \Delta$ be a morphism with $X$ canonical, and denote the central fiber by $X_0 = f^{-1}\{0\}$. Then $X_0$ is semi-log canonical.*

Someone can prove this later.

**Corollary 2.7.** *Let $X^\times \to \Delta^\times$ be a family of canonically polarized surfaces over the punctured disk. Then (after a base change) you can complete it to a (necessarily unique) $\mathbb{Q}$-Gorenstein family $X \to \Delta$ with $X_0$ semi-log canonical and $K_X$ ample.*

We will see that if we drop $\mathbb{Q}$-Gorenstein we lose separatedness (i.e. uniqueness).

## New Functor

We should now consider the functor of *stable surfaces*

$$\overline{M_h^N} : B \mapsto \Big\{ f : \mathscr{X} \to B : f \text{ has semi-log canonical geometric fibers of index } N \text{ and } \Big(\omega^{**}_{\mathscr{X}_b^{\otimes n}} = \omega\mathscr{X}_b^{[N]}\Big)$$

We consider the following standard construction.

**Example 2.8.** Degeneration of a variety to the cone over the hyperplane section. Let $X \subset \mathbb{P}^n$ be a projective variety and consdier the cone $C(X) \subset \mathbb{P}^{n+1}$. The generic pencil of hyperplane sections is a family $Y \to \mathbb{P}^1$ with generic fiber $X$ and special fiber $C(X \cap H)$ (here $H$ is a hyperplane). ◇

**Example 2.9.** Consider the cubic scroll $S_{1,2} \subset \mathbb{P}^4$: Fix an isomorphism between $\mathbb{P}^1$ and a conic, and take the space of lines between isomorphic points. Alternatively, $S_{1,2} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(2))$. now consider $C(S_{1,2}) \subset \mathbb{P}^5$ as a family over $\mathbb{P}^1$. Its generic fiber is smooth, and the central fiber is $C(S \cap H)$. Also, $S \cap H$ is degree 3 in $\mathbb{P}^3$ and is thus a rational normal curve, which we denote by $C_3$.

**Claim**: the cone $C(C_3)$ is $\mathbb{Q}$-Gorenstein and semi-log canonical.

*Proof.* Consider the desingularization $f : Z \to C(C_3)$, $Z = Bl(C(C_3)) \cong \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(3))$, and we have $K_Z = f^* K_{C(C_3)} + aE$ where $E$ is the exceptional divisor. Then

$$E \cdot K_Z = E \cdot (aE) = aE^2,$$

and also

$$E \cdot K_Z = K_E - E^2, E^2 = -3, \deg K_E = -2.$$

The canonical divisor $K_{C(C_3)}$ is also ample, simply because... ◇

Thus the cone over the rational normal cubic is a stable surface. Nonetheless, there are two things that are wrong with this family

(1) $K^2$ is not constant on the fibers.

(2) It is not $\mathbb{Q}$-Gorenstein.

**Lemma 2.10.** *The total space $C(S)$ is not $\mathbb{Q}$-Gorenstein at the vertex.*

*Proof.* We have the rational map $C(S) \to S \to \mathbb{P}^1$, wher ethe first map is the projection. Call the composition $f$.

$$\begin{array}{ccccc} & & \Gamma = \overline{\Gamma_f} \subset C(S) \times \mathbb{P}^1 & & \\ & \swarrow^{\pi_1} & & \searrow^{\pi_2} & \\ C(S) & & & & \mathbb{P}^1 \end{array}$$

Then a moment of reflection will convince you that what you add is

$$\pi_1^{-1}(0) = \mathbb{P}^1;$$

in particular this is a small contraction. Once we show that the graph $\Gamma = \overline{\Gamma_f}$ is smooth we lose $\mathbb{Q}$-Gorenstein. But under the map $\Gamma \xrightarrow{\pi_2} \mathbb{P}^1$ $\pi_2^{-1}(b) = C(F_b) \cong \mathbb{P}^2$, thus $\Gamma$ is smooth. Now think about the fibers of $\Gamma$ over $\Delta$. The generic fiber is $S = \mathbb{F}_1$. The special fiber is a blow up of a cone over $C_3$. The conclusion is that $\Gamma \to \Delta$ has semi-log canonical fibers and is Gorenstein, and $C(S)$ is not even $\mathbb{Q}$-Gorenstein.

