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The Complexity of the Fixed Point Property

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Abstract. It is NP-complete to determine whether a given ordered set has a fixed point free order-preserving self-map. On the way to this result, we establish the NP-completeness of a related problem: Given ordered sets P and Q with t-tuples $(p_1, \ldots, p_t)$ and $(q_1, \ldots, q_t)$ from P and Q respectively, is there an order-preserving map $f\colon P \to Q$ satisfying $f(p_i) \geqslant q_i$ for each $i = 1, \ldots, t$?

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1. Introduction

A (partially) ordered set X has the *fixed point property* if for every order-preserving map $f\colon X \to X$ there is an element $x \in X$ such that $f(x) = x$. The fixed point property, in the context of ordered sets, has widest circulation in the theorems of Knaster [8], Tarski [13], and Davis [3]: in the class of lattices, the lattices that are complete are precisely those with the fixed point property.

For finite structures, the characterization problem appears first to have been raised in Crawley and Dilworth [2]. It has attracted a good deal of attention and was included in the *Order* problem list [12]: characterize those finite ordered sets with the fixed point property. While there are few results, those that exist provide good characterizations within special classes. For ordered sets of length one, Rival showed that an ordered set has the fixed point property if and only if its covering graph is a tree [11]. In the class of width-two orders, Fofanova and Rutkowski show that the fixed point property is equivalent to the absence of a tower of four-element crowns as a retract [6]. The same condition is equivalent to the fixed point property for N-free dimension-two ordered sets [5]. Finally, consider those ordered sets which are obtained from semimodular lattices by removing the maximum and minimum elements. A member of this class has the fixed point property if and only if the original lattice is noncomplemented [1].

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In all of these special cases the fixed point property is equivalent to dismantlability (see [12], for instance, for a discussion of dismantlability). With the aid of a result from [4], it is easily seen that whether or not a finite ordered set is dismantlable can be determined in polynomial time. Indeed, determining the complexity of the related decision problem is an obvious step in attempts to characterize the fixed point property. Here is a statement of the problem which places it in the class NP. Call an ordered set *fixed point free* if it has an order-preserving map without fixed points, that is, a *fixed point free map*.

Fixed Point Free Map (FPFMAP)
Instance: An ordered set P.
Question: Is there is an order-preserving map $f\colon P \to P$ such that $f(x) \neq x$ for all $x \in P$?

The main purpose of this paper is to prove

THEOREM 1.1. FPFMAP *is* NP-*complete.*

In Section 2, we discuss the literature related to this decision problem and provide terminology and some preliminary lemmas. We note that the NP-completeness of the corresponding decision problem for automorphisms of ordered sets follows directly from a well-known result for graphs. Section 3 contains the construction of ordered sets with 'well-behaved' fixed point free order-preserving maps. In Section 5, this construction, together with the main steps in transforming 3SAT to a decision problem called OPEXT (given in Section 4), provides the basis of our proof of Theorem 1.1.

2. Preliminaries and Background

The only results in the complexity literature directly related to FPFMAP were obtained by Lubiw [10] in her study of the complexity of graph isomorphism. She established the NP-completeness of the following decision problem. (An *automorphism* of a graph is a bijection on the set of vertices preserving adjacency and nonadjacency.)

Fixed Point Free Automorphism
Instance: A graph G.
Question: Does G have a fixed point free automorphism?

In fact, the complexity of the analogous ordered set decision problem can be obtained as a corollary. (As is usual, an *automorphism* of an ordered set is an order-preserving bijection on the underlying set whose inverse is also order-preserving.)

Ordered Set Fixed Point Free Automorphism (FPFAUT)
Instance: An ordered set P.
Question: Does P have a fixed point free automorphism?

One can amend Lubiw's transformation from 3SAT to FIXED POINT FREE AUTOMORPHISM to obtain the NP-completeness of FPFAUT (as is done in [14]). Alternatively, one can employ a variant of a standard ordered set construction – this is outlined here.

