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# Lattices arising in categorial investigations of Hedetniemi's conjecture

Dwight Duffus[a,1], Norbert Sauer[b,*,2]

[a] *Department of Mathematics and Computer Science, Emory University, Atlanta, Georgia 30322 USA*
[b] *Department of Mathematics and Statistics, University of Calgary, 2500 University Drive N.W., Calgary, Alberta, T2N 1N4, Canada*

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**Abstract**

We discuss Hedetniemi's conjecture in the context of categories of relational structures under homomorphisms. In this language Hedetniemi's conjecture says that if there are no homomorphisms from the graphs $G$ and $H$ to the complete graph on $n$ vertices then there is no homomorphism from $G \times H$ to the complete graph. If an object in some category has just this property then it is called multiplicative. The skeleton of a category of relational structures under homomorphisms forms a distributive lattice which has for each of the objects $K$ of the category a pseudocomplementation. The image of the distributive lattice under such a pseudocomplementation is a Boolean lattice with the same meet as the distributive lattice and the structure $K$ is multiplicative if and only if this Boolean lattice consists of at most two elements. We will exploit those general ideas to gain some understanding of the situation in the case of graphs and solve completely the Hedetniemi-type problem in the case of relational structures over a unary language.

*Keywords*: Hedetniemi conjecture; Product; Multiplicative structures in categories

## Section 0

Let $\mathscr{L}$ be a relational language. That is, $\mathscr{L}$ is a set of relation symbols together with the arities associated with those relation symbols. A *model* $A$ of $\mathscr{L}$ is a set $A$ together with a relation of the appropriate arity for each of the relation symbols in $\mathscr{L}$. If $A$ and $B$ are two models of $\mathscr{L}$ then the function $\alpha: A \to B$ is a *homomorphism* from $A$ to $B$ iff for each of the relations $R \in \mathscr{L}$ and each sequence of elements $a_1, a_2, \dots, a_n$ of $A$,

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$R(a_1, a_2, \ldots, a_n)$ implies $R(\alpha(a_1), \alpha(a_2), \ldots, \alpha(a_n))$. If $\alpha$ is a homomorphism from $A$ to $B$ we will write $A \xrightarrow{\alpha} B$ and express the fact that there is a homomorphism from $A$ to $B$ by writing $A \rightarrow B$. If there is no homomorphism from $A$ to $B$ we will write $A \not\rightarrow B$ and will denote by $\boldsymbol{M}_{\mathscr{L}}$ the category of $\mathscr{L}$-models under $\rightarrow$ as morphisms. The category $\boldsymbol{M}_{\mathscr{L}}$ is a category with *product* $\times$. For two $\mathscr{L}$-models $A$ and $B$, $A \times B$ is the $\mathscr{L}$-model defined on the cartesian product of the sets $A$ and $B$ such that $R((a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n))$ in $A \times B$ iff $R(a_1, a_2, \ldots, a_n)$ in $A$ and $R(b_1, b_2, \ldots, b_n)$ in $B$. The category $\boldsymbol{M}_{\mathscr{L}}$ has a coproduct, *sum* $+$, given by the 'disjoint union' of $\mathscr{L}$-models. That is if $A$ and $B$ are two $\mathscr{L}$-models then $A + B$ is the $\mathscr{L}$-model defined on the disjoint union of the sets $A$ and $B$ and if $R$ is an $n$-ary relation symbol of $\mathscr{L}$, then $R(x_1, x_2, \ldots, x_n)$ in $A + B$ if and only if all of the elements $x_1, x_2, \ldots, x_n$ are in $A$ or in $B$ and then either $R(x_1, x_2, \ldots, x_n)$ in $A$ or $R(x_1, x_2, \ldots, x_n)$ in $B$. The category $\boldsymbol{M}_{\mathscr{L}}$ is a category with *exponentiation*. If $A$ and $B$ are two $\mathscr{L}$-models then $A$ power $B$, $A^B$, is an $\mathscr{L}$-model defined on the set of all functions from the base set of $B$ to the base set of $A$. If $f_1, f_2, \ldots, f_n$ are functions from $B$ to $A$ and $R$ is an $n$-ary relation symbol of $\mathscr{L}$ then $R(f_1, f_2, \ldots, f_n)$ in $A^B$ if and only if for all sequences $a_1, a_2, \ldots, a_n$ of elements of $A$, $R(a_1, a_2, \ldots, a_n)$ implies $R(f_1(a_1), f_2(a_2), \ldots, f_n(a_n))$. For any relational language $\mathscr{L}$ we will denote by *one*, or $one_{\mathscr{L}}$, the $\mathscr{L}$ model which has as base set a single element set $a$ and for every relation symbol $R \in \mathscr{L}$, $R(a, a, \ldots, a)$.

