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Endomorphisms of Partially Ordered Sets

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It is shown that every partially ordered set with n elements admits an endomorphism with an image of a size at least $n^{1/7}$ but smaller than n. We also prove that there exists a partially ordered set with n elements such that each of its non-trivial endomorphisms has an image of size $O((n \log n)^{1/3})$.

1. Introduction

Let P be a partially ordered set. A function $\phi : P \to P$ is an *endomorphism* of P if for every two elements x, y of P, the inequality $\phi(x) \leqslant \phi(y)$ holds whenever $x \leqslant y$. Obviously, the identity mapping is a (*trivial*) endomorphism. Here, however, we will be interested in endomorphisms with an image of size less than $|P|$, *i.e.* endomorphisms which are not automorphisms of P. We will refer to them as *non-surjective*. Define

$$\eta(P) = \max\{|\phi(P)| \, : \, \phi \text{ is a non-surjective endomorphism of } P\}$$

and

$$g(n) = \min\{\eta(P) \, : \, |P| = n\}.$$

The only known estimates for $g(n)$ are due to Grant, Nowakowski and Rival, who introduced the problem in [2] and proved that

$$(1 + o(1))\frac{\log n}{\log\log n} < g(n) = O(\sqrt{n}).$$

In this note we improve both the above bounds for $g(n)$, showing the following two

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results. Here and below $C(x) = \{y \in P : y \geqslant x \text{ or } y \leqslant x\}$ denotes the set of all elements comparable with x, and $D = D(P) = \max_{x \in P} |C(x)|$.

Theorem 1.1. *Let P be a partially ordered set of size $|P| = n$.*

(i) *If P contains an element comparable to each element of P then $\eta(P) = n - 1$.*

(ii) *If P has no such element then*

$$\eta(P) \geqslant \max\left\{D, \left(\frac{n}{D^4}\right)^{1/3}\right\}.$$

In particular, $g(n) \geqslant n^{1/7}$.

Theorem 1.2. *There is a constant $c > 0$ such that for every n there exists a partially ordered set P of size n with the property that the only endomorphism ϕ of P with $|\phi(P)| \geqslant c(n \log n)^{1/3}$ is the identity. In particular, for every n, $g(n) \leqslant c(n \log n)^{1/3}$.*

Remark. One can easily modify the proofs of Theorems 1.1 and 1.2 to get similar bounds for a function analogous to $g(n)$ and defined for graphs, provided we allow an endomorphism of a graph to map a pair of adjacent vertices into one vertex (or, equivalently, we assume that there is a loop at each vertex of a graph).

2. Proof of Theorem 1.1

Throughout this section, P denotes an n-element ordered set. We say that x *covers* y (or y *is covered by* x) if $x \neq y$, $x \geqslant y$ and, for every z, if $y \leqslant z \leqslant x$ then either $z = x$ or $z = y$. First, if there is some $x \in P$ with $C(x) = P$, let y be any element covered by or covering x. The map ϕ which sends y to x and fixes all other elements is an endomorphism with $|\phi(P)| = n - 1$. This proves (i).

Assuming P has no element comparable with all others, choose $x \in P$ with $|C(x)| = D$ and define ϕ_x on P by

$$\phi_x(y) = \begin{cases} y, & \text{if } y \in C(x), \\ x, & \text{otherwise.} \end{cases}$$

Then, $|\phi_x(P)| = C(x) = D < n$ and ϕ_x is an endomorphism.

Thus, to prove part (ii) of Theorem 1, it is enough to construct a non-surjective endomorphism with image of size at least $(n/D^4)^{1/3}$. Without loss of generality we may assume that P is connected, *i.e.* that it is not possible to split P into two subsets of mutually incomparable elements. Indeed, if P has components P_i with $n_i = |P_i|$ and $D_i = D(P_i)$, $i = 1, \ldots, s$, and ϕ is an endomorphism of P with $(n_i/D_i^4)^{1/3} \leqslant |\phi(P_i)| < n_i$, $i = 1, \ldots, s$, then

$$|\phi(P)| = \sum_i |\phi(P_i)| \geqslant \left(\sum_i n_i/D_i^4\right)^{1/3} \geqslant (n/D^4)^{1/3}$$

since $D_i \leqslant D$. Let $\max(P)$ ($\min(P)$) denote the set of maximal elements of P (minimal elements of P, respectively), and let $\text{ext}(P) = \min(P) \cup \max(P)$. We call a subset $X \subseteq P$

an *up-set* (*down-set*) whenever $y \leqslant x$ ($y \geqslant x$) and $y \in X$ imply $x \in X$. In particular, if $x \in \min(P)$ ($\max(P)$) then the singleton set $\{x\}$ is a down-set (up-set). Given an up-set X and a down-set Y of P, let

$$N(X) = \{z \in P : z \leqslant x \quad \text{for some } x \in X\}$$
$$N(Y) = \{z \in P : z \geqslant y \quad \text{for some } y \in Y\}.$$

Then $N(X)$ is a down-set and $N(Y)$ is an up-set. We define sets $N_k(X)$ and $N_k(Y)$ recursively, setting

$$N_0(X) = X, \text{ and } N_k(X) = N\left(N_{k-1}(X)\right), \text{ for } k = 1, 2, \ldots,$$

and write $N(x)$, $N_k(x)$ instead of $N(\{x\})$, $N_k(\{x\})$.

