ABELIAN CHABAUTY

DAVID ZUREICK-BROWN

Abstract. These are expanded lecture notes for a series of four lectures at the Arizona Winter School on “Nonabelian Chabauty”, held March 7-11, 2020 in Tucson, Arizona.

Last update: February 17, 2020

Contents

1. Introduction 1
   1.1. Course outline, and how to read these notes 2
   1.2. Acknowledgements 3
2. Abelian Chabauty 3
3. The uniformity conjecture 4
   3.4. Evidence and records 4
   3.5. Chabauty–Coleman bounds 5
   3.7. A new hope 5
4. Bad Reduction 5
5. Rank Favorable bounds 6
   5.3. Stoll’s proof of Theorem 5.1 7
   5.4. The rank of a divisor 9
   5.6. Chip firing 9
   5.7. Rank favorable bounds for curves with totally degenerate reduction 10
   5.8. Refined ranks 10
6. Tropical Geometry, Berkovich spaces, and Chabauty 10
References 11

1. Introduction

Let $K$ be a number field and let $X/K$ be a nice\footnote{smooth, projective, and geometrically integral} curve of genus $g > 1$ whose Jacobian has rank $r := \text{rank} \ J(K)$.

The Method of Chabauty–Coleman (alternatively: “Chabauty’s method”, “Abelian Chabauty”, or just plain, vanilla, “Chabuaty”) is among the most successful and widely applicable techniques for analyzing (either theoretically or explicitly) the set $X(K)$ of $K$-points of a low rank ($r < g$) curve $X$, and is an essential part of the “explicit approaches to rational points” toolbox. In particular, with some luck, Chabauty’s method allows one to
• explicitly determine, with proof, the set $X(K)$, or
• determine an upper bound on $\#X(K)$.

Mordell conjectured in the 20’s that the set $X(K)$ of $K$-points of $X$ was finite. This was famously proved in the 80’s by Faltings [Fal86] (for which he was awarded a Fields medal), with subsequent independent proofs by Vojta and Bombieri [Voj91, Bom90].

Chabauty [Cha41], building on a idea of Skolem [Sko34], gave the first substantial progress toward Mordell’s conjecture. Chabauty’s method, which used $p$-adic techniques to produce $p$-adic “locally analytic” functions, relies on the hypothesis that $r < g$. This technique sat somewhat dormant until the 80’s, when Coleman’s seminal paper [Col85a] resurrected and improved Chabauty’s idea.

**Machine Computation.** Computer aided computational tools took some time to catch up. The first big bottleneck was to improve the techniques for compute ranks of Jacobians of curves. To execute Chabauty’s method for a particular, explicit curve, one needs to know the rank $r$ of its Jacobian (to check the $r < g$ condition), and a basis for the Mordell–Weil group of $J(K)$ (or at least a finite index subgroup). An early highlight is due to Gordon and Grant [GG93]; building on work of Cassels and Flynn [Fly90, Cas89, Cas83] they work out the special case of two-descent on the Jacobian of a genus 2 hyperelliptic curve with rational Weierstrass points, and (with the help of a SUN Sparcstation) provably compute the rank of a couple examples.

This ushered in a golden era of computer assisted approaches to rational points on curves (and higher dimensional varieties). Even today there are substantial conceptual and practical improvements; one highlight is [BPS16].

**Applications.** It is worth highlighting a few of the numerous applications of Chabauty’s method.

- McCallum made substantial progress towards a proof of Fermat’s Last Theorem, via a careful study of certain quotients of the Fermat curve $x^p + y^p = z^p$; for instance, [McC94] proves the “second case” of Fermats Last Theorem for regular primes;
- arithmetic statistics: Poonen and Stoll use Chabauty’s method to prove that 100% of odd degree hyperelliptic curves have only one rational point [PS14];
- analysis of rational points on modular curves, and Mazur’s “Program B” [RZB15];
- resolution of various generalized Fermat equations [PSS07].