The *split* of a graph $G = (V, E)$ is the two-level ordered set with maximals V, minimals E, and comparabilities $v > e$ if $v \in e$. The 2-*split* of G has maximals V, minimals $(E \times \{0\}) \cup (E \times \{1\})$ and comparabilities $v > (e, 0)$ and $v > (e, 1)$ if $v \in e$. (There is a technical difficulty if G has isolated vertices, as these are both maximal and minimal elements of the split and the 2-split. In this case, replace 'minimal' by 'nonmaximal' in the sequel.)

THEOREM 2.1. FPFAUT *is* NP-*complete.*

This theorem is an immediate consequence of the following lemma.

LEMMA 2.2. *Let $G = (V, E)$ be a graph and let P denote the 2-split of G. Then G has a fixed point free automorphism if and only if P has a fixed point free automorphism.*

Proof. Let φ be an automorphism of G and let $\hat{\varphi}$ denote the map on P defined by $\hat{\varphi}|_V = \varphi$ and for $e = \{u, v\} \in E$,

$$\hat{\varphi}(e, 0) = (\{\varphi(u), \varphi(v)\}, 1), \qquad \hat{\varphi}(e, 1) = (\{\varphi(u), \varphi(v)\}, 0).$$

It is obvious that $\hat{\varphi}$ is an automorphism of P and that if φ is fixed point free then $\hat{\varphi}$ is fixed point free.

The converse is also straightforward. □

It is worth noting that FPFAUT and FPFMAP have different complexity for the class of length one ordered sets, provided that P $\neq$ NP. FPFMAP, restricted to length one instances, is in the complexity class P, as already noted. Since the split of a graph is a length one ordered set, FPFAUT is NP-complete even when restricted to length one instances.

We conclude this section with two elementary observations, concerning fixed point free maps, that will be used repeatedly in the following sections.

OBSERVATION 1. Let X be a finite ordered set. If X has a fixed point free map then there is a fixed point free map f on X satisfying

$$f[\min(X)] \subseteq \min(X) \text{ and } f[\max(X)] \subseteq \max(X)$$

(where $\min(X)$ denotes the minimals of X and $\max(X)$ denotes the maximals of X). Say g is fixed point free on X. Define f by letting it be g off $\max(X) \cup$

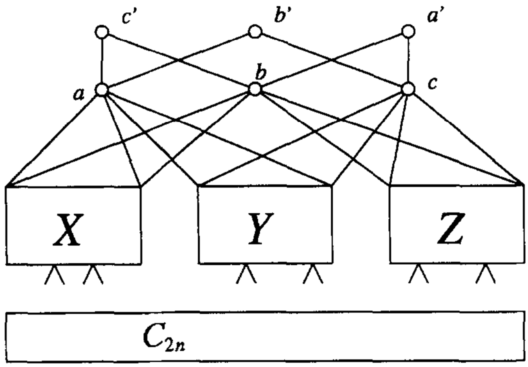

Figure 1. General construction.

min(X) and by using these rules: for $x \in \max(X)$, let $f(x)$ be any maximal above $g(x)$; for $x \in \min(X)$, let $f(x)$ be any minimal below $g(x)$. It is obvious that f is order-preserving. Since g is fixed point free, $g(y)$ is incomparable to y for all $y \in X$ (as is the case for any fixed point free map on a finite ordered set). Thus, f is fixed point free.

OBSERVATION 2. For a positive integer n, let C_{2n} denote the crown on $2n$ elements, that is; C_{2n} is the 2-level ordered set whose covering graph is a $2n$-cycle. The only fixed point free order-preserving maps on C_{2n} are automorphisms. This is well known; see, for instance, [11].

3. A General Construction

Let R be defined on the following sets, with the comparabilities as indicated (see Figure 1).

$$R = C_6 \cup X \cup Y \cup Z \cup C_{2n},$$

where C_6 is the 6-crown on a, b, c, a', b', c' with comparabilities

$$a < b', c', \qquad b < a', c' \quad \text{and} \quad c < a', b'$$

and X, Y and Z are arbitrary nonempty ordered sets, all of whose elements dominate at least 2 maximal elements of the $2n$-crown C_{2n} ($n \geqslant 3$), and which satisfy

$$X < \{a, b\}, \qquad Y < \{a, c\} \quad \text{and} \quad Z < \{b, c\}.$$

Finally, each element in C_{2n} is less than a, b, and c.