Two $\mathscr{L}$-models $A$ and $B$ are *similar*, $A \sim B$, iff $A \rightarrow B$ and $B \rightarrow A$. Clearly $\sim$ is an equivalence relation. Note that if $A \rightarrow A_1$ and $B \rightarrow B_1$ then $A + B \rightarrow A_1 + B_1$ and $A \times B \rightarrow A_1 \times B_1$. Hence if $A \sim A_1$ and $B \sim B_1$ then $A + B \sim A_1 + B_1$ and $A \times B \sim A_1 \times B_1$. If $\boldsymbol{C}$ is a small subcategory of $\boldsymbol{M}_{\mathscr{L}}$ then $\boldsymbol{C}$ is called a *proper $\mathscr{L}$-category* if $\boldsymbol{C}$ is closed under $\times$, $+$, exponentiation and contains the $\mathscr{L}$-models *one* and the empty model. Let $\boldsymbol{C}$ be a proper $\mathscr{L}$-category and $K$ an element of $\boldsymbol{C}$, then $K^{\boldsymbol{C}}$ is the set of all elements of the form $K^A$ for some element $A$ of $\boldsymbol{C}$. Let **Sim** be the functor on $\boldsymbol{C}$ which associates with every $\mathscr{L}$-structure $A$ of $\boldsymbol{C}$ the equivalence class **Sim**$(A)$ which consists of all of the $\mathscr{L}$-structures $B$ of $\boldsymbol{C}$ such that $A \sim B$. Observe by straightforward verification that for any proper $\mathscr{L}$-category $\boldsymbol{C}$ the $\sim$ equivalence classes form a distributive lattice **Sim**$(\boldsymbol{C})$ with $\times$ as meet and $+$ as join. If $G$ and $H$ are two elements of the lattice **Sim**$(\boldsymbol{C})$ then $G \leqslant H$ in **Sim**$(\boldsymbol{C})$ if and only if for some element $A$ of $G$ and some element $B$ of $H$, $A \rightarrow B$. Clearly then for all elements $A$ in $G$ and $B$ in $H$, $A \rightarrow B$. We will sometimes also write $G \rightarrow H$. The one of the lattice **Sim**$(\boldsymbol{C})$ which we will denote by $\overset{\circ}{1}$ is the equivalence class containing the element *one* and the zero, 0, of this lattice is the class containing the empty $\mathscr{L}$-model. An $\mathscr{L}$-model $M$ is *multiplicative* in a category $\boldsymbol{C}$ of $\mathscr{L}$-models if whenever $A \not\rightarrow M$ and $B \not\rightarrow M$ for two $\mathscr{L}$-models $A$ and $B$ of $\boldsymbol{C}$ then $A \times B \not\rightarrow M$. Note that if $M \sim M_1$ then $M$ is multiplicative in $\boldsymbol{C}$ if and only if $M_1$ is multiplicative in $\boldsymbol{C}$. An element $A$ of the lattice **Sim**$(\boldsymbol{C})$ is called *multiplicative* if one of the $\mathscr{L}$-models and hence all of the $\mathscr{L}$-models in the equivalence class $A$ of $\mathscr{L}$-models are multiplicative.

It follows that an element $M$ of the distributive lattice **Sim**$(\boldsymbol{C})$ is multiplicative if whenever $A \nleqslant M$ and $B \nleqslant M$ then the meet $A \times B \nleqslant M$. Clearly, if $M$ is multiplicative then $M$ is meet-irreducible. (A lattice element $z$ is called *meet irreducible* if $x \wedge y = z$

implies $x = z$ or $y = z$). The converse, that is if $M$ is not multiplicative then $M$ is not meet irreducible, is also true. For assume that $A \not\leqslant M$, $B \not\leqslant M$ but $A \times B \leqslant M$. Then $M < A + M$ and $M < B + M$ but $(A + M) \times (B + M) = A \times B + A \times M + M \times B + M \times M \leqslant M$. We conclude tha an element of the distributive lattice $\mathbf{Sim}(\boldsymbol{C})$ is multiplicative if and only if it is meet irreducible and that an element $M$ of a *proper* $\mathscr{L}$-category $\boldsymbol{C}$ is multiplicative in $\boldsymbol{C}$ if and only if the similarity class of $\mathbf{Sim}(\boldsymbol{C})$ containing $M$ is meet irreducible in the distributive lattice $\mathbf{Sim}(\boldsymbol{C})$.

Exponentiation is a contravariant functor. That is, if $G$, $H$ and $K$ are $\mathscr{L}$-models with $G \to H$ then $K^H \to K^G$. In order to see this assume that $G \xrightarrow{\alpha} H$. Let $\alpha^*$ be the function from $K^H$ into $K^G$ such that $\alpha^*(f)(a) = f(\alpha(a))$. Clearly, $\alpha^*$ is a homomorphism from $K^H$ into $K^G$. It follows that if $G \sim H$ then $K^G \sim K^H$. If $G \xrightarrow{\alpha} K$, then $\alpha \in K^G$ has the property that for every relation symbol $R$ of $\mathscr{L}$, $R(\alpha, \alpha, \ldots, \alpha)$ and hence $K^G \sim$ *one*.