The following basic construction will be crucial in defining a non-surjective endomorphism with large image. Let elements $f_0, \ldots, f_k$ of P (not necessarily distinct) form a *fence*, *i.e.* for each $i = 1, \ldots, k-1$ they satisfy $f_{i-1} \leqslant f_i \geqslant f_{i+1}$ or $f_{i-1} \geqslant f_i \leqslant f_{i+1}$. Let us assume that $f_0 \leqslant f_1$. Then, for any $v \in \min(P)$, the mapping

$$\phi(A_i) = f_{k-i}\,, \quad i = 0, 1, \ldots, k, \tag{2.1}$$

where $A_0 = \{v\}$, $A_1 = N_1(v) \setminus \{v\}$, $A_2 = N_2(v) \setminus N_1(v), \ldots, A_{k-1} = N_{k-1}(v) \setminus N_{k-2}(v)$, $A_k = P \setminus N_{k-1}(v)$, is an endomorphism. This is because if $y \leqslant z$ then $\{y, z\} \subset A_i \cup A_{i+1}$ for some $i = 0, \ldots, k-1$, and therefore the elements y, z are mapped either to the same image f_j or to two consecutive elements of the fence, always preserving the order.

Now, we shall find integers k and r, elements $v_1, \ldots, v_r$ and fences $F_1, \ldots, F_r$ with common final vertex, such that the sets $N_k(v_i)$ will be pairwise disjoint, and mapping each set $N_k(v_i)$ on one of the fences in the above prescribed manner will create a non-surjective endomorphism with large image.

For all $x \in \text{ext}(P)$, let $\deg(x) = |N(x)|$ and $d = \max\{\deg(x) : x \in \text{ext}(P)\}$. (Note that $d \leqslant D$.) Then, for an up-set X and a down-set Y,

$$|N(X)| \leqslant d|\max(P) \cap X| \leqslant d|X| \quad \text{and, similarly,} \quad |N(Y)| \leqslant d|Y|\,. \tag{2.2}$$

By taking $X = \max(P)$ and $Y = \min(P)$, it follows that $|\max(P)| \geqslant n/d$ and $|\min(P)| \geqslant n/d$. Set

$$t(x) = \max\left\{k : |N_k(x)| < \left(\frac{n}{d}\right)^{1/3}\right\}. \tag{2.3}$$

Note that $N_0(x) = \{x\}$ while, since P is connected, $|N_k(x)| = n$ for some k. Thus $t(x)$ always exists. Observe that, by (2.2) with $X = N_{t(x)}(x)$, it follows that

$$|N_{t(x)}(x)| \geqslant \frac{1}{d}\left(\frac{n}{d}\right)^{1/3} = \left(\frac{n}{d^4}\right)^{1/3}. \tag{2.4}$$

Choose x_0 such that $t_0 = t(x_0)$ minimizes $t(x)$ over $\text{ext}(P)$. The set $N_{t_0}(x_0)$ will serve as a source of fences with common origin x_0 and length t_0. There are four similar options with respect to the location of x_0 ($\min(P)$ or $\max(P)$) and the parity of t_0. Without loss of generality we thus assume that, say, $x_0 \in \min(P)$ and t_0 is even.

By the definition of t_0 and by (2.3), for all $v \in \min(P)$ we have $|N_{t_0}(v)| \leqslant (n/d)^{1/3}$, and so

$$|N_{2t_0}(v)| = \left| \bigcup_{u \in N_{t_0}(v) \cap \min(P)} N_{t_0}(u) \right| \leqslant \left(\frac{n}{d}\right)^{2/3}.$$

Since, in addition, $|\min(P)| \geqslant n/d$, one can greedily find $l \geqslant (n/d)^{1/3}$ elements $v_1, \ldots, v_l \in \min(P)$ such that $N_{t_0}(v_i) \cap N_{t_0}(v_j) = \emptyset$ for all $i \neq j$. In fact, we will only need $v_1, \ldots, v_r$, where $r = |N_{t_0}(x_0)| < (n/d)^{1/3}$. Let us associate to each v_i a fence F_i of the form

$$x_0 = f_{i,0} \leqslant f_{i,1} \geqslant f_{i,2} \leqslant \ldots \geqslant f_{i,t_0} = u_i,$$

where u_i runs through all the elements of $N_{t_0}(x_0)$, and where the $f_{i,j}$s need not be distinct.

Now we construct a mapping which maps each set $N_{t_0}(v_i)$ onto the fence F_i, $i = 1, \ldots, r$, as in (2.1). Formally, define ϕ on P by

$$\phi(v_i) = u_i \quad \text{for} \quad i = 1, 2, \ldots, r,$$

$$\phi(N_j(v_i) \setminus N_{j-1}(v_i)) = f_{i,t_0-j} \quad \text{for} \quad i = 1, 2, \ldots, r, \quad \text{and} \quad j = 1, 2, \ldots, t_0 - 1, \quad \text{and}$$

$$\phi(y) = x_0 \text{ for all } y \notin \left(\bigcup_{i=1}^{r} N_{t_0-1}(v_i) \right).$$

Because $N_{t_0}(v_i) \cap N_{t_0}(v_j) = \emptyset$ for all $i \neq j$, the map ϕ is well defined. Since $v_i \in \min(P)$, each $x \in N_1(v_i) \setminus \{v_i\}$ satisfies $x \geqslant v_i$, and

$$\phi(x) = f_{i,t_0-1} \geqslant f_{i,t_0} = u_i = \phi(v_i).$$

That ϕ is an endomorphism follows by similar argument. It is important to notice that for all $i \neq j$ and all $x \in N_{t_0}(v_i)$ and $y \in N_{t_0}(v_j)$, the elements x and y are incomparable, and we do not need to worry about the relation between $\phi(x)$ and $\phi(y)$ for such pairs. Furthermore, we have clearly

$$|\phi(P)| = |N_{t_0}(v_i)| = r,$$

and, by (2.2) and (2.3),

$$\left(\frac{n}{D^4}\right)^{1/3} \leqslant \left(\frac{n}{d^4}\right)^{1/3} \leqslant r < \left(\frac{n}{d}\right)^{1/3} < n.$$

Thus, $|\phi(G)| \geqslant \left(n/D^4\right)^{1/3}$, yielding the required lower bound for $\eta(P)$.