CLAIM. *If R has a fixed point free order-preserving map then R has a fixed point free order-preserving map f such that*

(i) $f|_{C_6}$ *and* $f|_{C_{2n}}$ *are both automorphisms, and*
(ii) $f(a) = b,\ f(b) = c,\ f(c) = a$, *and* $f[X] \subseteq Z,\ f[Y] \subseteq X,\ f[Z] \subseteq Y$,
or
$f(a) = c,\ f(b) = a,\ f(c) = b$, *and* $f[X] \subseteq Y,\ f[Y] \subseteq Z,\ f[Z] \subseteq X$.

By Observation 1, we may assume that $f[\{a', b', c'\}] \subseteq \{a', b', c'\}$. Suppose that the image has size less than 3, say $f(a') = b',\ f(b') = c'$, and $f(c') = b'$. Since $f(a) < b', c'$ and $f(a) \not\leqslant a$, $f(a) \in Z$. Also, $f(c) < b', c'$, and $f(c) \not\leqslant c$, so $f(c) = a$ or $f(c) \in X$. We may assume that $f(c) = a$. Now choose $y \in Y$ (thus $y < a$ and $y < c$, but $y \not\leqslant b$). Then $y > u, u'$ for distinct maximal elements $u, u' \in C_{2n}$. Also, $f(y) \leqslant f(a), f(c)$, so $f(y) \leqslant b, c, a$; thus $f(y) \in C_{2n}$.

Also note that for any $u \in C_{2n}$, $u < a, b, c$, so $f(u) < b, c, a$, meaning that $f[C_{2n}] \subseteq C_{2n}$. By Observation 2, $f|_{C_{2n}}$ is an automorphism; but then $f(y)$ cannnot belong to C_{2n} and dominate the distinct maximal elements $f(u)$ and $f(u')$. This contradiction implies that $f[\{a', b', c'\}] = \{a', b', c'\}$.

So without loss, we may assume that $f(a') = b',\ f(b') = c',\ f(c') = a'$. Then $f(c) < b', c'$ and $f(c) \not\leqslant c$, so $f(c) \notin Y \cup Z \cup C_{2n}$; that is, $f(c) = a$ or $f(c) \in X$. In any case, we may assume that $f(c) = a$. (Just redefine f; it will remain order-preserving and fixed point free.) Similarly, we may assume $f(a) = b$ and $f(b) = c$. It follows that $f[C_{2n}] \subseteq C_{2n}$, so, again using Observation 2, $f|_{C_{2n}}$ is an automorphism. Now let $x \in X$. Then $x > u, u'$ for distinct maximal elements u and u' of C_{2n}. It follows that $f(x)$ dominates distinct maximals of C_{2n}, so $f(x) \notin C_{2n}$. Since $x < a, b$, the image $f(x) < b, c$, so $f(x)$ is an element of Z. Similarly, $f[Y] \subseteq X$ and $f[Z] \subseteq Y$, completing the verification of the claim.

4. Order-Preserving Extension (OPEXT)

While trying to determine the complexity status of FPFMAP, we were led to consider the following decision problem.

Order-Preserving Extension (OPEXT)
Instance: Ordered sets P and Q with t-tuples $(p_1, \ldots, p_t)$ and $(q_1, \ldots, q_t)$ from P and Q respectively.
Question: Is there is an order-preserving map $f\colon P \to Q$ satisfying $f(p_i) \geqslant q_i$ for each $i = 1, \ldots, t$?

This decision problem may equivalently be thought of as being an instance of ordered sets P and Q with a partial map defined by the conditions $f(p_1) \geqslant q_1, \ldots, f(p_t) \geqslant q_t$. The question then becomes: is there an order-preserving map $f\colon P \to Q$ extending the given partial map?