◇

Thus you lose uniqueness. ◇

To get an example of general type you need to change this slightly.

Expository article on stable surfaces

# 3 Boundedness of Stable surfaces

Today's speaker is Brian.

**Definition 3.1.** We say that a set $A \subset \mathbb{R}$ is a **DCC set** if it satisfies the descending chain condition. ◇

The setup is as follows. Suppose $A \subset [0,1]$ is a DCC set. Let $S$ be a surface with log-terminal singularities, $B$ a divisor with coefficients in $A$ and $(S, B)$ is a log-canonical pair. Then $K_S + B$ is ample.

**Theorem 3.2.** *The following are true.*

*(1) The set of values $\{(K_S + B)^2\}$ is also a DCC set.*

*(2) The family with $(K_S + B)^2 = C$ is bounded.*

We have defined a functor $M_{N,H}$ which sends a base $T$ to flat families over $T$ whose fibers are stable surfaces such that $K_S$ is ample, $K_S^2 = C$ is fixed. Here $N$ is the index of the total space $X \to T$.

Eventually we want to show that the coarse moduli space is projective. We want to use Kollar's criterion, and to this end we need things to be bounded.

**Theorem 3.3.** *Let $X \to T$ be stable degeneration of surfaces of general type, and let $S$ be be general fiber. Then the index of $X$ is bounded by $K_S^2$.*

*Proof.* Suppose $Y$ is a singular fiber of the map $X \to T$. Since this is only stable, $Y$ may not be normal (just semi-log canonical). Let $Y^\nu = \cup Y_m$, and define $B_m$ on $Y_m$ to be the double intersection locus.

**Key fact**: Locally around the singular fiber of $Y$, the index of $X$ is the lcm of the indices of the $Y_m$. It turns out that the pairs $(Y_m, B_m)$ are log canonical and $K_S^2 = K_Y^2 = \sum (K_{Y_m} + B_m)^2$. (This is why we included the $B_m$'s in the first place). By the first theorem there are only finitely many numbers occurring at $(K_{Y_m} + B_m)^2$. By the other part of the first theorem we are done. □

Recall the following definition.

**Definition 3.4.** We say that a family $X \to T$ is **bounded** if any fiber can be found as a fiber of a fixed map $T' \to T''$ of finite type schemes. ◇

**Example 3.5.** For smooth surfaces, with $K_S$ ample, $K_S^2$ fixed, it is true that $5K_S$ defines an embedding. So for this we can take the universal family over a subscheme of a Hilbert scheme. ◇

**Question 3.6.** What are the barriers to doing this for singular surfaces. ◇

The $B$'s in the first theorem will play the same role as marked points in the curve case.

What are the problems with extending these things to stable surfaces?

**Problem 3.7.** Blowing up causes the ample curves to become nef instead of ample. This is why ruled surfaces are not bounded; you can fix $K_S^2$ and keep blowing up. ◇

The solution is to look at $\rho(\tilde{S})$ of the desingularization.

**Problem 3.8.** SLC is hard. ◇

But it's not too hard.

**Idea 3.9.** SLC normalizes to LC, which is a limit of LT. ◇

**Problem 3.10.** Everything we are doing is taking place over $\mathbb{Q}$ instead of $\mathbb{Z}$. ◇

**Idea 3.11.** Scale coefficients of $B$ downwards and use induction. ◇

This is sort of like the minimal model program.

## Proofs

These lemma's will let us work with the boundary of the nef cone.