1.1. **Course outline, and how to read these notes.** There is already an excellent survey by McCallum and Poonen [MP12]. This is short (16 pages) and a great entry point. I recommend my survey with Katz and Rabinoff [KRZB] for the connections between $p$-adic and tropical techniques, and the survey [BJ15] by Baker and Jensen for a more geometric and combinatorial perspective.

These notes will focus on the ideas from [KZB13] and [KRZB], and the papers [LT02a, Sto06, Sto19, Bak08] which inspired our work. In particular, my discussion of the foundations of Abelian Chabauty, and discussion of tropical techniques, will be abridged, and these notes are somewhat of an advertisement for [MP12] and [KRZB16].

**Abelian Chabauty.** We will start with a detailed discussion of the method of Chabauty and Coleman, and will address various points of view (theoretical, practical, computational);
this section is mostly an abridged version of [MP12], and it is recommended to read their
survey alongside this section and before reading future sections.

Exhibiting Abelian Chabauty as a special case of Nonabelian Chabauty is not completely
straightforward. These notes do not address this, and we instead recommend Poonen’s

**Bad reduction.** One avenue to improve on Coleman’s bound is to generalize the
framework of Chabauty and Coleman’ arguments to the case of bad reduction. We will
discuss the advantage of working at bad primes and the difficulties and tradeoffs that arise,
starting with the work of Lorenzini–Tucker [LT02a].

**Rank favorable bounds.** When the rank is strictly smaller than \( g - 1 \), there are
more “inputs” to Chabauty’s method and one expects this extra flexibility to lead to
improvements to the method, giving rise to “rank favorable” bounds. We’ll discuss the
setup, and the translation to the notion of a “rank” of a divisor (due to Stoll [Sto06]). For
a curve with good reduction, this notion of rank will be the classical one, and the improved
bounds will follow from Clifford’s Theorem [Har77, Theorem 5.4]. In the case of bad
reduction, reducible reduction, ranks are no longer as well behaved; instead, we introduce
Baker’s notion of “numerical rank” [Bak08] and explain how to repair Stoll’s argument in
the special case of a curve with totally degenerate reduction.

**Tropical Geometry and Berkovich spaces.** For a curve with bad reduction at a
prime \( p \), it had been well understood that “monodromy” and “analytic continuation” of
\( p \)-adic integrals was an issue. Coleman proved that in the case of good reduction, there is no
“monodromy” and the various ways of analytically continuing \( p \)-adic integrals all coincide.
In the case of bad reduction, they generally do not coincide (we will discuss a simple example
which illustrates this).

Stoll [Sto19] discovered that, while choices of analytic continuation genuinely do differ,
they do so in a fairly controlled manner (linear, even), and was able to exploit this to prove
a uniformity result for hyperelliptic curves of small rank.

These results all argue in the framework of rigid geometry (in the sense of Tate). Great
clarification arose from the systematic reformulation via Berkovich spaces, which fill in the
“missing” points of rigid spaces and which, at least in the case of curves, are fairly concrete
and manageable topological spaces (they’re even Hausdorff). I’ll discuss Chabauty in the
setting of Berkovich and tropical geometry and explain how modern tools (e.g., Berkovich’s
contraction theorem and Thuillier’s slope formula, exposited in [BPR13]) give a clean ex-
planation of Coleman’s “good reduction” theorem, and will discuss my work with Katz and
Rabinoff [KRZB] which give uniform bounds for arbitrary (but still small rank) curves.

1.2. **Acknowledgements.** The author would like to thank Jackson Morrow and John
Voight for useful discussions. The author was supported by National Science Foundation
CAREER award DMS-1555048.

2. **Abelian Chabauty**

As before, let \( K \) be a number field with ring of integers \( \mathcal{O}_K \) and let \( X/K \) be a nice curve
of genus \( g > 1 \). Let \( r := \text{rank } J(K) \) be the rank of the Jacobian of \( X \). Fix a prime \( p \) and a
prime $p$ of $\mathcal{O}_K$ above $p$.