THEOREM 4.1. OPEXT *is* NP-*complete.*

Proof. The decision problem OPEXT $\in$ NP because verification that, for a given instance, the answer to OPEXT is 'yes' is provided by an order-preserving map extending the given partial map. Such a map is obviously of size polynomial in the size of the instance.

To show that OPEXT is NP-complete, we exhibit a transformation from 3SAT. Let $C = \{C_1, \ldots, C_m\}$ be a set of clauses over the variables $x_1, \ldots, x_n$. For notational purposes in the construction that follows, we place an ordering on the terms of each clause; that is, $C_i = \{C_i(1), C_i(2), C_i(3)\}$. Construct graphs G_P and G_Q with vertex and edge sets as follows ($i = 1, \ldots, n$, $j = 1, \ldots, m$):

$$\begin{aligned}
V(G_P) &= \left\{x_i^a, x_i^*, x_i^b\right\} \cup \left\{C_j^0, C_j^*\right\} \\
E(G_P) &= \left\{\left\{x_i^a, x_i^*\right\}, \left\{x_i^*, x_i^b\right\}\right\} \cup \left\{\left\{C_j^0, C_j^*\right\}\right\} \\
&\quad \cup \left\{\left\{C_j^*, x_i^*\right\}\colon\ x_i \in C_j \text{ or } \bar{x}_i \in C_j\right\} \\
V(G_Q) &= \left\{x_i^a, x_i^T, x_i^F, x_i^b\right\} \cup \left\{C_j^0, C_j^1, C_j^2, C_j^3\right\} \\
E(G_Q) &= \left\{\left\{x_i^a, x_i^T\right\}, \left\{x_i^a, x_i^F\right\}, \left\{x_i^T, x_i^b\right\}, \left\{x_i^F, x_i^b\right\}\right\} \\
&\quad \cup \left\{\left\{C_j^0, C_j^k\right\}\colon\ k = 1, 2, 3\right\} \\
&\quad \cup \left\{\left\{C_j^k, x_i^T\right\}\colon\ (C_j(k) = x_i) \text{ and } k = 1, 2, 3\right\} \\
&\quad \cup \left\{\left\{C_j^k, x_i^F\right\}\colon\ (C_j(k) = \bar{x}_i) \text{ and } k = 1, 2, 3\right\} \\
&\quad \cup \left\{\left\{C_j^k, x_i^F\right\}, \left\{C_j^k, x_i^T\right\}\colon\ (x_i \in C_j \text{ or } \bar{x}_i \in C_j) \text{ and}\right. \\
&\qquad \left.(C_j(k) \neq x_i, \bar{x}_i) \text{ and } k = 1, 2, 3\right\}.
\end{aligned}$$

(Figure 2 shows the construction with $C_1 = \{x_1, \bar{x}_2, x_3\}$.)

Let P and Q be the splits of G_P and G_Q respectively. An instance of OPEXT is then given by the ordered sets P and Q and the partial map $f(x_i^a) = x_i^a, f(x_i^b) = x_i^b$ ($i = 1, \ldots, n$), and $f(C_j^0) = C_j^0$ ($j = 1, \ldots, m$). Notice that this partial map is equivalent to the conditions $f(x_i^a) \geqslant x_i^a$, $f(x_i^b) \geqslant x_i^b$ ($i = 1, \ldots, n$), and $f(C_j^0) \geqslant C_j^0$ ($j = 1, \ldots, m$) because the specified elements are maximals of P and Q.

Suppose that $f\colon P \to Q$ is an order-preserving map extending the described partial map. Then f must satisfy $f(x_i^*) \in \{x_i^T, x_i^F\}$; thus, f gives rise to a truth assignment in the natural way. As well, $f(C_j^*) \in \{C_j^1, C_j^2, C_j^3\}$. Let $j \in \{1, \ldots, m\}$ and assume $f(C_j^*) = C_j^k$. Let us see that the clause C_j is satisfied. For some $i = 1, \ldots, n$, $C_j(k) \in \{x_i, \bar{x}_i\}$; for illustration, say $C_j(k) = \bar{x}_i$ (the case $C_j(k) = x_i$ is handled similarly). Because x_i^* and C_j^* have a common lower bound in P, $f(x_i^*)$ and $f(C_j^*)$ must have a common lower bound in Q; but then we must have $f(x_i^*) = x_i^F$ as x_i^T and C_j^k do not have a common lower bound