**Lemma 3.12.** *Let $X$ be a $\mathbb{Q}$-factorial surface and let $(X, B)$ be a log canonical pair such that $K_X + B == 0$ (== means numerically equivalent). Write $B = \sum b_i B_i$. Then $\sum b_i \leq \rho(X) + 2$.*

**Lemma 3.13.** *Let $X$ be a nonsingular surface, and let $B = \sum b_i B_i$ with $0 \leq b_i \leq 1 - \epsilon$ for some $\epsilon > 0$. Suppose that $K_X + B == 0$. Then $\rho(X) \leq \frac{128}{\epsilon^5}$.*

The following will be the most careful proof we do.

*Proof of Lemma 2.* (1) Suppose that $K_X == 0$. Then $K_X$ is nef. Thus $X$ is $K3$ or Enriques, and $\rho(X) \leq 22$.

(2) $K_X == -B$. Suppose $X$ is not rational. We know that for a ruled surface, $K_X^2 = 8(1-g)$. Since $K^2 = B^2 \geq 0$, $g = 1$ and no blowups are allowed, so $X$ is ruled over an elliptic curve.

(3) Suppose $X$ is rational. Then $X \to \mathbb{F}_n$ is the blowup of a Hurziwitz surface. Let $\bar{B}_j$ be the transform of $B_j$ on $\mathbb{F}_n$ (i.e. ignore it if it is contracted). Then We still have $K_{\mathbb{F}_n} + \bar{B} == 0$, and we use adjunction. I.e. if $C$ is a distinguished section, then

$$-2 = C(K_{\mathbb{F}_n} + C) = (K_{\mathbb{F}_n} + \bar{B}) \cdot C - \bar{B} \cdot C + C^2 = 0 + C^2 - (1-\epsilon) \cdot C^2 = -\epsilon n$$

and thus $n \geq \frac{2}{\epsilon}$.

Now we need to bound the Picard number. We want to bound the number of blow ups. We have $q : X \to \mathbb{F}_n$. **Fact**. We can do intermediate blow-ups along $p_i$ so that the multiplicity of $\bar{B}$ at $\rho_i$ is non-increasing. We can bound the number of parts with multiplicity $> \epsilon/2$. Consider

$$\bar{B}^2 - q^*(\bar{B})^2.$$

This number decreases by at most $\epsilon/4$. Then parts with multiplicity $< \epsilon/2$. This means that the exceptional curve you get when you blow up does not satisfy the log condition. When you count this carefully you get the lemma. □

All of the arguments in this paper are pretty similar. The ruled surface case will be handled by adjunction.

## Scaling coefficients

**Lemma 3.14.** *Let $K + B$ be big and log canonical on a smooth scheme $X$ and suppose $\theta < b_j < 1 - \epsilon$. Then there exists an 'indexing' subset $J' \subset J$ and $N(\theta, \epsilon)$ so that $|J'| < N$ and $L + \sum_{J'} b_i B_i$ is big.*

*Proof.* By scaling components. □

## Our first boundedness theorem

**Remark 3.15.** Let $\mathscr{S} = (S, H)$ be a family of $\mathbb{Q}$-polarized normal surfaces such that $NH$ is Cartier for some fixed $N$ and $H^2 \leq C$ and $H \cdot K \leq C$ for some $C$. Then $\mathscr{S}$ is bounded. Furthermore, if we include a divisor $\mathscr{S} = (S, H, B)$ is also bounded if $H \cdot B \leq C'$ ◇

**Theorem 3.16** (First Boundedness)**.** *Fix $C > 0$ and a DCC set $\mathcal{A} \subset [0, 1]$ with $1 \in \mathcal{A}$. Then there exists a bounded class $(Z, D)$ of surfaces with divisors such that for every $X$ with $(X, B)$ with $K + B$ big, nef, and log canonical such that $B$ has coefficients in $\mathcal{A}$ and $(K + B)^2 \leq C$, there exists a diagram*

$$\begin{array}{ccc} Y & \xrightarrow{g} & Z \\ \downarrow{\scriptstyle f} & & \\ X & & \end{array}$$

*with such that*

*(1) $Y$ is a minimal resolutoin of $X$.*

*(2) $D = q(Suppf^*(K_X + B) \cup Exc(f))$*

This does not prove boundedness, it is just a related result.