Under the assumption $r < g$, there exist **locally analytic** functions $f_\omega$ on $X(K_p)$ (arising as a $p$-adic integral of a differential $\omega$) which vanish on $X(K)$, but not on $X(K_p)$. This (more or less) is enough to conclude finiteness (and is roughly the original argument of Chabauty [Cha41]). Better, though, is that “locally analytic” means that on (residue) discs, they have $(p$-adic) power series expansions, and are amenable to fairly explicit study via tools from $p$-adic analysis (Newton polygons, or in more complicated situations, tropical geometry); in Coleman’s original analysis ([Col85a Lemma 3] or [MP12 Lemma 5.1]), one can bound the zeros of $f_\omega$ in a residue disc.

One can summarize the local analysis as a ‘$p$-adic Rolle’s theorem.’ In the classical (good reduction) case, with the right conditions, you get Rolle’s theorem on the nose. But in general, you want a way to bound the zeroes of a function in terms of the zeroes of its derivative.

“Global coordination”. One needs some way to coordinate the different, initially independent, local bounds. Typically one needs some “global” theorem such as Riemann–Roch or Clifford’s theorem.

See [MP12] for a more thorough survey.

3. **The uniformity conjecture**

The uniformity conjecture is one of the outstanding open conjectures in arithmetic and diophantine geometry.

**Conjecture 3.1** ([CHM97]). Let $K$ be a number field and let $g \geq 2$ be an integer. There exists a constant $B_g(K)$ such that for every smooth curve $X$ over $K$ of genus $g$, the number $\#X(K)$ of $K$-rational points is at most $B_g(K)$.

**Remark 3.2.** The uniformity conjecture famously follows [CHM97, Theorem 1.1] from the conjecture of Lang–Vojta (the higher dimension analogue of Faltings’ theorem).

The Lang–Vojta conjecture says the following.

**Conjecture 3.3** ([Lan86][Voj87]). Let $X$ be a smooth proper variety of general type. Then there exists a proper closed subscheme $Z$ of $X$ such that $X(K) = Z(K)$.

Actually, one only needs to know Lang–Vojta for symmetric powers of the universal curve over $\overline{M}_{g,n}$.

3.4. **Evidence and records.** The following table gives the best known lower bounds on the constant $B_g(\mathbb{Q})$.

<table>
<thead>
<tr>
<th>$g$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>10</th>
<th>45</th>
<th>$g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_g(\mathbb{Q})$</td>
<td>642</td>
<td>112</td>
<td>126</td>
<td>132</td>
<td>192</td>
<td>781</td>
<td>$16(g + 1)$</td>
</tr>
</tbody>
</table>

The record so far is due to Michael Stoll, who found (searching systematically through several families of curves constructed by Noam Elkies) the following:

$$y^2 = 82342800x^6 - 470135160x^5 + 52485681x^4 + 2396040466x^3 + 567207969x^2 - 985905640x + 247747600$$
It has at least 642 rational points, and rank at most 22. See [http://www.mathe2.uni-bayreuth.de/stoll/recordcurve.html](http://www.mathe2.uni-bayreuth.de/stoll/recordcurve.html) for a full list of the known points.

The families constructed by Elkies arise in the following way: he studied K3 surfaces of the form

\[ y^2 = S(t, u, v) \]

with lots of rational lines, such that \( S \) restricted to such a line is a perfect square.

3.5. **Chabauty–Coleman bounds.** The proofs of Mordell due to Faltings, Vojta, and Bombieri [Fal97, Voj91, Bom90] give upper bounds on \(#X(K)\). These bounds tend to be astronomical, and are not explicit in their original proofs; moreover, it is unclear (to me) how they depend on \( X \) and \( K \).

One application of Chabauty’s method is to give uniform bounds on small rank curves. Coleman’s original theorem is the following.

**Theorem 3.6 (Coleman, [Col85b]).** Let \( X \) be a curve of genus \( g \) and let \( r = \text{rank}_\mathbb{Z} \text{Jac}_X(\mathbb{Q}) \). Suppose \( p > 2 g \) is a prime of good reduction. Suppose \( r < g \). Then

\[ \#X(\mathbb{Q}) \leq \#X(\mathbb{F}_p) + 2g - 2. \]

See [Col85a, Lemma 3] or [MP12, Theorem 5.3] for a proof.