We verify $g(n) \geqslant n^{1/7}$ easily: the inequalities $D < n^{1/7}$ and $\left(n/D^4\right)^{1/3} < n^{1/7}$ contradict each other. □

3. Proof of Theorem 1.2

The proof of Theorem 1.2 involves a natural relation between bipartite graphs and 2-level partial orders. A partial order P such that $P = \min(P) \cup \max(P)$ is called *2-level*. The Hasse diagram of a 2-level partial order is a bipartite graph $G(P)$. Conversely, every bipartite graph G with bipartition (V_1, V_2) can be viewed as a partial order $P(G)$ on the set $V_1 \cup V_2$, in which $x \geqslant y$ if and only if $x = y$ or $x \in V_1$ and $\{x, y\}$ is an edge of G. (Throughout we will be assuming that V_1 is the set of maximal and V_2 of minimal elements of P.)

From now on we set, for convenience,

$$m = m(n) = (n \log n)^{1/3}.$$

For further reference observe that $m < n^{2/5}$ as well as $m < n^{7/18}$ (here and throughout this section we tacitly assume that all the inequalities are only claimed to hold for sufficiently large n).

Let $v_i(H) = |V_i \cap V(H)|$, $i = 1, 2$, for any subgraph H of G. Furthermore, let $\deg_G(v)$ denote the degree of a vertex v in a graph G and $N_G(S)$ stand for the neighbourhood of a subset of vertices S in a graph G. (Note that if P is 2-level then, for any $v \in P$, $N_{G(P)}(v) = N(v) \setminus \{v\}$, where $N(v)$ was defined in Section 2.)

To prove Theorem 1.2, we will show the existence of a bipartite graph G on $n + n$ vertices such that the partial order $P(G)$ satisfies the inequality $\eta(P(G)) \leqslant 100 \times 2^{27} m$.

It is shown (probabilistically) in Lemma 4.1 that almost all bipartite graphs G with bipartition (V_1, V_2), $|V_1| = |V_2| = n$, which belong to a certain space of random graphs, satisfy all the properties (1–6) listed below. Thus the next lemma follows.

Lemma 3.1. *For every n large enough, there exists a bipartite graph G with bipartition (V_1, V_2), $|V_1| = |V_2| = n$, satisfying all the properties* (1–6) *listed below.*

(1) *There is no subgraph H of G with $|V(H)| < 80$ and $|E(H)| > \frac{3}{2}|V(H)|$.*

(2) *For each vertex v of G,*

$$|\deg_G(v) - m| \leqslant n^{1/3}.$$

(3) *Every subgraph H of G such that $\min\{v_1(H), v_2(H)\} \leqslant n^{2/5}$ contains at most $3|V(H)|$ edges.*

(4) *Let S be a subset of vertices of G contained in one set of the bipartition.*

(a) *If $|S| \leqslant n^{7/18}$ then*

$$|N_G(S)| \geqslant |S| m(1 - 10^{-15});$$

(b) *if $|S| \leqslant n^{7/18}$ and H is a subgraph of G with bipartition (S, N) and such that $\deg_H(v) \geqslant 0.999m$ for every $v \in S$, then*

$$|N| \geqslant 0.999 |S| m(1 - 10^{-15});$$

(c) *if $|S| = \lfloor n^{2/3} \rfloor$ and H is a subgraph of G with bipartition (S, N) and such that $\deg_H(v) \geqslant 0.9m$ for every $v \in S$, then*

$$|N| > 0.8n;$$

(d) *if $8m \leqslant |S| \leqslant n - 8m^2$ then*

$$|N_G(S)| \geqslant |S| + 7m^2.$$

(5) *For each $i = 1, 2$ and for every pair of disjoint sets $S', S'' \subseteq V_i$ of $\lfloor 2m \rfloor$ vertices each, there exists $w \in V_{3-i}$ which has neighbours in both S' and S''.*

(6) *No two edge-disjoint, induced subgraphs H', H'' of G with*

$$v_1(H') = v_2(H') = v_1(H'') = v_2(H'') = \lceil n^{3/4} \rceil$$

are isomorphic.

In the proof of Theorem 1.2 we shall also need the following lemma, which is proved in Section 4.

Lemma 3.2. *Let G be a bipartite graph with bipartition (V_1, V_2), such that, for some natural numbers $d, k \geqslant 2$, the following holds.*
(i) $|V_1| \geqslant 2^{4d+3}k$;
(ii) $\deg_G(v) \leqslant k/8d$ *for every vertex v of V_2;*
(iii) $|E(G)| \leqslant d|V_1|$.
Then there are two disjoint subsets W', W'' of V_1, of k elements each, such that no vertex u of V_2 has neighbours in both W' and W''.

The proof of Theorem 1.2 consists of several steps in which we gradually reduce possible forms of endomorphisms with large image to that of the identity. Let us start with a simple observation which will allow us to consider only endomorphisms of a special type. We call an endomorphism $\phi : P \to P$ of a 2-level partial order P *straight* if $\phi(V_1) \subseteq V_1$ and $\phi(V_2) \subseteq V_2$.

Claim 1. Let P be a 2-level partial order such that the minimum degree of the graph $G(P)$ is at least 2. Then for every non-surjective endomorphism $\phi : P \to P$ there exists a straight, non-surjective endomorphism $\phi' : P \to P$ such that $|\phi'(P)| \geqslant |\phi(P)|$.

Proof. Let us define ϕ' setting

$$\phi'(x) = \begin{cases} \phi(x) & \text{if } \; x, \phi(x) \in V_1 \;\text{ or }\; x, \phi(x) \in V_2, \\ x' & \text{otherwise,} \end{cases}$$

where x' is any element of $N_{G(P)}(\phi(x))$.