If $A$, $B$, $G$ are three $\mathscr{L}$-models with $A \xrightarrow{\alpha} B$ then $A^G \xrightarrow{\alpha^1} B^G$. The homomorphism $\alpha^1$ is defined as follows: if $f \in A^G$ and $x \in G$, then $(\alpha^1 f)(x) = \alpha(f(x))$. Hence if $G \sim G_1$ and $H \sim H_1$, then $G^H \sim G_1^{H_1}$. This implies that if $\boldsymbol{C}$ is a proper $\mathscr{L}$-category and $K$ and $A$ are two elements of $\mathbf{Sim}(\boldsymbol{C})$ then $K^A$ is well defined as that element of $\mathbf{Sim}(\boldsymbol{C})$ which contains the $\mathscr{L}$-model $K_1^{A_1}$ for some $K_1 \in K$ and some $A_1 \in A$. By $K^{\mathbf{Sim}(\boldsymbol{C})}$ we will denote the set of all elements $H$ of $\mathbf{Sim}(\boldsymbol{C})$ such that for some element $X$ of $\mathbf{Sim}(\boldsymbol{C})$, $H = K^X$. Note that $K^{\mathbf{Sim}(\boldsymbol{C})} = \mathbf{Sim}(K^{\boldsymbol{C}})$.

Our interest in this general setting stems from a vexing problem in graph theory originally due to Hedetniemi [8]. Given two graphs $A$ and $B$ both having chromatic number $n$, what is the chromatic number of $A \times B$? It is easy to see that the chromatic number of $A \times B$ is at most $n$. (The graph $A \times B$ can be colored with $n$ colors by choosing some $n$-coloring of $A$ and by coloring the pair $(a, b)$ with the color of the vertex $a$). Hedetniemi's conjecture says that the chromatic number of the graph $A \times B$ is $n$ if the graphs $A$ and $B$ have chromatic number $n$. It is known that the conjecture is true for $n \leqslant 4$, [4], and not true for infinite chromatic numbers, [7]. For all we know the following might be true: 'For any arbitrarily large finite number $n$ there exists two $n$ chromatic graphs $A$ and $B$ such that the chromatic number of the graph $A \times B$ is at most 9' [14, 15]. It is known that the conjecture is true for several special classes of finite graphs. See the references at the end of this paper [20, 3].

Observe that the chromatic number of a graph $A$ is $n$ if and only if $A \to K_n$ and $A \not\to K_{n-1}$. ($K_n$ denotes the complete graph on $n$ vertices and by misuse of notation also the $\sim$ equivalence class of graphs containing $K_n$.) If for two graphs $A$ and $B$, $A \sim B$ holds then the chromatic number of $A$ is equal to the chromatic number of $B$. Let $\mathbf{FG}$ be the proper category of finite graphs and $\mathbf{Sim}(\mathbf{FG})$ the distributive lattice of $\sim$ equivalence classes of finite graphs. We can then speak of the chromatic number of an element of $\mathbf{Sim}(\mathbf{FG})$ as the chromatic number of one of the graphs it contains. Hedetniemi's conjecture is equivalent to the statement that the complete graphs are multiplicative in the category of finite graphs or equivalently that the $\sim$ classes containing the complete graphs are multiplicative, that is meet irreducible, in the distributive lattice $\mathbf{Sim}(\mathbf{FG})$. We will prove (Lemma 4, see also [4]) that the complete graph $K_n$ is multiplicative iff for every graph $G$ with $G \nrightarrow K_n$, $K_n^G \to K_n$.

Different aspects of this categorial setting of Hedetniemi's conjecture are quite widely known as 'folklore' and have been independently discovered by several authors. See [12], for instance, for an example of this viewpoint. We know from personal communications that certainly Lovász and Walker knew about this and that this connection between homomorphisms of graphs and Hedetniemi's conjecture has led to investigations into the category of graphs under homomorphisms by J. Nešetřil and others. Welzl, for example [21], proved tha the lattice **Sim(FG)** is dense. Hell introduced the notion of a multiplicative element of a category. Starting with [6], and continuing in [9] and [19], multiplicative objects in various categories have been studied. As far as we know the connection between Hedetniemi's conjecture, multiplicativity and distributive lattices has not been commented on and the formalisms of the abstract setting for Hedetniemi's conjecture and its connection with the Boolean algebra $K^{\mathbf{Sim}(C)}$ do not appear in the literature.

As can be seen from the last paragraph, the 'categorial' perspective on Hedetniemi's conjecture has quite a long history. Let us also mention here that exponentiation was defined in [11] by Lovasz. G. Sabidussi, (D.J. Millers advisor), was also thinking along these lines, [10]. Lemma 4 and several of the statements in Lemma 1 can be viewed as folklore. They are implicit in [4] and [6] explicitly proved for one binary relation. We included them here for completeness. Our notion of exponentiation in lattices agrees with the one of Birkhoff in [1] and [2].