Various authors have worked to weaken Coleman’s hypotheses and to improve the bound; see Theorems 4.1, 5.1, 5.2, 6.1, and 6.2 below and the surrounding discussion.

3.7. **A new hope.** The recent work of Dimitrov, Gao and Habegger [DGH19, DGH20] give bounds on \(#X(F)\) which only depend on \( g(X), \text{deg} F, \) and rank \( J(F) \). Combined with the conjectural boundedness of ranks of Jacobians of curves of fixed genus over a fixed number field (as predicted by [PPVW19] and [Poo18, Section 4.2]) this would prove uniformity. Their approach is in the spirit of Vojta’s original proof [Voj91], and relies on their recent other work on improved height bounds.

4. **BAD REDUCTION**

One avenue to improve on Coleman’s bound is to generalize the Chabauty framework and Coleman’s arguments to the case of bad reduction. We will discuss the advantage of working at bad primes and the difficulties that arise, starting with the work of Lorenzini–Tucker [LT02a].

Coleman’s original bound (3.6) relies on an initial choice of a prime of good reduction. The first such prime could be arbitrarily large (e.g., consider a hyperelliptic curve \( y^2 = f(x) \), and twist it by \( d \), where \( d \) is the product of the first million primes; or, pick singular curves \( X_p \) over \( \mathbb{F}_p \), for the first million primes \( p \), and use the Chinese Remainder Theorem to construct a curve \( X \) over \( \mathbb{Q} \) that \( X_{\mathbb{F}_p} \cong X_p \) for each such prime \( p \). The problem with this is that the Hasse bound only provides that \(#X(\mathbb{F}_p) \leq 2g\sqrt{p} + p + 1\); so as \( p \) increases, Coleman’s bound becomes increasingly worse, and in particular is not uniform.

Whence the appeal of the following theorem of Lorenzini and Tucker.
Theorem 4.1 (Lorenzini, Tucker, [LT02b], Corollary 1.11). Suppose $p > 2g$ and let $\mathcal{X}$ be a proper regular model of $X$ over $\mathbb{Z}_p$. Suppose $r < g$. Then
\[
\# X(\mathbb{Q}) \leq \# \mathcal{X}_{\mathbb{F}_p}^{\text{sm}} + 2g - 2
\]
where $\mathcal{X}_{\mathbb{F}_p}^{\text{sm}}$ is the smooth locus of the special fiber $\mathcal{X}_{\mathbb{F}_p}$. 

The utility of the proper regular model $\mathcal{X}$ of $X$ is that the reduction map
\[r: \mathcal{X}(\mathbb{Q}) \to \mathcal{X}(\mathbb{F}_p)\]
takes values in the smooth locus $\mathcal{X}_{\mathbb{F}_p}^{\text{sm}}$. In Chabauty’s method, one thus only needs to consider residue classes $r^{-1}(Q)$ of points $Q \in \mathcal{X}_{\mathbb{F}_p}^{\text{sm}}$. Such residue classes are ($p$-adically analytically) isomorphic to discs; this makes the setup of Chabauty easier, and makes the “local analysis” much easier.

The “$2g - 2$” term in Coleman’s bound (3.6) is derived from Riemann–Roch on $X_{\mathbb{F}_p}$, and is the rationale for the “good reduction” hypothesis. Lorenzini and Tucker recover the $2g - 2$ term via Riemann–Roch on $X_{\mathbb{Q}_p}$ and a more involved $p$-adic analytic argument. A later, alternative proof [MP12, Theorem A.5] instead recovers the $2g - 2$ term via arithmetic intersection theory on $\mathcal{X}$ and adjunction.

The drawback is that $\mathcal{X}_{\mathbb{F}_p}$ could contain an arbitrarily long chain of $\mathbb{P}^1$’s. (For example, if $X$ is an elliptic curve with semistable reduction at $p$ and $v_p(j(X)) = -n$, then $\mathcal{X}_{\mathbb{F}_p}$ is an $n$-gon of $\mathbb{P}^1$’s. Once again, the size of $\mathcal{X}_{\mathbb{F}_p}^{\text{sm}}$ could be arbitrarily large, giving non-uniform bounds.