It is not hard to see that ϕ' is indeed an endomorphism of P and that it is straight. Furthermore, if $x \in V_i$ but $\phi(x) = z \in V_{3-i}$, for some $i = 1, 2$, then for all $y \in N(x)$, $y \in \phi^{-1}(z)$. Thus, $\phi^{-1}(z)$ contains elements from V_{3-i} and $\phi'(P) \supseteq \phi(P)$. Moreover, as $|N(x)| \geqslant 2$, ϕ is not an automorphism. □

Now, let G be a bipartite graph satisfying properties (1–6) above and let ϕ be a straight endomorphism of $P = P(G)$ with $|\phi(P)| \geqslant 100 \times 2^{27}m$. In Claim 2 we show that $|\phi(V_i)| \geqslant 50 \times 2^{27}m$ for both $i = 1$ and $i = 2$. As a result of Claim 3 we conclude that $|\phi^{-1}(v)| \leqslant n^{7/18}$ for all $v \in P$. Claim 4 reduces ϕ to an automorphism, and finally, in Claim 5 we prove that ϕ must be the identity.

Claim 2. If $\phi : P \to P$ is a straight endomorphism of P such that $|\phi(V_i)| \geqslant 50 \times 2^{27}m$ for $i = 1$ or $i = 2$, then $|\phi(V_i)| \geqslant 50 \times 2^{27}m$ for both $i = 1$ and $i = 2$.

Proof. Let us assume that $|\phi(V_2)| < 50 \times 2^{27}m \leqslant |\phi(V_1)|$. Then the graph G_ϕ spanned in $G(P)$ by the set of vertices $\phi(P)$ has, by property (3), at most $6|\phi(V_1)|$ edges.

Thus we can apply Lemma 3.2 to the graph G_ϕ, with $d = 6$ and $k = 50m$, to deduce the existence of subsets $W', W'' \subseteq \phi(V_1)$ such that $|W'| = |W''| \geqslant 50m$ and no vertex

$w \in \phi(V_2)$ has neighbours in both W' and W''. But then the sets $\phi^{-1}(W')$ and $\phi^{-1}(W'')$ are disjoint subsets of V_1 of at least $50m$ vertices each, which have no common neighbours in V_2, contradicting property (5). □

Claim 3. Let $\phi : P \to P$ be a straight endomorphism of P such that $|\phi(V_i)| \geqslant 50 \times 2^{27}m$, $i = 1, 2$. Then $|\phi^{-1}(v)| \leqslant n^{7/18}$ for all $v \in P$.

Proof. Let $\phi : P \to P$ be a straight endomorphism of a partially ordered set P such that $|\phi(V_i)| \geqslant 50 \times 2^{27}m$ for $i = 1, 2$. We classify elements of $\phi(P)$ into two categories: we call $v \in \phi(P)$ *large* if $|\phi^{-1}(v)| \geqslant 2m$ and *small* otherwise.

Let us first prove that each set of the bipartition contains at most three large elements. Suppose that v_1, v_2, v_3, v_4 are large elements which belong to, say, V_1.

By property (5), for each $i = 1, 2, 3, 4$, the vertex v_i must be connected by a path of length two to all except at most $2m$ vertices of the set $\phi(V_1)$. Indeed, let $S \subset \phi(V_1)$, $|S| \geqslant 2m$. Then the sets $\phi^{-1}(v_i)$ and $\phi^{-1}(S)$ are disjoint and of size at least $2m$ each. By property (5) there is a vertex $w \in V_2$ adjacent to both these sets. But then its image $\phi(w)$ is adjacent to both v_i and S. Hence, all except at most $8m$ vertices of $\phi(V_1)$ are each connected by a path of length two to each of v_1, v_2, v_3, v_4.

Let U be the set of all elements of V_2 which are adjacent to at least two vertices from v_1, v_2, v_3, v_4. By property (1), $|U| \leqslant 12$, since otherwise there would be a 17-vertex subgraph with density $26/17 > 3/2$.

Since the maximum degree of G is smaller than $1.1m$, all except at most $(1.1 \times 12 + 8)m$ vertices of $\phi(V_1)$ are connected to each of vertices v_1, v_2, v_3, v_4 by paths of length two not containing vertices from U, *i.e.* by internally disjoint paths.

More precisely, if for such a vertex x the paths from x to v_1, v_2, v_3, v_4 go through $u_1^x, \ldots, u_4^x$, then the vertices $u_1^x, \ldots, u_4^x$ are distinct.

Moreover, one can find, say, 15 vertices $x_1, \ldots, x_{15}$ with all 60 vertices $u_i^{x_j}$, $i = 1, \ldots, 4$, $j = 1, \ldots, 15$, distinct. This is because every 4-tuple $u_1^x, \ldots, u_4^x$ excludes (*i.e.* have edges to) at most $4.4m$ vertices of $\phi(V_1)$, and there are still plenty of them available.

But these 15 vertices create a subgraph of G with $4 + 5 \times 15 = 79$ vertices and $15 \times 8 = 120$ edges, which is prohibited by property (1) of G. Thus, each level of P contains at most three large vertices.

Let us denote the set of all large elements contained in V_1 and V_2 by L_1 and L_2 respectively, and set $U_i = N_{G[\phi(P)]}(L_{3-i})$, $i = 1, 2$. Finally, for $i = 1, 2$, set $R_i = \phi(V_i) \setminus (L_i \cup U_i)$.

We have $|L_i| \leqslant 3$, $i = 1, 2$, and, without loss of generality, we can assume that $|\phi^{-1}(R_1)| \geqslant |\phi^{-1}(R_2)|$.

Note that there is no edge between L_2 and R_1 and thus $N_G(\phi^{-1}(R_1)) \subseteq \phi^{-1}(R_2) \cup \phi^{-1}(U_2)$. Moreover, since U_2 consists of at most $3.3m$ vertices, all of which are small, $|\phi^{-1}(U_2)| \leqslant 6.6m^2$ and consequently

$$|N_G(\phi^{-1}(R_1))| \leqslant |\phi^{-1}(R_2)| + |\phi^{-1}(U_2)| < |\phi^{-1}(R_1)| + 7m^2.$$

Thus, according to property (4d), either $|\phi^{-1}(R_1)| \leqslant 8m$, or $|\phi^{-1}(R_1)| \geqslant n - 8m^2$. However, $|\phi^{-1}(R_1)| \geqslant |R_1| \geqslant |\phi(V_1)| - |L_1| - |U_1| > 8m$ and the first option is ruled out.