Given a relational language $\mathscr{L}$ and a proper $\mathscr{L}$-category $\boldsymbol{C}$ a desirable solution to what might be called the 'generalized Hedetniemi problem' would be to give a structural characterization of the multiplicative elements of $\boldsymbol{C}$ and a description of the subsets $K^{\mathbf{Sim}(C)}$ for elements $K$ of $\mathbf{Sim}(\boldsymbol{C})$. We will prove, Theorem 2, that for every proper $\mathscr{L}$-category $\boldsymbol{C}$ and element $K$ of $\mathbf{Sim}(\boldsymbol{C})$ the set $K^{\mathbf{Sim}(C)}$ contains at most two elements. In a certain sense the size of $K^{\mathbf{Sim}(C)}$ determines the 'degree' to which $K$ fails to be multiplicative. In the case of graphs for example, if the chromatic number of the graphs in $K_n^{\mathbf{FG}}$ is at most $m$, then it follows that if $A$ and $B$ are two at least $m$-chromatic graphs then the chromatic number of $A \times B$ is larger than $n$. The statement: For every $n$, $K_n^{\mathbf{Sim(FG)}}$ is finite, implies that there is a function $f(n)$ such that whenever two graphs $A$ and $B$ have chromatic number larger than $f(n)$ then the chromatic number of $A \times B$ is larger than $n$ (Theorem 5).

In the case when the language $\mathscr{L}$ contains only unary relations we will provide a complete description of the situation (Theorem 8 and Corollaries 1 and 2).

For the reader interested in the formal categorial aspects of the theory we mention the following without going into details. The operations of product, sum and exponentiation are categorial as the diagrms in Fig. 1 commute.

The object *one* is terminal and the empty $\mathscr{L}$-model is initial in the category $\boldsymbol{M}_{\mathscr{L}}$. The category $\boldsymbol{M}_{\mathscr{L}}$ has pushouts and pullbacks and hence $\boldsymbol{M}_{\mathscr{L}}$ is finitely complete and finitely cocomplete ([10 Theorem 23.7], [12]). Let $S$ be a two element $\mathscr{L}$-model with the property that for every positive $n$, relation symbol $R$ of $\mathscr{L}$ of arity $n$, and every

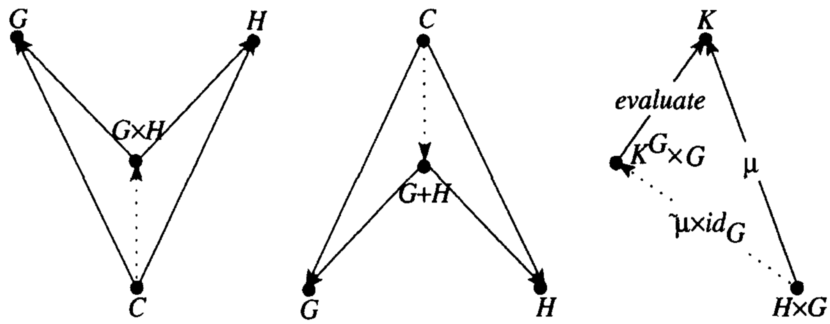


Fig. 1.

sequence $s_1, s_2, \ldots, s_n$ of elements of $S$, $R(s_1, s_2, \ldots, s_n)$ in $S$. It is not difficult to see that $S$ is a subobject classifier of $\boldsymbol{M}_{\mathscr{L}}$ and hence $\boldsymbol{M}_{\mathscr{L}}$ is a topos. Of course, $\boldsymbol{M}_{\mathscr{L}}$ induces a topos structure on certain subcategories, for example the category **FG** is a topos.

It is not difficult to see that exponentiation in the distributive lattice $\mathbf{Sim}(\boldsymbol{C})$ for a proper $\mathscr{L}$-category is *relative pseudo-complementation*, hence the lattice $\mathbf{Sim}(\boldsymbol{C})$ is a relatively pseudocomplemented lattice and because it contains a zero, 0, it is a *Heyting algebra* [17, 16].

## Section 1

In Lemma 1 we will list some formal properties of $+$, $\times$ and exponentiation of $\mathscr{L}$-models. Inspection shows that those properties are actually properties of the underlying distributive lattice. In order to establish Lemma 1 we could have taken a more abstract route by first proving that the distributive lattice $\mathbf{Sim}(\boldsymbol{C})$ is relatively pseudocomplemented for every proper $\mathscr{L}$-category $\boldsymbol{C}$ and then used the formal rules for relatively pseudocomplemented lattices derived in [17] and [16]. We did not choose to take this route because we wanted to gain an understanding of the concrete homomorphisms on the $\mathscr{L}$-models which give rise to the assertions of Lemma 1.

Theorem 2 establishes the fact that for every element $K \in \mathbf{Sim}(\boldsymbol{C})$, $K^{\mathbf{Sim}(\boldsymbol{C})}$ is, under the ordering in $\mathbf{Sim}(\boldsymbol{C})$, a Boolean lattice with the same meet as in $\mathbf{Sim}(\boldsymbol{C})$ and the join given by $A \oplus B = K^{(K^A \wedge K^B)}$. In this case we use a theorem of Grätzer [5, p. 58] on pseudocomplemented lattices. We think that not much could be gained from a more direct but longer argument, even though it would not be difficult to produce:

**Lemma 1.** *Let $G$, $H$ and $K$ be three elements of a proper $\mathscr{L}$-category $\boldsymbol{C}$ for some relational language $\mathscr{L}$, then*

$$G + H \sim H + G, \qquad G \times H \sim H \times G, \tag{1}$$

$$G \times (H + K) \sim G \times H + G \times K, \tag{2}$$

$$K^{G+H} \sim K^G \times K^H, \tag{3}$$

$$(K^G)^H \sim K^{G \times H}, \tag{4}$$

$$one \times G \sim G, \qquad K^{one} \sim K, \tag{5}$$

$$G \times H \to K \Rightarrow G \to K^H, \tag{6}$$

$$K^G \times G \to K, \quad K \to K^G, \tag{7}$$

$$K \to G \Rightarrow K^G \times G \sim K, \tag{8}$$

$$G \to H \Rightarrow K^{(K^G)} \to K^{(K^H)}, \quad G \to K^{(K^G)}, \tag{9}$$

$$G \in K^C \Leftrightarrow K^{(K^G)} \sim G, \tag{10}$$

$$one \to K^G \Leftrightarrow G \to K. \tag{11}$$

**Proof.** Assertations (1) and (2) follow by straightforward verification. In order to see that $K^{G+H} \sim K^G \times K^H$ associate with each function $f$ of $K^{G+H}$ the pair of functions $f|V(G)$ and $f|V(H)$ of $K^G$ and $K^H$, respectively. This association is a bijection between the elements of $K^{G+H}$ and $K^G \times K^H$. For $R$ an $n$-ary relation symbol, $f_1, f_2, \ldots, f_n$ functions of $K^{G+H}$, $R(f_1, f_2, \ldots, f_n)$ iff for all sequences $x_1, x_2, \ldots, x_n$ of elements of $G$ such that $R(x_1, x_2, \ldots, x_n)$, $R(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))$ in $K$ iff for all sequences $x_1, x_2, \ldots, x_n$ of elements of $G$ such that $R(x_1, x_2, \ldots, x_n)$ and for all sequences $x_1, x_2, \ldots, x_n$ of elements of $H$ such that $R(x_1, x_2, \ldots, x_n)$, $R(f_1(x_1), f_2(x_2), \ldots, f_n(x_n))$ in $K$ iff $R(f_1|V(G), f_2|V(G), \ldots, f_n|V(G))$ and $R(f_1|V(H), f_2|V(H), \ldots, f_n|V(H))$.

In order to establish $(K^G)^H \sim K^{G \times H}$ associate with each function $f$ of $(K^G)^H$ and element $b$ of $H$ the function $f_b = f(b)$ of $K^G$ and the function $f^*$ of $K^{G \times H}$ given by $f^*(a,b) = f_b(a)$. If $g$ is function in $K^{G \times H}$ we associate with $g$ and every $b$ of $H$ the function $g_b$ of $K^G$ given by $g_b(a) = g(a,b)$ and then the function $g^\circ$ of $(K^G)^H$ given by $g^\circ(b)(a) = g_b(a)$. Observe that * and $^\circ$ are two-sided inverses of each other and hence bijections. For $R$ an $n$-ary relation symbol, and $f_1, f_2, \ldots, f_n$ a sequence of elements of $(K^G)^H$, $R(f_1, f_2, \ldots, f_n)$ if for all sequences $b_1, b_2, \ldots, b_n$ of elements of $H$ such that $R(b_1, b_2, \ldots, b_n)$, $R(f_1(b_1), f_2(b_2), \ldots, f_n(b_n))$. Now $R(f_1(b_1), f_2(b_2), \ldots, f_n(b_n))$ iff for all sequences $a_1, a_2, \ldots, a_n$ of elements of $G$ such that $R(a_1, a_2, \ldots, a_n)$, $R(f_1(b_1)(a_1), f_2(b_2)(a_2), \ldots, f_n(b_n)(a_n))$ iff $R(f_1^*(a_1, b_1), f_2^*(a_2, b_2), \ldots, f_n^*(a_n, b_n))$.

The first part of (5) is easy to check and for $K^{one} \sim K$ let *one* be that $\mathscr{L}$ model consisting of the single element $a$ and for every relation $R$ of $\mathscr{L}$, $R(a, a, \ldots, a)$. The association between $f$ of $K^{one}$ and $f(a)$ of $K$ is an isomorphism between $K^{one}$ and $K$.

To see assertion (6) assume that $G \times H \xrightarrow{\alpha} K$. Then we associate with $g \in G$ the function $g^*: H \to K$ given by $g^*(h) = \alpha(g,h) = \alpha(g,h)$. Straightforward verification shows that * is a homomorphism from $G$ to $K^H$.

If $(f,a) \in V(K^G \times G)$, then the *evaluator* which maps $(f,a)$ to $f(a)$ is a homomorphism. For each element $x$ of $K$, the map which associates with $x$ the constant map from $G$ to $x$ is a homomorphism from $K$ to $K^G$. Assertion (8) follows immediately from assertion (7).