Stoll [Sto19] had the bold idea to work with a non-regular, minimal model: one contacts each chain of $\mathbb{P}^1$’s into a single node. Such a model is no longer regular, but the number of components is bounded, and the genus of each component is bounded (exercise: verify this). Since the model is no longer regular, rational points no longer reduce to smooth points (exercise: give an example), and might reduce to a node. The residue class of a node is now an annulus (explain in an example). This creates multiple problems: an annulus admits “monodromy” and integrals no longer admit a unique analytic continuation, and local expansions are now laurent, rather than power, series. See Theorem 6.1 below and the surrounding discussion.

5. Rank Favorable bounds

Lorenzini and Tucker [LT02b] ask if one can refine Coleman’s bound (Theorem 3.6) when the rank $r$ is small (i.e., $r \leq g - 2$). This was subsequently answered by Stoll.

Theorem 5.1 (Stoll, [Sto06], Corollary 6.7). With the hypothesis of Theorem 3.6,
\[
\# X(\mathbb{Q}) \leq \# X(\mathbb{F}_p) + 2r
\]

The space of differentials “suitable” for Chabauty’s method has dimension $g - r$. When $r < g - 1$, there are thus more “inputs” to Chabauty’s method and this extra flexibility can be exploited to improve the $p$-adic analysis. Indeed, Stoll’s idea is: instead of using a single integral, to taylor the choice of integral to each residue class. The additional “global geometric input” is the (classical) notion of “rank of a divisor”, and after translating his setup into this language, improved bounds follow from Clifford’s Theorem.
The application of Clifford’s theorem is also the source of the “good reduction” hypothesis. Using ideas from the “discrete case” of tropical geometry (in particular “chip-firing”), Eric Katz and I generalized Stoll’s theorem to arbitrary reduction types.

**Theorem 5.2** (Katz, Zureick-Brown, [KZB13]). Let \( X/Q \) be a curve of genus \( g \) and let \( r = \text{rank} \text{Jac}_X(Q) \). Suppose \( p > 2r + 2 \) is a prime, that \( r < g \), and let \( \mathcal{X} \) be a proper regular model of \( X \) over \( \mathbb{Z}_p \). Then

\[
#X(Q) \leq \#\mathcal{X}^\text{sm}_p(F_p) + 2r.
\]

Unlike Lorenzini and Tucker’s generalization of Coleman’s theorem, where they replace Coleman’s use of Riemann–Roch on \( X_{\mathbb{F}_p} \) with Riemann–Roch on \( X_{\mathbb{Q}_p} \), it does not seem possible to replace Stoll’s use of Clifford’s theorem on \( X_{\mathbb{F}_p} \) with Clifford’s theorem on \( X_{\mathbb{Q}_p} \). Matt Baker suggested that it might be possible to generalize Stoll’s theorem to curves with bad, totally degenerate reduction (i.e., \( X \) is a union of rational curves meeting transversely) using ideas from tropical geometry (see the recent survey [BJ15] on tropical geometry and applications), in particular the notion of “chip firing”, Baker’s combinatorial definition of rank, and Baker–Norine’s [BN07b] combinatorial Riemann–Roch and Clifford theorems. Baker was correct, and in fact an enrichment of his theory led to the following common generalization of Stoll’s and Lorenzini and Tucker’s theorems.

In this section, I’ll discuss the special case of a “Mumford curve” (i.e., a curve with “totally degenerate” reduction, in that the reduction is a collection of components of the reduction having higher genus.