Hence

$$|\phi^{-1}(R_1)| \geqslant n - 8m^2, \tag{3.1}$$

which implies two further inequalities:

$$|\phi^{-1}(L_1)| \leqslant 8m^2 \tag{3.2}$$

and

$$|N_G(\phi^{-1}(R_1))| \geqslant n - 9m, \tag{3.3}$$

the second one by property (4a) applied to the set $S = V_2 \setminus N_G(\phi^{-1}(R_1))$. (If this set is bigger than $9m$ then, by (4a), the set $N_G(S)$ is bigger than $8m^2$, contradicting (3.1).)

Inequality (3.3) implies, in turn, that

$$|\phi^{-1}(L_2)| \leqslant 9m. \tag{3.4}$$

On the other hand,

$$|N_G(\phi^{-1}(L_1))| \leqslant |\phi^{-1}(L_2)| + |\phi^{-1}(U_2)| \leqslant 7m^2,$$

and, again by property (4d) and in view of (3.2), we conclude that

$$|\phi^{-1}(L_1)| \leqslant 8m. \tag{3.5}$$

The upper bounds (3.4) and (3.5) we just obtained and the definition of the sets L_1 and L_2 imply together that $|\phi^{-1}(v)| \leqslant 9m < n^{7/18}$ for every element $v \in P$. □

Claim 4. If $\phi : P \to P$ is a straight endomorphism of P such that for every $v \in P$, $|\phi^{-1}(v)| \leqslant n^{7/18}$, then ϕ is an automorphism of P, i.e. $|\phi^{-1}(v)| = 1$ for every $v \in P$.

Proof. Let $\phi : P \to P$ be a straight endomorphism such that $s = \max_{v\in P} |\phi^{-1}(v)| \leqslant n^{7/18}$ and let $v_0 \in \phi(P)$ be such that $|\phi^{-1}(v_0)| = s$. Without loss of generality we may assume that $v_0 \in V_1$.

In the rest of this proof we shall frequently recall the expanding property (4). First, by (4a), since $N_G(\phi^{-1}(v_0)) \subseteq \bigcup_{w\in N_{G[\phi(P)]}(v_0)} \phi^{-1}(w)$,

$$\sum_{w\in N_{G[\phi(P)]}(v_0)} |\phi^{-1}(w)| \geqslant |N_G(\phi^{-1}(v_0))| \geqslant (1 - 10^{-15})sm. \tag{3.6}$$

But the number of terms in the sum in (3.6) is at most $m + n^{1/3}$ (property (2)), and each of them is not larger than s. Thus, by simple counting argument, the vertex v_0 has in $\phi(P)$ at least $0.9999999m$ neighbours w for which $|\phi^{-1}(w)| \geqslant 0.9999999s$ (otherwise, the sum in (3.6) would be less than $0.9999999ms + 0.0000001m \times 0.9999999s = (1 - 10^{-14})sm$ – a contradiction). Let us denote the set of such neighbours of v_0 by N_1.

An analogous argument shows that each $w \in N_1$ has at least $0.999m$ neighbours u with $|\phi^{-1}(u)| \geqslant 0.999s$. Here we apply (4a) with $S = \phi^{-1}(w)$, so the respective sum is at least, say, $0.9999995sm$ and if there were less than $0.999m$ such neighbours this sum would be at most $0.999ms + 0.001 \times 0.999s$, not reaching its lower bound – a contradiction. Let us denote the set of these neighbours u of w by N_w. In turn, by a similar argument, for each $w \in N_1$, each $u \in N_w$ has at least $0.9m$ neighbours z with $|\phi^{-1}(z)| \geqslant 0.9s$. This set is denoted by N_u.

Now, we apply (4b) to the subgraph H_1 consisting of all $w \in N_1$ on one side and $\bigcup_{w \in N_1} N_w$ on the other, concluding that the set $N_2 = N_{H_1}(N_1)$ has size at least $0.99m^2$.

Next, consider the graph H_2 with vertex set N_2' being any subset of N_2 with $\lfloor n^{2/3} \rfloor$ elements on one side, and the set $\bigcup_{u \in N_2'} N_u$ on the other.

Applying to this graph the property (4c) yields $|N_{H_2}(N_2')| > 0.8n$, *i.e.* more than $0.8n$ vertices which lie at a distance of three from v_0 have their preimages of size at least $0.9s$ (and, of course, all these preimages are disjoint). But all vertices which are at distance three from v_0 belong to V_2 and, as ϕ is straight, their preimages are subsets of V_2. Hence, the sum of their preimages must not be greater than n, *i.e.* we must have $0.72sn < n$, and consequently, $s = 1$. □

Claim 5. The only automorphism of P is the identity.

Proof. Let $\sigma : P \to P$ be an automorphism of P. Let us denote by W the set of all fixed points of σ and let $\overline{W}$ stand for the complement of W.

Note that σ must be straight, and suppose that $|\overline{W} \cap V_i| \geqslant 3n^{3/4}$ for some $i = 1, 2$. Then one can find in this $\overline{W} \cap V_i$ two disjoint subsets S_1 and S_2 such that $|S_1| = |S_2| = \lceil n^{3/4} \rceil$ and $\sigma(S_1) = S_2$. Simply, for each cycle of the permutation σ, include every second element to S_1 and all the remaining ones, except one in case the cycle is odd, to S_2.

Then, for any subset $T \subset V_{3-i}$ of size $|T| = \lceil n^{3/4} \rceil$, the subgraphs H', H'' of G spanned by the pairs of sets (S_1, T) and $(S_2, \sigma(T))$, respectively, satisfy $v_i(H') = v_i(H'') = \lceil n^{3/4} \rceil$, $i = 1, 2$, and are edge-disjoint. Moreover, since σ is an automorphism of the graph G, subgraphs H' and H'' are isomorphic, which contradicts property (6).