We already observed that the function which maps $G$ to $K^G$ is a contravariant functor. Hence if $G \to H$ then $K^H \to K^G$ and $K^{(K^G)} \to K^{(K^H)}$. Let us denote by $* : G \to K^{(K^G)}$ the function such that for $a \in V(G)$ and $f \in V(K^G)$, $a^*(f) = f(a)$. Observe that $*$ is a homomorphism from $G$ into $K^{(K^G)}$ and hence $G \to K^{(K^G)}$. Applying this rule to the graph $K^G$ we get $K^G \to K^{(K^{(K^G)})}$. On the other hand, using the fact that exponentiation is a contravariant functor we obtain from $G \to K^{(K^G)}$, $K^{(K^{(K^G)})} \to K^G$. Apply the latter to $G = K^H \in K^C$. Thus $K^{(K^{(K^H)})} \to K^H$, that is $K^{(K^G)} \to G$. We obtain then that $G \sim K^{(K^G)}$ whenever $G \in K^C$.

For (11) note that if $one \xrightarrow{\alpha} K^G$ then the function $\alpha(one)$ from the elements of $G$ to the elements of $K$ is a homomorphism from $G$ to $K$. On the other hand, if $f$ is a homomorphism from $G$ to $K$ then there is a homomorphism from $one$ to $K^G$ whose image is $f$. $\square$

We have mentioned earlier that if $\boldsymbol{C}$ is a proper $\mathscr{L}$-category then $\mathbf{Sim}(\boldsymbol{C})$ is a distributive lattice. Also, $\mathbf{Sim}(A \times B) = \mathbf{Sim}(A) \wedge \mathbf{Sim}(B)$, $\mathbf{Sim}(A + B) = \mathbf{Sim}(A) + \mathbf{Sim}(B)$, $\mathbf{Sim}(A^B) = \mathbf{Sim}(A)^{\mathbf{Sim}(B)}$ and $\mathbf{Sim}(K^C) = (\mathbf{Sim}(K))^{\mathbf{Sim}(C)}$.

Let $D$ be a distributive lattice. Suppose that for every two elements $x$ and $y$ of $D$ the element $\sup\{z\colon y \wedge z \leqslant x\}$ exists and has the property that $(\sup\{z\colon y \wedge z \leqslant x\}) \wedge y \leqslant x$. We then say that $D$ is a distributive lattice with *exponentiation* and put $x^y = \sup\{z\colon y \wedge z \leqslant x\}$ and $x^D = \{x^y\colon y \in D\}$.

**Lemma 2.** *Let $D$ be a distributive lattice with exponentiation and with $x, u, v \in D$, $\mathring{1}$ the one of $D$, $y$ and $z$ in $x^D$. Then*

$$z \wedge y \leqslant x \Rightarrow z \leqslant x^y, \tag{12}$$

$$x \leqslant x^u, \tag{13}$$

$$y \wedge x^y = x, \tag{14}$$

$$x^{\mathring{1}} = x, \qquad x^x = \mathring{1}, \tag{15}$$

$$y \leqslant z \Rightarrow x^z \leqslant x^y, \tag{16}$$

$$x^y \wedge x^z = x^{y \vee z}, \tag{17}$$

$$y \leqslant x^{(x^y)}, \tag{18}$$

$$u = x^{(x^u)} \Leftrightarrow u \in x^D, \tag{19}$$

$$x^{u \wedge v} \leqslant (x^u)^v. \tag{20}$$

**Proof.** It follows from the definition that $z \wedge y \leqslant x \Rightarrow z \leqslant x^y$.

If $v \wedge u \leqslant x$ then $(v \vee x) \wedge u = (v \wedge u) \vee (x \wedge u) \leqslant x$. It follows that $x \leqslant x^u$ and that for every element $y \in x^D$, $x \leqslant y$.

We deduce from the definition of $x^y$ that $y \wedge x^y \leqslant x$. And hence because $x \leqslant y$ and $x \leqslant x^y$, $y \wedge x^y = x$. It is also immediate from the definition of $x^y$ that $x^{\mathring{1}} = x, x^x = \mathring{1}$ and $y \leqslant z \Rightarrow x^z \leqslant x^y$.

It follows from assertion (16) that $x^y \geqslant x^{y \vee z}$ and $x^z \geqslant x^{y \vee z}$, hence $x^y \wedge x^z \geqslant x^{y \vee z}$. In order to prove that $x^y \wedge x^z \leqslant x^{y \vee z}$ it is sufficient to prove that $(x^y \wedge x^z) \wedge (y \vee z) \leqslant x$. We calcuate:

$$x^y \wedge x^z \wedge (y \vee z) = x^y \wedge ((x^z \wedge y) \vee x) = x^y \wedge (x^z \wedge y) = (x^y \wedge y) \wedge z^z = x \wedge x^z = x$$

Assertion (18) follows directly from the definition of exponentiation. Obviously if $y = x^{(x^y)}$ then $y \in x^D$. Also from the definition of exponentiation if $u \leqslant v$ then $x^u \geqslant x^v$. Hence if $y \in x^D$, that is if there is a $z \in D$ such that $y = x^z$, then $z \leqslant x^{x^z} := p(z)$ and

$$y = x^z \geqslant x^{p(z)} = x^{x^y} .$$

Because $y \leqslant x^{x^y}$, it follows that $y = x^{x^y}$.