*Proof.* By the previous lemma we scale $B$ to $B'$ such that

(1) $K + B'$ is big.

(2) The coefficients of $B'$ belong to a finite set of rational numbers.

(3) The coefficients of $B'$ are $< 1 - \epsilon$.

Now we apply the previous lemma to toss out components of $B'$ leading to a $B''$ with $K_X + B''$ big (but maybe no longer log canonical). Now we run a general kind of minimal model program (more generally than log canonical) on $K_X + B''$ and we get a surface $Z$ where the we can apply the previous remark. You need to be a little more careful, but this is more or less it. □

## Boundedness

How do we actually get boundedness out of this. There could be many $(X, B)$ giving us the same $(Z, D)$. How do we work backwards? Fix $(Z, D)$. Then we get a diagram

insert picture

Such that $(V, B^V) \to (X, B)$ are bounded.

We show that there exists a $V$ with $h_Y^* \left(K_V + D^V\right) \leq K^Y + B^Y$. This implies that essentially since $\left(K_V + B^V\right)^2$ is some kind of global minimum, every map $Y \to X$ factors through $V$.

Once we know this the two original theorems follow pretty easily.

## References

(1) V. Alexeev. Boundedness and $K^2$ for Log Surfaces. On the Arxiv.

(2) V. Alexeev, S. Mori. Bounding Surfaces of General Type. Not published, its on kollar's website.

(3) Kollár. Projectivity of Complete Moduli, Log Surfaces - Some conjecture, and Threefolds and deformation of surface singularities.

# 4 Valentino: Stability for Polarized Varieties

**GIT**: Let $G = SL(m, \mathbb{C})$ and $W \cong \mathbb{C}^{n+1}$ and $\mathbb{P}(W) \cong \mathbb{P}^n$. Consider an action of $G$ on $GL_{n+1}(\mathbb{C})$.

**Definition 4.1.** A point $v \in \mathbb{P}(W)$ is **semistable** if for some (all) $\hat{v} \in W$ above $v$, $0 \notin G \cdot \hat{v}$. We say that $v$ is **polystable** if the orbit of $\hat{v}$ is closed in $W$, and $v$ is **stable** if the orbit is closed and the stabilizer is finite. ◇

Consider a 1-parameter subgroup $\rho : \mathbb{C}^\times \to G$, non-trivial. Then if $v$ is semistable, then $0 \notin \rho(\mathbb{C}^\times)\hat{v}$. We have a converse.

**Theorem 4.2** (Hilbert-Mumford criterion)**.** *If $0 \notin \rho(\mathbb{C}^\times)\hat{v}$ for all non-trivial $\rho$, then $v$ is semi-stable.*

Now we want to get a numerical criteria.

**Remark 4.3. Numerical criterion**: Fix a basis $(a_1, \ldots, a_{n+1})$ of $W$ such that the action of $\rho(t)$ is diagonal, say with diagonal elements $t^{w_i}$, $w_i \in \mathbb{Z}$. We define the weight to be $\sum w_i$. The $w_i$ are integers because $\rho$ is a morphism of schemes, and $\sum w_i = 0$ by the SL condition. Fix $v \in \mathbb{P}(W)$, $\hat{v} = \sum_i v_i a_i$ and $\rho(t)\hat{v} = \sum_i t^{w_i} v_i a_i$. We want to take a limit as $t \to 0$, so we define $\mu = \min\{w_i | v_i \neq 0\}$ and $\hat{v_0} = \lim_{t\to 0} t^\mu \rho(t) \cdot \hat{v}$ exists (since $\mathbb{P}$ is proper) so set $v_0 = \lim_{t\to 0} \rho(t) \cdot v$. We then find that $v_0$ is a fixed point of the $\mathbb{C}^\times$ action. ◇