5.3. **Stoll’s proof of Theorem 5.1** Let \( p > 2 \). Denote by \( V_{\text{chab}} \) the vector space of all \( \omega \in H^0(X_{\mathbb{Q}_p}, \Omega^1_{X_{\mathbb{Q}_p}/\mathbb{Q}_p}) \) such that \( \int_{P_i} \omega = 0 \) for all \( P_1, P_2 \in X(Q) \). Then \( \dim V_{\text{chab}} \geq g - r \). For each \( \omega \in V_{\text{chab}} \), scale \( \omega \) by a power of \( p \) so that the reduction \( \tilde{\omega} \) of \( \omega \) is non zero, and denote by \( \tilde{V}_{\text{chab}} \) the set of all such reductions; we note that \( \dim F, \tilde{V}_{\text{chab}} = \dim Q_{\mathbb{Q}_p} V_{\text{chab}} \).

For each \( Q \in X(F_p) \) and \( \omega \in \tilde{V}_{\text{chab}} \), set

\[
n_Q(\omega) := \deg (\text{div} \omega|_Q) \quad \text{and} \quad n_Q := \min_{\omega \in \tilde{V}_{\text{chab}}} n_Q(\omega)
\]

(where we recall that \( |Q| \) denotes the tube or residue class of \( Q \), that is, the set of all points of \( X(Q_p) \) which reduce to \( Q \). Since \( \text{div} \) is compatible with reduction mod \( p \), \( n_Q(\omega) \) is equal to the valuation \( \nu_Q(\tilde{\omega}) \) (i.e., the order of vanishing of \( \tilde{\omega} \) at \( Q \)).

By the “\( p \)-adic Rolle’s theorem”, the number of zeroes of \( \int \omega \) in \( |Q| \) is at most \( 1 + n_Q(\omega) \), so

\[
X(Q) \leq \sum_{Q \in X(F_p)} (1 + n_Q) = \sum_{Q \in X(F_p)} 1 + \sum_{Q \in X(F_p)} n_Q = X(F_p) + \deg (D_{\text{chab}}),
\]
where we define \( D_{\text{chab}} \) to be the divisor
\[
D_{\text{chab}} := \sum_{Q \in X} n_Q Q \in \text{Div } X_{\mathbb{F}_p}.
\]
By Riemann–Roch, \( \deg D_{\text{chab}} \leq 2g - 2 \), recovering the bound
\[
X(\mathbb{Q}) \leq X(\mathbb{F}_p) + 2g - 2.
\]
We claim that, in fact, \( \deg D_{\text{chab}} \leq 2r \), which suffices to prove Theorem 5.1. (When \( r = g - 1 \), \( 2r = 2g - 2 \).) Stoll’s main observation is that
\[
\widetilde{V}_{\text{chab}} \subset H^0(X_{\mathbb{F}_p}, \Omega^1_{X_{\mathbb{F}_p}}(-D_{\text{chab}})),
\]
and in particular,
\[
\dim H^0(X_{\mathbb{F}_p}, \Omega^1_{X_{\mathbb{F}_p}}(-D_{\text{chab}})) \geq \dim \widetilde{V}_{\text{chab}} \geq g - r.
\]
To justify Equation 5.3.1, given an effective divisor \( E = \sum n_P P \) and a line bundle \( \mathcal{L} \) on a curve \( X \), recall that \( H^0(X, \mathcal{L}(-E)) \) is the subspace of sections of \( H^0(X, \mathcal{L}) \) that have at least a zero of order \( n_p \) at \( P \). A differential \( \tilde{\omega} \in \widetilde{V}_{\text{chab}} \) thus satisfies \( v_P(\omega) \geq n_P \) by definition of \( n_p \).