In order to prove assertion (20) observe first that $x \geqslant x^{u \wedge v} \wedge (u \wedge v) = (x^{u \wedge v \wedge v}) \wedge u$. This implies by the definition of $x^u$ that $x^{u \wedge v} \wedge v \leqslant x^u$, which in turn implies that $x^{u \wedge v} \leqslant (x^u)^v$. □

**Lemma 3.** *For every element $x \in D$, $x$ is the smallest element in $x^D$, $\mathring{1}$ is the largest element of $x^D$, the set $x^D$ is closed under $\wedge$ and hence is a meet semilattice.*

**Proof.** Immediate from Lemma 2. □

Following [5] an element $a^*$ is a *pseudocomplement* of $a \in L$ if $a \wedge a^* = 0$ and $a \wedge x = 0$ implies that $x \leqslant a^*$. A *pseudocomplemented lattice* is one in which every element has a pseudocomplement. Note that Lemmas 2 and 3 imply that if $D$ is a distributive lattice with exponentiation then for every element $x \in D$, $x^D$ is a meet semilattice with smallest element $x$, largest element $\mathring{1}$ and for every element $y \in x^D$ the element $x^y$ is a pseudocomplement of the element $y$.

**Theorem 1** (Grätzer [5]). *Let $L$ be a pseudocomplemented meet-semilattice, $S(L) = \{a^*; a \in L\}$. Then the partial ordering of $L$ partially orders $S(L)$ and makes $S(L)$ into a Boolean lattice. For $a, b \in S(L)$ we have $a \wedge b \in S(L)$, and the join $\oplus$ in $S(L)$ is described by*

$$a \oplus b = (a^* \wedge b^*)^* .$$

Hence the following theorem follows.

**Theorem 2.** *If $D$ is a distributive lattice with exponentiation, largest element $\mathring{1}$, and if $x \in D$, then the partial ordering of $D$ partially orders $x^D$ and makes $x^D$ into a Boolean lattice. For $z, y \in x^D$ we have $z \wedge y$ in $x^D$, and the join $\oplus$ in $x^D$ is described by*

$$z \oplus y = x^{x^z \wedge x^y} .$$

Let $\boldsymbol{C}$ be a proper $\mathscr{L}$-category then we conclude from assertion (6) that $\mathbf{Sim}(\boldsymbol{C})$ is a distributive lattice with exponentiation. Assertion (3) implies that $L = \mathring{K}^{\mathbf{Sim}(C)}$ is a meet semilattice. Because $K^K \sim one$ and $K^{one} \sim K$ the elements $\mathbf{Sim}(one) = \mathring{1}$ and $\mathbf{Sim}(K)$ are elements of $L$ and the largest and smallest elememts of $L$, respectively. Hence the next theorem follows directly from Theorem 2.

**Theorem 3.** *Let $C$ be a proper $\mathscr{L}$-category and $K$ an element of* $\mathbf{Sim}(\boldsymbol{C})$. *Then $\mathring{K}^{\mathbf{Sim}(C)}$ is a Boolean lattice with smallest element. $K$, largest element $\mathring{1}$, for each $A \in K^{\mathbf{Sim}(C)}$ the complement of $A$ in $K^{\mathbf{Sim}(C)}$ is $K^A$, the meet in $K^{\mathbf{Sim}(C)}$ is $\times$ and the join in $K^{\mathbf{Sim}(C)}$ is given by $A \oplus B = K^{(K^A \wedge K^B)}$.*

Note that under the conditions of Theorem 3, $K^{(K^A \wedge K^B)} = K^{K^{A \vee B}}$ holds. This follows from Lemma 1 (3).

**Lemma 4.** *Let $C$ be a proper $\mathscr{L}$-category and $K$, $G$ be two elements of $\boldsymbol{C}$ with $G \nrightarrow K$. Then*

$$K^G \to K \Leftrightarrow \forall (H \nrightarrow K)(G \times H \nrightarrow K)\,.$$

**Proof.** Assume that $K^G \to K$. If for some $H \in \boldsymbol{C}$ with $H \nrightarrow K$, $G \times H \to K$ then it follows from assertions (6), that $G \to K^H$. Hence by assertion (9) and the fact that exponentiation is contravariant,

$$H \to K^{K^H} \to K^G \to K\,,$$

in contradiction to the assumption that $H \nrightarrow K$.

Assume that for all $H \in \boldsymbol{C}$ with $H \nrightarrow K$, $G \times H \nrightarrow K$. Then in particular if $K^G \nrightarrow K$, $G \times K^G \nrightarrow K$ in contradiction to assertion (7). $\square$

We call an element $G \in \boldsymbol{C}$ with $G \nrightarrow K$ and the property that for all $H \in \boldsymbol{C}$ with $H \nrightarrow K$, $G \times H \nrightarrow K$, *stable* with respect to $K$. (This notion of stable element is a generalization of the notion of 'nice' graph to general relational structures. The latter was studied in [20, 3], and implicitly also in [21].) Lemma 2 can then be restated to: The structure $G$ is stable with respect to $K$ if and only if $K^G \to K$. Note also that $K$ is multiplicative in $\boldsymbol{C}$ if and only if every structure $G \in \boldsymbol{C}$ with $G \nrightarrow K$ is stable with respect to $K$. Hence the structure $K \in \boldsymbol{C}$ is multiplicative in $\boldsymbol{C}$ if and only if for every structure $G \in \boldsymbol{C}$ either $G \to K$ or if $G \nrightarrow K$ then $K^G \to K$, that is, $G$ is stable with respect to $K$. This leads us to the next theorem.