A more intrinsic way to state this is the following. We get a $\mathbb{C}^\times$-action on $\mathcal{O}_{\mathbb{P}(w)}(-1)\big|_{V_0}$, so we get a weight which turns out to be $\mu = \mu(v, \rho)$. Notice that in $W$, the limit $\lim_{t\to 0} \rho(t) \cdot \hat{v}$ exists iff $\mu(v, \rho) \geq 0$, and is zero iff $\mu(v, \rho) > 0$. But if this limit exists, then it is either $\hat{v}$, in which case the stabilizer contains $\mathbb{C}^\times$ and is not finite, or it is another point, which is then fixed by $\mathbb{C}^\times$. But then you have a point in the closure of the orbit which is not in the orbit; in particular the orbit is not closed.

We summarize.

**Theorem 4.4.** $0 \notin \rho(\mathbb{C}^\times) \cdot \hat{v}$ *iff* $\mu(v, \rho) \leq 0$*,* $\rho(\mathbb{C}^\times \cdot \hat{v})$ *is closed iff* $\mu(v, \rho) \leq 0$ *and* $= 0$ *implies that* $\rho$ *fixes* $v$*.* $\rho(\mathbb{C}^\times) \cdot \hat{v}$ *is closed iff* $\mu(v, \rho) < 0$ *and the stabilizer is finite.*

Reformulating the Hilbert-Mumford briterion gives semistable, polystable, and stable for these above three cases.

## Hilbert Stability

Let $(X, L)$ be a polarized smooth variety with ample line bundle $L$ and Hilbert polynomial $P(r) = \chi(X, L^\times)$. Pick $r$ large enough so that $H^i(X, L^r) = 0$ for $i > 0$ and $L^r$ very ample.

Then $X \to \mathbb{P} = \mathbb{P}\left(H^0\left(X, L^r\right)^*\right) \cong \mathbb{P}^{P(r)-1}$ has Hilbert polynomial $P(Kr) = P'(K)$. You thus get a point of a Hilbert scheme. And there thus exists an $K_0$ depending only on $P^1$ such that

$$0 \to I_X(K) \to H^0\left(\mathbb{P}, \mathcal{O}_P(K)\right) \to H^0\left(X, \mathcal{O}_{\mathbb{P}}(K|_X)\right) \to 0$$

where the middle term is isomorphic to $Sym^K(H^0\left(X, L^r\right))$. Then $I_X(K)$ is a linear subspace in $Sym^K H^0\left(X, L^r\right) \cong Sym^K \mathbb{C}^{P(r)}$ of codimension $P(Kr)$. It defines a point in $\mathbb{G} = Gr\left(Sym^K \mathbb{C}^{P(r)}, P(Kr)\right) \xrightarrow{plucker} \mathbb{P}\left(\Lambda^{P(Kr)} Sym^K \mathbb{C}^{P(r)^*}\right)$. Call the point $Hilb_{r,Kr}(X)$, and we have an action of $G = SL\left(P(r), \mathbb{C}\right)$ on $\mathbb{P}(\ldots)$, and we are in precisely in the situation of the beginning of the talk.

**Definition 4.5.** We say that $(X, L)$ is **Hilbert stable** with respect to $r$ if $Hilb_{r,Kr}(X)$ is GIT stable for all $K >> 0$, and **asympotoically** so if it is Hilbert stable with respect to $r >> 0$. ◇

**Remark 4.6.** Notice from the above that $\mathcal{O}_{\mathbb{G}}(-1)|_{Hilb_{r,Kr}} \cong \Lambda^{\max} Sym^K H^0\left(X, L^r\right) \otimes \left(\Lambda^{\max} H^0\left(X, L^{Kr}\right)\right)^*$. ◇