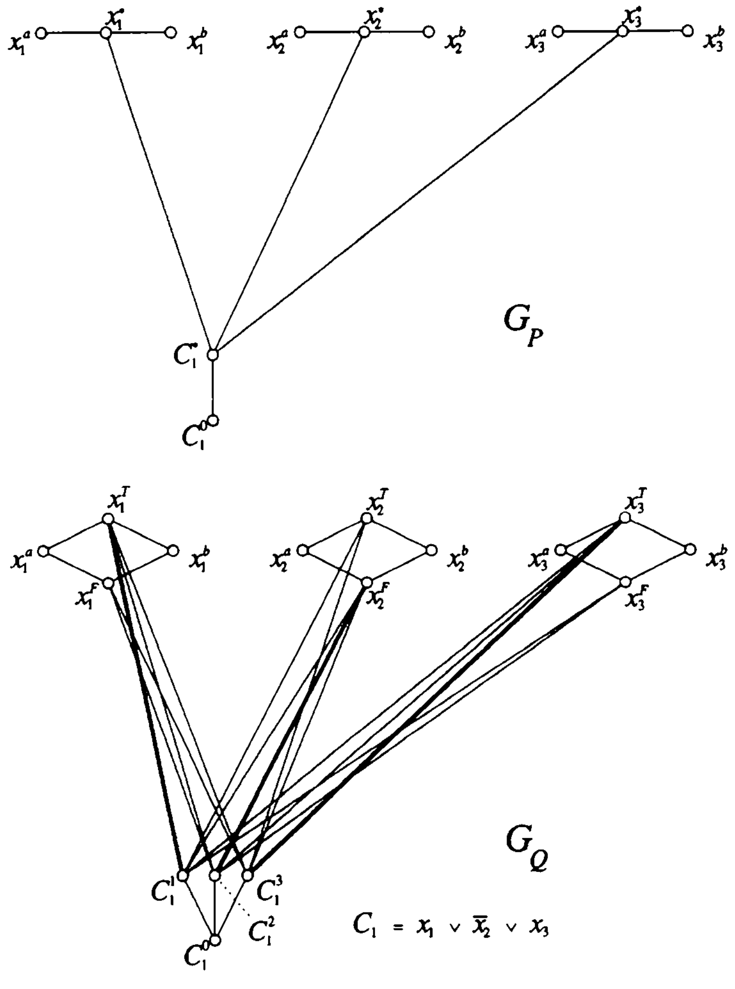

Figure 2. Detail of G_P and G_Q with $C_1 = \{x_1, \bar{x}_2, x_3\}$.

in Q. Hence the truth assignment induced by f satisfies C_j; consequently, the entire set of clauses, C, must be satisfied.

Conversely, given a satisfying assignment φ: $\{x_1, \ldots, x_n\} \to \{T, F\}$, set $f(x_i^*) = x_i^{\varphi(x_i)}$. For each $j = 1, \ldots, m$, some term $C_j(k)$ is satisfied by φ; set $f(C_j^*) = C_j^k$. The construction of the edge set of G_Q then easily allows f to be extended to the entire domain P. □

We are also interested in maps f: $Q \to P$ obeying the same restrictions $f(x_i^a) = x_i^a$, $f(x_i^b) = x_i^b$ $(i = 1, \ldots, n)$, and $f(C_j^0) = C_j^0$ $(j = 1, \ldots, m)$. In this case, there is no difficulty mapping Q into P; just set $f(C_j^k) = C_j^*$ $(k = 1, 2, 3$ and

$j = 1, \ldots, m$) and $f(x_i^T) = f(x_i^F) = x_i^*$ ($i = 1, \ldots, n$). Such a map can easily be extended to the minimals of Q and always exists regardless of whether C has a satisfying assignment or not. Moreover, this is the only map from Q to P satisfying the given restrictions.