On the other hand, Clifford’s Theorem [Har77, Theorem 5.4] implies that
\[
\dim H^0(X_{\mathbb{F}_p}, \Omega^1_{X_{\mathbb{F}_p}}(-D_{\text{chab}})) \leq \frac{1}{2} \deg \left( \Omega^1_{X_{\mathbb{F}_p}}(-D_{\text{chab}}) \right) + 1.
\]
Combining equations 5.3.2 and 5.3.3 gives
\[
g - r \leq \frac{1}{2} \deg \left( \Omega^1_{X_{\mathbb{F}_p}/\mathbb{Q}_p}(-D_{\text{chab}}) \right) + 1 = g - 1 - \frac{1}{2} \deg D_{\text{chab}} + 1
\]
and simplifying gives
\[
\deg D_{\text{chab}} \leq 2r.
\]
To justify Equation 5.3.3, we switch to the language of divisors. Recall that a divisor is special if \( \dim H^0(X, K - D) > 0 \), where \( K \) is a canonical divisor. (Equivalently, \( D \) is special if and only if it is a subdivisor of some canonical divisor.) For context: Riemann–Roch reads:
\[
H^0(X, D) = \dim H^0(X, K - D) + \deg D + 1 - g.
\]
This gives a formula for \( H^0(X, D) \) when the degree of \( D \) is large; indeed when \( \deg D > \deg K = 2g - 2 \), \( \deg K - D < 0 \), therefore \( \dim H^0(X, K - D) = 0 \) and \( H^0(X, D) = \deg D + 1 - g \). At the other extreme: if \( \deg D \leq 2g - 2 \), then it is still possible that \( \dim H^0(X, K - D) = 0 \), in which case, again, \( H^0(X, D) = \deg D + 1 - g \). If \( D \) is special, i.e., \( \dim H^0(X, K - D) > 0 \), then by Riemann–Roch, \( H^0(X, D) < \deg D + 1 - g \); in this case, Clifford’s Theorem gives the much stronger bound
\[
H^0(X, D) \leq \frac{1}{2} (\deg D) + 1.
\]
Let \( K = \text{div} \tilde{\omega} \). Then, by definition of \( n_Q \), \( D_{\text{chab}} \) is a subdivisor of the canonical divisor \( K \) (since \( v_Q(D_{\text{chab}}) := v_Q(\tilde{\omega}) = n_Q(\omega) \geq n_Q \)); in particular, \( D_{\text{chab}} \) is special.
5.4. The rank of a divisor. In the proof of Stoll’s theorem, we implicitly used the notion of rank of a line bundle (or divisor). One can simply define the rank \( r_X(\mathcal{L}) \) of a divisor \( \mathcal{L} \in \text{Pic}(X) \) to be

\[ r_X(\mathcal{L}) := \dim H^0(X, \mathcal{L}). \]

This has the following alternative interpretation over an algebraically closed field: \( r_X(\mathcal{L}) \) is the largest number of independent and generic “vanishing conditions” one can impose on sections of \( \mathcal{L} \). Recall that for a closed point \( P \), \( H^0(X, \mathcal{L}(-P)) \) is the subspace of sections of \( H^0(X, \mathcal{L}) \) that have at least a simple zero at the point \( P \). More generally, for an effective divisor \( E = \sum n_P P \), \( H^0(X, \mathcal{L}(-E)) \) is the subspace of sections of \( H^0(X, \mathcal{L}) \) that have at least a zero of order \( n_p \) at \( P \). By \cite[Proof of Theorem 1.3]{Har77},

\[ \dim H^0(X, \mathcal{L}(-P)) \geq \dim H^0(X, \mathcal{L}) - 1, \]

and in particular,

\[ \dim H^0(X, \mathcal{L}(-E)) \geq \dim H^0(X, \mathcal{L}) - \deg E. \]

One can simply define the rank \( r_X(D) \) of a divisor \( D \in \text{Div}(X) \) to be

\[ r_X(D) := \dim H^0(X, D). \]

This has the following alternative interpretation over an algebraically closed field: \( r_X(D) \) is the largest number of points of \( X \) (allowing for multiplicity) one can remove from \( D \) before forcing \( D \) is no longer equivalent to some effective divisor. Equivalently, the rank is the largest number of points that you can demand occurs as a subdivisor of some effective \( E \in \lvert D \rvert \) (i.e., some effective divisor \( E \) equivalent to \( D \).

More formally, we make the following definitions.