## Chow Stability

Let $X \to \mathbb{P} = \mathbb{P}^{P(r)-1} = \mathbb{P}^N$. Let $Z_X = \left\{\Lambda \in Gr\left(\mathbb{P}^N n-1\right) | \Lambda \cap X \neq \emptyset\right\}$. This is a divisor in $\mathbb{G}' = Gr\left(\mathbb{P}^N, n-1\right)$. Let $d = \deg L = \frac{(L^n)}{n!} = \deg X$. A classical computation which is in Harris's book says that $Z_X$ is also of degree $d$. There thus exists $f \in H^0\left(\mathbb{G}', \mathcal{O}_{\mathbb{G}}(d)\right)$, unique up to scaling, such taht $Z_X = \{f = 0\}$. Thus $Chow_r(X) = [f] \in \mathbb{P}\left(H^0\left(\mathbb{G}', \mathcal{O}_{\mathbb{G}}(d)\right)\right)$ and $G = SL\left(P(r), \mathbb{C}\right)$ acts on this. Thus we can talk about stability.

**Definition 4.7.** We say that $(X, L)$ is **Chow stable** with respect to $r$ if $Chow_r(X)$ is GIT stable. We say that it is **asymtotocally** Chow stable if it is Chow stable with respect to $r >> 0$. ◇

## Geometric visualization of 1-paramater subgroups

**Definition 4.8.** A **test configuration** for $(X, L^r)$ is the following data

(1) $\mathscr{X}$ a projective scheme with a $\mathbb{C}^\times$-action and amap $\pi : \mathscr{X} \to \mathbb{C}$ which is $\mathbb{C}^\times$ equivariant.

(2) A $\mathbb{C}^\times$ equivariant line bundle $\mathscr{L} \to \mathscr{X}$ which is ample over all fibers of $\pi$ such that for some (and hence any) $t \neq 0$ the pair $(X, L^r) \cong (\mathscr{X}_t, \mathscr{L}|_{\mathscr{X}_t})$.

We call it a **product configuration** if the total space is a product $\mathscr{X} \cong X \times \mathbb{C}$, and **trivial** if it is a product and $\mathbb{C}^\times$ only acts on the second factor. ◇

These test configurations are the same as 1-parameter families.

**Remark 4.9.** Let $X \to \mathbb{P}$, $\rho$ a 1-parameter family in $G = SL\left(P(r), \mathbb{C}\right)$. Then we get a correspondence

$$\{testconfig\} \leftrightarrow \{\text{1param.}\}$$

given by as follows. We get a map $\mathbb{C}^\times \to Hilb$ given by $\rho$. Inside you get a universal family inside $Hilb \times \mathbb{P}$. Hilb is proper so you can fill in the map to a map $\mathbb{C} \to Hilb$ and pulling back the universal family gives you a test-configuration. The inverse map is given by setting $L_0 = \mathscr{L}|_{X_0}$. This inherits a $\mathbb{C}^\times$ action. You thus get an action on $H^0\left(X_0, L_0\right)$ (at least when there is no higher cohomology, else you need to take the push forward). Then

$$h^0\left(X_0, L_0\right) = \chi\left(X_0, L_0\right) = \chi\left(X_t, L_t\right) \cong \chi\left(X, L^r\right) = P(r).$$

You need the weight of the one-parameter subgroup to be zero, so we modify the test-config. Let the weight of the $\mathbb{C}^\times$-action on $H^0\left(X_0, L_0^K\right)$ be $w(Kr) = w(k), k = Kr$. First I pull back $\mathscr{X}$ via the map $\mathbb{C} \to \mathbb{C}$ given by $t \mapsto t^{rP()r}$. The weight on $H^0\left(X_0, L_0\right)$ becomes $w(r)rP(r)$. Next I compare the $\mathbb{C}^\times$ action with scaling along the fibers of $\mathscr{L}$ by $t^{rw(r)}$, and the weight scales by $-rw(r)P(r)$, and we can force the weight to be 0. ◇