Thus, we may assume that $|\overline{W} \cap V_1| \leqslant |\overline{W} \cap V_2| < 3n^{3/4}$. Suppose $\overline{W}$ is non-empty. Then, by property (4),

$$N_G(\overline{W} \cap V_2) \geqslant 7|\overline{W} \cap V_2| + 1$$

and, by the pigeon-hole principle, there is $v \in \overline{W}$ with at least seven neighbours in W.

The image of v, $\phi(v)$ is different from v and is adjacent to the same seven neighbours of v. Thus, a copy of $K_{2,7}$ is present in G: a contradiction with property (1). Hence $\overline{W}$ is empty, *i.e.* σ is the identity automorphism of P. □

4. Random graphs and the probabilistic method

The bipartite graph we employ in the proof of Theorem 1.2 has properties 'typical' for a random bipartite graph $\mathscr{G}(n, n; p)$. Indeed, we shall show in this section that 'almost all' graphs from the space $\mathscr{G}(n, n; p)$ satisfy all properties (1–6) listed in Lemma 3.1.

At the end of this section we provide a probabilistic proof of Lemma 3.2 which was used in the proof of Theorem 1.2.

The random bipartite graph $\mathscr{G}(n, n; p)$ is the graph with vertex set $V = V_1 \cup V_2$, where $|V_1| = |V_2| = n$ and each pair of vertices $\{v_1, v_2\}$, where $v_1 \in V_1$ and $v_2 \in V_2$, appears in $\mathscr{G}(n, n; p)$ with probability p, independently for each such a pair. If $p = p(n)$ is a given function of n and the probability that $\mathscr{G}(n,n;p)$ has a given graph property $\mathscr{A}$ tends to 1 as $n \to \infty$, we say that $\mathscr{G}(n,n;p)$ has property $\mathscr{A}$ *almost surely.*

Throughout the whole section we set the probability of the existence of an edge to be

$$p = p(n) = (\log n/n^2)^{1/3}.$$

Lemma 4.1. *Almost surely, $\mathscr{G}(n,n;p)$ possesses all the properties* (1–6) *listed in Lemma 3.1.*

Proof. We prove properties (1–6) one by one. Having proved some of them, we will be, explicitly or not, assuming that they hold when proving the next properties. Formally, we will be using on such occasions the standard argument that, for two sequences of events $\mathscr{A}_n$ and $\mathscr{B}_n$, if $\Pr(\mathscr{B}_n) \to 1$ then $\Pr(\mathscr{A}_n) - \Pr(\mathscr{A}_n \cap \mathscr{B}_n) = o(1)$.

All convergences and $o(\cdot)$ terms are with respect to $n \to \infty$. The dependence on n of several random variables will be suppressed.

(1) We prove property (1) using the first moment method – a basic tool in the theory of random structures. Let X count the forbidden subgraphs H. Then X is the sum of at most $\sum_H n^{|V(H)|}$ indicator random variables, where each of them equals 1 with probability $p^{|E(H)|}$. Thus, by the linearity of expectation,

$$\mathrm{E}X \leqslant \sum_H n^{|V(H)|} p^{|E(H)|} = o(1)$$

and, by Markov's inequality,

$$\Pr(X > 0) \leqslant \mathrm{E}(X) = o(1).$$

In the forthcoming proofs we shall utilize the following estimate for the binomial distribution $B(n,p)$ (see, for instance, Bollobás [1], Corollary I.4ii).

Chernoff's inequality. *Let X be a random variable with the binomial distribution $B(n,p)$. If n, p and ϵ are such that* $\max\{100p, 100/\sqrt{np}\} < \epsilon < 1/6$ *then*

$$\Pr(|X - pn| \geqslant \epsilon pn) \leqslant \exp(-\epsilon^2 pn/4). \qquad \square$$

(2) To see property (2) it is enough to notice that Chernoff's and Markov's inequalities imply that the probability that there exists a vertex of $\mathscr{G}(n,n;p)$, the degree of which differs by more than $n^{1/3}$ from its expected value $(n-1)p = m + o(1)$, is bounded from above by

$$n \exp\left(-n^{1/3}/5 \log^{1/3} n\right) \to 0.$$

(3) Let X be the random variable which counts subgraphs H of $\mathscr{G}(n,n;p)$ such that, with $v_1 = v_1(H)$ and $v_2 = v_2(H)$, $\min(v_1, v_2) \leqslant n^{2/5}$ and $|E(H)| \geqslant 3(v_1 + v_2)$.

Then, using the well known inequality

$$\binom{n}{k} < \left(\frac{en}{k}\right)^k,$$

we obtain

$$\begin{aligned}
\mathrm{E}X &\leqslant 2\sum_{v_1=1}^{\lfloor n^{2/5}\rfloor}\sum_{v_2=v_1}^{n}\binom{n}{v_1}\binom{n}{v_2}\binom{v_1v_2}{3(v_1+v_2)}p^{3(v_1+v_2)}\\
&\leqslant 2\sum_{v_1=1}^{\lfloor n^{2/5}\rfloor}\sum_{v_2=v_1}^{n}\left(\frac{en}{v_1}\right)^{v_1+v_2}\left(\frac{ev_1v_2p}{3(v_1+v_2)}\right)^{3(v_1+v_2)}\\
&\leqslant 2\sum_{v_1=1}^{\lfloor n^{2/5}\rfloor}\sum_{v_2=v_1}^{n}\left(\frac{en}{v_1}(v_1p)^3\right)^{(v_1+v_2)}\\
&\leqslant 2\sum_{v_1=1}^{\lfloor n^{2/5}\rfloor}\sum_{v_2=v_1}^{n}\left(en^{-1/5}\log n\right)^{(v_1+v_2)} \to 0,
\end{aligned}$$

and property (3) follows.