**Definition 5.5.** Let \( D \in \text{Div}X \) be a divisor. The linear system associated to \( D \) is the collection \( \lvert D \rvert \) of effective divisors linearly equivalent to \( D \). We define the rank \( r_X(D) \) of \( D \) to be -1 if \( \lvert D \rvert \) is empty (i.e., if \( D \) is not equivalent to an effective divisor). Otherwise, we define

\[ r(D) := \max\{k \in \mathbb{Z}_{\geq 0} : \lvert D - E \rvert \neq \emptyset, \forall E \in \text{Div}^k(X)\}, \]

where \( \text{Div}^k(X) \) is the subset of \( \text{Div}(X) \) of effective divisors of degree \( k \).

The linear system \( \lvert D \rvert \) is naturally isomorphic to the projective space \( \text{Proj} H^0(X, D) \cong \mathbb{P}^r - 1 \), where \( r = \dim H^0(X, D) \).

5.6. Chip firing. If you’ve never see or heard of Chip Firing, I recommend taking a quick look at Baker’s short expository article available here

[http://people.math.gatech.edu/~mbaker/pdf/g4g9.pdf](http://people.math.gatech.edu/~mbaker/pdf/g4g9.pdf)

For a short selection of other references: the papers \cite{Bak08,BN07a} by Baker and collaborators are my preferred starting point.

One can associate to such a singular curve with transverse crossings its dual graph as in Figure 1. Component curves become nodes, and intersections correspond to edges.
5.7. **Rank favorable bounds for curves with totally degenerate reduction.** In this subset section, I’ll discuss the special case of a “Mumford curve” (i.e., a curve with “totally degenerate” reduction, in that the reduction is a collection of $\mathbb{P}^1$’s meeting transversely, and in particular represents an isolated point on the moduli space of curves); for such curves, one only needs Baker’s original notion of rank (which we call “numerical rank”); for a discussion of the “abelian rank”, see our paper [KZB13].

5.8. **Refined ranks.** In a certain sense, the numerical rank only “sees” the component group of the Néron model. The enriched notion of abelian rank from [KZB13] and ?? sees the abelian part of the Néron model. One can ask if there is a corresponding notion of “toric” or “unipotent” rank. In [KZB13, Subsection 3.3], we define a “toric” rank, and in [KZB13, Example 5.5] demonstrate that these ranks differ; we have yet to find a useful application.

### 6. Tropical Geometry, Berkovich spaces, and Chabauty

A recent breakthrough [Sto13] fully removed, in the special case of hyperelliptic curves, the dependence on a regular model and derived a uniform bound on $\#X(\mathbb{Q})$ for small ($r \leq g-3$) rank curves.

**Theorem 6.1** (Stoll, [Sto13]). *Let $X$ be a smooth hyperelliptic curve of genus $g$ and let $r = \text{rank}_\mathbb{Z} \text{Jac}_X(\mathbb{Q})$. Suppose $r \leq g - 3$. Then*

$$\#X(\mathbb{Q}) \leq 8(r + 4)(g - 1) + \max\{1, 4r\} \cdot g.$$  

A main ingredient in Stoll’s proof is to understand the discrepancy between the different flavors of integration. Eric Katz noticed that this discrepancy “factored through the tropicalization of the torus part of the Berkovich uniformization of $X$”. After a thorough reinterpretation of the method of Chabuaty and Coleman via Berkovich spaces, and harnessing the full catalogue of tropical and non-Archimedean analytic tools, we were able to improve Stoll’s result to arbitrary curves of small rank.

**Theorem 6.2** (Katz–Rabinoff–Zureick-Brown, [KRZP]). *Let $X$ be any smooth curve of genus $g$ and let $r = \text{rank}_\mathbb{Z} \text{Jac}_X(\mathbb{Q})$. Suppose $r \leq g - 3$. Then*

$$\#X(\mathbb{Q}) \leq 84g^2 - 98g + 28.$$  

For more details, see our survey [KRZB16].
References


[CHM97] Lucia Caporaso, Joe Harris, and Barry Mazur, Uniformity of rational points, J. Amer. Math. Soc. 10 (1997), no. 1, 1–35. MR1325796 (97f:14033) ↑4


11


Thoralf Skolem, *Ein verfahren zur behandlung gewisser exponentierter gleichungen und diophantischer gleichungen*, C. r **8** (1934), 163–188. ↑2