**Question 4.10.** What is the Mumford weight $\mu$ of this 1 parameter subgroup on $Hilb_{r,Kr}(X)$? ◇

Well, set

$$I_{X_0}(K) = \lim_{t\to 0} \rho(t) \cdot I_X(K).$$

Then we get an exact sequence

$$0 \to I_{x_0}(K) \to Sym^K H^0\left(X_0, L_0\right) \to H^0\left(X_0, L_0^K\right) \to 0$$

and thus

$$\mathcal{O}_{\mathbb{G}}(-1)|_{(I_{X_0}(K))} \cong \Lambda^{\max} Sym^K H^0\left(X_0, L_0\right) \times \left(\Lambda^{\max} H^0\left(X_0, L_0^K\right)\right).$$

The weight of $\rho$ is 0, so

$$-w(k)rP(r) + krw(r)P(Kr) = -w(k)rP(r) + w(r)kP(k)$$

and we define $\tilde{w}_{r,k} = \mu$.

**Corollary 4.11.** *$(X, L)$ is asymptotically Hilbert stable iff for all $r >> 0$ and for every nontrivial test configuration for $(X, L^r)$, $\tilde{w}_{r,k} < 0$ for all $k >> 0$.*

For Chow, we first use Riemann Roch. We know that $P(r)$ is a polynomial of degree $n$. By equivariant Riemann Roch, $w(k)$ is a polynomial of degree $n+1$ when $k >> 0$. We can thus expand

$$\tilde{w}_{r,k} = \sum_{i=1}^{n+1} e_i(r) k^i$$

and each term is

$$e_i(r) = \sum_{j=0}^{n+1} e_{i,j} r^j, r, k >> 1.$$

**Theorem 4.12** (Mumford, 1977)**.** *The weight for the action of* $\rho$ *on* $Chow_r(X)$ *is* $e_{n+1}(r)$*.*

This is not obvious.

**Corollary 4.13** (Fogarty)**.** *Asymptotically Chow stable implies asymptotically Hilbert stable which implies A.H.ss. which implies Ass. C. ss.*

*Proof.* By H-M, $(X, L)$ is asym. Chow S. iff for all $r >> 0$ and for all nontrivial $(X, L^r)$, $e_{n+1}(r) < 0$. This implies that $w_{r,k} < 0$ for large $k$. □

**Theorem 4.14.** *T. Mabuchi proved in 2006 that Asymptotically H.S implies Asym. C.S.*

## One step further

This is introduced by Tian and Donaldson. It is immediate that $e_{n+1,n+1}$ is always zero (because of our normalization). The next term $e_{n+1,n}$ is called the **Futaki invariant** of the test configuration.

**Definition 4.15.** We say that $(X, L)$ is $K$-stable if for large $r$ and all non-trivial test configurations $(X, L^r), e_{n+1,n} < 0$. ◇

Now we see that Asym. Chow ss. implies $K$-ss.

**Question 4.16.** Does K. ss implies Asym. Chow stable? ◇

**Question 4.17** (Yau, Tian, Donaldson)**.** $(X, L)$ is K-polystable iff there exists a Kahler metric $\omega \in c_1(L)$ with constant scalar curvature. ◇

**Theorem 4.18** (Donaldson, 2001, 2005)**.** *If there exists csc K then you have K ss. if Aut(X,L) is discrete. If there exists cscK then you have Asym Chow Stable.*

**Theorem 4.19** (Yau 1978)**.** *If* $L = K_X$ *is ample and if there exists a csc K then we have Ascym Chow ss. If* $K_X$ *is trivial, L ample then there exists csc K.*

Finally, if $L$ is $K_X$ and is ample (i.e. Fano), then this is open (i.e. the existence of a Kahler metric).

For curves, this says that $g \geq 1$ implies Asym. Chow ss and $K$-stable. For $g = 0$, we get $K$-polystable.

# References