(4) Both expanding properties (4a) and (4b) can be derived from the same fact which can be formulated as follows.

Let S be a subset of vertices of $\mathscr{G}(n,n;p)$ contained in one of the sets of the bipartition with $|S| = s \leqslant n^{7/18}$, and let M_S denote the set of all vertices of the graph $\mathscr{G}(n,n;p)$ which have at least two neighbours in S. The probability that, for some S, $|M_S| \geqslant sn^{1/4}$ is bounded from above by

$$\begin{aligned}
\sum_{s=1}^{\lfloor n^{7/18}\rfloor} 2\binom{n}{s}\sum_{\ell=\lfloor sn^{1/4}\rfloor}^{n}\binom{n}{\ell}(s^2p^2)^{\ell} &\leqslant \sum_{s=1}^{\lfloor n^{7/18}\rfloor} 2\left(\frac{en}{s}\right)^{s}\sum_{\ell=\lfloor sn^{1/4}\rfloor}^{n}\left(\frac{en}{\ell}s^2p^2\right)^{\ell}\\
&\leqslant \sum_{s=1}^{\lfloor n^{7/18}\rfloor} 2\left(\frac{en}{s}\right)^{s}\sum_{\ell=\lfloor sn^{1/4}\rfloor}^{n} n^{-\ell/6} \to 0.
\end{aligned}$$

Thus, almost surely, for all such S we have $|M_S| \leqslant sn^{1/4}$, and, because of (3), the number of edges between S and M_S is bounded from above by $4sn^{1/4}$. This and (2) yield (4a) and (4b).

Now, for a set $S \subseteq V_1$, $|S| = \lfloor n^{2/3}\rfloor$, let U_S denote the set of all vertices of V_2 which have in S less than $0.999\log n^{1/3}$ neighbours. The probability that a given vertex v has less than $0.999\log n^{1/3}$ neighbours is, by Chernoff's inequality, smaller than $\exp(-10^{-7}\log n^{1/3})$. Thus, the probability that $|U_S| \geqslant n/\log\log n$ for some S, is bounded from above by

$$\binom{n}{\lfloor n^{2/3}\rfloor}2^n\exp(-10^{-7}n\log n^{1/3}/\log\log n) \to 0.$$

Hence, almost surely, for all $S \subseteq V_1$, $|S| = \lfloor n^{2/3}\rfloor$, all except at most $n/\log\log n$ vertices of V_2 have each at least $0.999\log n^{1/3}$ neighbours in S. Now, consider the complement N^c of the set N. It sends at least $(|N^c| - n/\log\log n)\,0.999\log n^{1/3}$ edges to S, which, by property (2) and the assumption that $\deg_H(v) \geqslant 0.9m$ for every $v \in S$, cannot be more than $|S|(.1)m(1+o(1))$. This yields that $|N| > 0.8n$ and (4c) is proved.

The proof of property (4d) is truly deterministic, as it can be derived directly from (4a) and (4c). If $8m \leqslant |S| \leqslant n^{7/18}$ then we deduce from (4a) that

$$|N_G(S)| \geqslant 0.99|S|m \geqslant |S| + 7m^2.$$

For $n^{7/18} \leqslant |S| \leqslant n^{13/19}$ the inequality (4d) follows from the fact that the neighbourhood of any subset of S with $\lfloor n^{7/18} \rfloor$ is, again due to (4a), larger than $n^{13/18}$.

Similarly, for $n^{2/3} \geqslant |S| \leqslant 3n/5$ property (4c) ensures that $|N_G(S)| \geqslant 4n/5$ and so (4d) holds.

If $|S| \geqslant 3n/5$ then note that (4d) holds if it holds for the set T of all vertices not adjacent to S which belong to the other set of the bipartition. Since $|S| \geqslant 3n/5$, and thus $|N_G(S)| \geqslant 4n/5$, we have $|T| \leqslant n/5$ and the previous argument applies.

(5) To prove property (5), note first that for two disjoint subsets $S', S'' \subseteq V_1$, where $|S'| = |S''| = t = \lfloor 2m \rfloor$, the probability that a given vertex $u \in V_2$ have neighbours in both S' and S'' is at least

$$\left[1-(1-p)^t\right]^2 \geqslant \left[tp - \binom{t}{2} p^2\right]^2 \geqslant t^2p^2(1-tp).$$

Thus, the expected number of pairs of such subsets which have no common neighbours is, for n large enough, bounded from above by

$$\begin{aligned}\binom{n}{t}^2 \left(1 - t^2p^2(1-tp)\right)^{n-2t} &\leqslant \left(\frac{en}{t}\right)^{2t} \exp\left(-t^2p^2n(1+o(1))\right)\\ &\leqslant n^{4t/3} \exp\left(-2t\log n\right) \to 0.\end{aligned}$$

(6) We first show that, almost surely, for every subgraph H of $\mathscr{G}(n,n;p)$ with $\min\{v_1(H), v_2(H)\} \geqslant n^{3/4}$,

$$||E(H)| - v_1(H)v_2(H)p| \leqslant n^{-1/40}v_1(H)v_2(H)p. \tag{4.1}$$

Indeed, let Y be the number of subgraphs H such that $v_1 = v_1(H)$, $v_2 = v_2(H)$, $v_1, v_2 \geqslant n^{3/4}$, and for which inequality (4.1) does not hold. Then, using Chernoff's bound, one can estimate the expectation of Y by

$$\begin{aligned}\mathrm{E}Y &\leqslant \sum_{v_1=\lceil n^{3/4}\rceil}^{n} \sum_{v_2=\lceil n^{3/4}\rceil}^{n} \binom{n}{v_1}\binom{n}{v_2} \exp\left(-v_1v_2pn^{-1/20}/4\right)\\ &\leqslant \sum_{v_1=\lceil n^{3/4}\rceil}^{n} \sum_{v_2=\lceil n^{3/4}\rceil}^{n} n^{2\max\{v_1,v_2\}} \exp\left(-\max\{v_1,v_2\}n^{1/60}/4\right) \to 0.\end{aligned}$$

Thus, if there is a pair of subgraphs H', H'' which violate (6), then we may assume that both H' and H'' have ℓ edges, for some $\ell > pk^2/2$, where $k = \lceil n^{3/4} \rceil$.