5. FPFMAP Is NP-Complete

The two preceding sections are now brought together to provide a proof of our main theorem.

THEOREM. FPFMAP *is* NP-*complete.*

Proof. We begin by constructing a special case of the ordered set defined in Section 3. Let X, Y, and Z be ordered sets containing the subsets $\{x_1, \ldots, x_t\}$, $\{y_1, \ldots, y_t\}$, and $\{z_1, \ldots, z_t\}$ respectively. Let

$$R = C_6 \cup X \cup Y \cup Z \cup C_{6t+18}.$$

Let the 6-crown C_6 with maximals $\{a', b', c'\}$, minimals $\{a, b, c\}$, and comparabilities $a < b', c'$, $b < a', c'$, and $c < a', b'$, satisfy $\{a, b\} > X$, $\{a, c\} > Y$, and $\{b, c\} > Z$. Cyclically enumerate the maximals of the $(6t + 18)$-crown by

$$\{m_1, x_0, x_0', \hat{x}_1, \ldots, \hat{x}_t, m_2, y_0, y_0', \hat{y}_1, \ldots, \hat{y}_t, m_3, z_0, z_0', \hat{z}_1, \ldots, \hat{z}_t\}.$$

Include exactly the comparabilities $\{a, b, c\} > C_{6t+18}$, $X > \{x_0, x_0'\}$, $Y > \{y_0, y_0'\}$, $Z > \{z_0, z_0'\}$, and, for each $i = 1, \ldots t$, $x_i > \hat{x}_i$, $y_i > \hat{y}_i$, and $z_i > \hat{z}_i$ (and those following by transitivity).

Note that R satisfies the conditions of Section 3. Thus, if R is fixed point free, there is an order-preserving map F: $R \to R$ so that F permutes the subsets X, Y, and Z in a three-cycle, say, $F[X] \subseteq Y$, $F[Y] \subseteq Z$, and $F[Z] \subseteq X$.

Since $F[X] \subseteq Y$, $F[Y] \subseteq Z$, $F[Z] \subseteq X$, and $F|_{C_{6t+18}}$ is an automorphism, we must have

$$\begin{aligned} F[\{x_0, x_0', \hat{x}_1, \ldots, \hat{x}_t\}] &\subseteq \{y_0, y_0', \hat{y}_1, \ldots, \hat{y}_t\}, \\ F[\{y_0, y_0', \hat{y}_1, \ldots, \hat{y}_t\}] &\subseteq \{z_0, z_0', \hat{z}_1, \ldots, \hat{z}_t\} \quad \text{and} \\ F[\{z_0, z_0', \hat{z}_1, \ldots, \hat{z}_t\}] &\subseteq \{x_0, x_0', \hat{x}_1, \ldots, \hat{x}_t\}. \end{aligned}$$

Indeed, because $F|_{C_{6t+18}}$ is an automorphism, it must be the case that for each $i = 1, \ldots t$, $F(\hat{x}_i) = \hat{y}_i$, $F(\hat{y}_i) = \hat{z}_i$, and $F(\hat{z}_i) = \hat{x}_i$. Then, for each $i = 1, \ldots t$, $F(x_i) \geqslant F(\hat{x}_i)$, $F(y_i) \geqslant F(\hat{y}_i)$, and $F(z_i) \geqslant F(\hat{z}_i)$. Thus, by the construction of R, $F(x_i) \geqslant y_i$, $F(y_i) \geqslant z_i$, and $F(z_i) \geqslant x_i$ (for each $i = 1, \ldots t$).