Now, let X be the random variable which counts the pairs of edge-disjoint, induced and isomorphic subgraphs H', H'' of $\mathscr{G}(n,n;p)$ for which

$$v_1(H') = v_2(H') = v_1(H'') = v_2(H'') = \lceil n^{3/4} \rceil$$

and $\ell > pk^2/2$.

To estimate the expectation of X, we denote by $\mathscr{H}_k$ the family of all pairwise nonisomorphic bipartite graphs with $k+k$ vertices and ℓ edges, where $\ell > pk^2/2$.

For a given graph $H \in \mathscr{H}_k$ and for two pairs of sets of size k, (S_1, T_1) and (S_2, T_2), such that $S_1, S_2 \subset V_1$ and $T_1, T_2 \subset V_1$, let $\pi(H, S_1, T_1)$ be the probability that the induced subgraph $\mathscr{G}(n, n; p)$ $[S_1 \cup T_1]$ is isomorphic to H, and let $\pi(H, S_1, T_1, S_2, T_2)$ be the conditional probability that $\mathscr{G}(n, n; p)$ $[S_2 \cup T_2]$ contains a copy of H, edge-disjoint from $\mathscr{G}(n, n; p)$ $[S_1 \cup T_1]$, given that $\mathscr{G}(n, n; p)$ $[S_1 \cup T_1]$ is isomorphic to H.

Then $\pi(H, S_1, T_1, S_2, T_2) \leqslant (2k)!p^{pk^2/2}$ and $\sum_{H \in \mathscr{H}_k} \pi(H, S_1, T_1) = 1 - o(1)$. Hence,

$$\mathrm{E}X \leqslant \sum_{S_1, S_2, T_1, T_2} \sum_{H \in \mathscr{H}_k} \pi(H, S_1, T_1, S_2, T_2)\pi(H, S_1, T_1) \leqslant n^{4k}(k!)^{-4}(2k)!p^{pk^2/2} \to 0,$$

and the proof of Lemma 4.1 is completed. □

We conclude the paper with a probabilistic proof of Lemma 3.2.

Lemma 3.2. *Let G be a bipartite graph with bipartition (V_1, V_2), such that, for some natural numbers $d, k \geqslant 2$, the following holds.*

(i) $|V_1| \geqslant 2^{4d+3}k$;

(ii) $\deg_G(v) \leqslant k/8d$ *for every vertex v of V_2;*

(iii) $|E(G)| \leqslant d|V_1|$.

Then there are two disjoint subsets W', W'' of V_1, of k elements each, such that no vertex u of V_2 has neighbours in both W' and W''.

Proof. Let G, d and k fulfil the assumptions of Lemma 3.2, and let

$$V_1' = \{v \in V_1 : \deg_G(v) \leqslant 2d\}.$$

Note that it follows from (i) and (iii) that $|V_1'| \geqslant |V_1|/2 \geqslant 2^{4d+2}k$.

Now, we shall show that for $t = 2^{4d}k$, there exist in V_1' some $2t$ vertices $x_1, \ldots, x_t$, $y_1, \ldots, y_t$ such that for each $i = 1, \ldots, t$ the set of common neighbours of x_i and y_i, $N_G(x_i) \cap N_G(y_i)$, is empty. Indeed, for $x \in V_1'$, the set $N_2(x)$ of the vertices at distance 2 from x satisfies the inequality

$$|N_2(x) \cup \{x\}| \leqslant 2d\frac{k}{8d} = \frac{k}{4}.$$

Thus the set $V_1' \setminus (N_2(x) \cup \{x\} \cup \{x_1, \ldots, x_i, y_1, \ldots, y_i\})$ is not empty as long as $i < t$ and one can find the next pair (x_{i+1}, y_{i+1}).

Now comes the probabilistic part of the proof. Let us choose a random subset $\mathscr{U}$ of V_2 in such a way that each vertex u of V_2 belongs to $\mathscr{U}$ independently, with probability $1/2$.

Let X denote the number of those pairs of vertices (x_i, y_i), $i = 1, \ldots, t$, for which the neighbours of x_i are in $\mathscr{U}$ and all the neighbours of y_i are not.

Note that X, in a natural way, can be represented as a sum of indicator random variables, *i.e.* $X = \sum_{i=1}^{t} X_i$, where

$$X_i = \begin{cases} 1 \text{ if } & N_G(x_i) \subseteq \mathscr{U} \quad \text{and} \quad N_G(y_i) \cap \mathscr{U} = \emptyset \\ 0 \text{ otherwise.} \end{cases}$$

Since, for each i, $N_G(x_i) \cap N_G(y_i) = \emptyset$ and $x_i, y_i \in V_1'$, we have

$$\mathrm{E}X_i = \Pr(X_i = 1) = 2^{-\deg_G(x_i) - \deg_G(y_i)} \geqslant 2^{-4d}$$

and by the linearity of expectation,

$$\mathrm{E}X = \sum_{i=1}^{t} \mathrm{E}X_i \geqslant 2^{-4d} t = k.$$

Thus, there exists an instance of $\mathcal{U}$, $U \subseteq V_2$, for which $X(U) \geqslant k$. This means that there are k disjoint pairs of vertices in V_1, $(x_{i_1}, y_{i_1}), \ldots, (x_{i_k}, y_{i_k})$ such that the set $W' = \{x_{i_1}, \ldots, x_{i_k}\}$ has all its neighbours inside U, while the set $W'' = \{y_{i_1}, \ldots, y_{i_k}\}$ has all its neighbours outside U. This is the required pair of sets and the proof of Lemma 3.2 is completed. □

References

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