Let us use the above as the foundation for a transformation of 3SAT to FPFMAP. Given an instance of 3SAT, $C = \{C_1, \ldots, C_m\}$, we construct R, an instance of FPFMAP, as follows. Let P and Q be the ordered sets obtained in Theorem 4.1. Set $X = P$, $Y = P$, and $Z = Q$. Fix a listing of $x_1^a, x_1^b, \ldots, x_n^a, x_n^b$, $C_1^0, \ldots, C_m^0$ in P and Q and let $x_1, \ldots, x_t$, $y_1, \ldots, y_t$, and $z_1, \ldots, z_t$ be relabellings of this list in X, Y, and Z respectively. The construction is completed

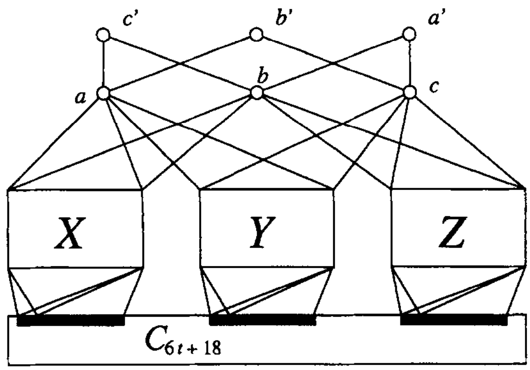

Figure 3. Specialized construction to house OPEXT.

by including the comparabilities with $x_0, x_0', \hat{x}_1, \ldots, \hat{x}_t$, $y_0, y_0', \hat{y}_1, \ldots, \hat{y}_t$, and $z_0, z_0', \hat{z}_1, \ldots, \hat{z}_t$ as described above.

Suppose that $C = \{C_1, \ldots, C_m\}$ has a satisfying assignment. As in the proof of Theorem 4.1 there is an order-preserving map f: $Y \to Z$ such that $f(y_i) = z_i$ $(i = 1, \ldots, t)$. Let g: $X \to Y$ be the identity and h: $Z \to X$ be the unique map satisfying $h(z_i) = x_i$ $(i = 1, \ldots, t)$ (as described after the proof of Theorem 4.1). We can then easily extend f, g, and h to a fixed point free map F: $R \to R$.

Conversely, if R has a fixed point free map F, then either $F[X] \subseteq Z$ or $F[Y] \subseteq Z$. In either case, we obtain (by restriction) an order-preserving map f: $P \to Q$ satisfying $f(x_i^a) = x_i^a$ and $f(x_i^b) = x_i^b$ for each $i = 1, \ldots, n$ and $f(C_i^0) = C_i^0$ for each $i = 1, \ldots, m$. This is precisely the instance of OPEXT constructed in Theorem 4.1, so C has a satisfying assignment.

We have shown that R has a fixed point free map if and only if C has a satisfying assignment. The construction of R is clearly polynomial in m and n, so it follows that FPFMAP is NP-complete. □

6. Related Results and Open Problems

We conclude with a few observations and questions.

As was stated in the introduction, it is possible to decide FPFMAP in polynomial time for ordered sets either with no three-element chain or with no three-element antichain. The construction used to show that FPFMAP is NP-complete (in general) produces ordered sets with six levels and arbitrarily large antichains. Is it NP-complete to decide FPFMAP for ordered sets with only three levels? What about ordered sets of width three, or simply of bounded width? What about the class of dimension-two orders (*N*-free or not)?

When one approaches a new decision problem, it is sometimes useful to seek a method of 'self-reduction', as all NP-complete problems possess one [9]. (It may not be a natural or immediate self-reduction algorithm as a detour through SAT may be required.) We are, to date, unable to solve this problem for FPFMAP, but the obvious self-reduction for OPEXT is one of the factors that gave us confidence in seeking a proof of its NP-completeness. We still wonder:

> Is there a natural self-reduction for FPFMAP; that is, is there a simple algorithm that can use an FPFMAP decision problem oracle to produce a fixed point free map in polynomial time?

Once a decision problem has been shown to be NP-complete, a next step is to check for #P-completeness (where #P is the class of counting problems associated with the decision problems of NP [9]). With a little amplification in Theorem 4.1 (expanding $\{C_j^k\colon\ k = 1,2,3\}$ to $\{C_j^k\colon\ k = 1,\dots,7\}$ in G_Q) it is possible to see that OPEXT, and consequently FPFMAP, are #P-complete. It turns out that R constructed in this way has exactly twice as many fixed point free maps as the Boolean expression has satisfying assignments.

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