**Theorem 4.** *Let $\boldsymbol{C}$ be a proper $\mathscr{L}$-category and $K$ an element of* $\mathbf{Sim}(\boldsymbol{C})$. *Then $K$ is multiplicative in* $\mathbf{Sim}(\boldsymbol{C})$ *if and only if $K^{\mathbf{Sim}(C)}$ consists of at most two elements.*

**Proof.** Assume that $K$ is multiplicative in $\mathbf{Sim}(\boldsymbol{C})$ then the above discussion has shown that for every $G \in \mathbf{Sim}(\boldsymbol{C})$ with $G \nrightarrow K$, $K^G \to I$ and hence by asseertion (7),

$K^G = K$. If $G \to K$ then according to assertion (11), $K^G = \mathring{1}$. Hence $K^{\mathbf{Sim}(C)}$ contains at most two elements.

Assume that $K^{\mathbf{Sim}(C)}$ contains at most two elements. Then by Theorem 3 the elements of $K^{\mathbf{Sim}(C)}$ are the largest element $\mathring{1}$ and the element $K$. If for $G \in \mathbf{Sim}(C)$, $K^G = \mathring{1}$ then by assertion (11), $G \to K$. This means that if $G \not\to K$, then $K^G = K$. Hence every element $G$ with $G \not\to K$ is stable and we conclude using the discussion above this theorem that $K$ is multiplicative. □

Even in the case when $C$ is a proper $\mathscr{L}$-structure which consists of finite structures only we cannot decide whether $K^{\mathbf{Sim}(C)}$ is always finite or not. This question is particularly interesting in light of the following. Let for a positive integer $m$, $g(m)$ be the smallest number such that if two graphs $A$ and $B$ have chromatic number larger than or equal to $m$, then the chromatic number of $A \times B$ is at least $g(m)$. Hedetniemi's conjecture says, (cf. [14]), that $g(m) = m$. Does $g(m)$ tend to infinity with $m$? The following is known [15]: $g(m)$ is either less than or equal to 9 or tends to infinity with $m$. Clearly if for every positive integer $n$ there is a number $f(n)$ such that whenever two graphs $A$ and $B$ have chromatic number larger than $f(n)$ the chromatic number of $A \times B$ is larger than $n$ then the function $g(m)$ tends to infinity with $m$. Remember that **FG** is the category of all finite graphs and $K_n$ is the complete graph or the equivalence class of all graphs equivalent to the complete graph.

**Theorem 5.** *If for every number $n$, $K_n^{\mathbf{Sim}(\mathbf{FG})}$ is finite then there is a function $f(n)$ such that whenever two graphs $A$ and $B$ have chromatic number larger than $f(n)$ the chromatic number of $A \times B$ is larger than $n$.*

**Proof.** Assume that $K_n^{\mathbf{Sim}(\mathbf{FG})}$ is finite. Then there is a number, say $f(n)$, such that for all elements $G$ of $K_n^{\mathbf{Sim}(\mathbf{FG})}$, the chromatic number of $G$ is at most $f(n)$. This implies that for every graph $A$ the chromatic number of $K_n^A$ is at most $f(n)$. If for some graph $B$ the chromatic number of $A \times B$ is less than or equal to $n$, that is if $A \times B \to K_n$, then by assertion (6) it follows that $B \to K^A$ and hence that the chromatic number of $B$ is at most $f(n)$. Of course the chromatic number of $A$ is then also at most $f(n)$. Hence if the chromatic number of two graphs is larger than $f(n)$ then the chromatic number of their product is larger than $n$. □

## Section 2

In the previous section we have seen that given a proper $\mathscr{L}$-category $C$ and two $\mathscr{L}$-structures $K$ and $G$ in $C$, then $G$ is stable with respect to $K$ if and only if $K^G \to K$. The $\mathscr{L}$-structure $K$ is multiplicative in $C$ if and only if every $\mathscr{L}$-structure $G \in C$ with $G \not\to K$ is stable with respect to $K$. The relevant information is contained in the distributive lattice $\mathbf{Sim}(C)$. The distributive lattice $\mathbf{Sim}(C)$ is a distributive lattice with exponentiation and the $\mathscr{L}$-structure $K$ is multiplicative in $C$ if and only if the Boolean

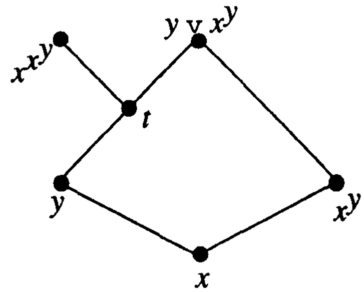


Fig. 2.

lattice $\mathbf{Sim}(K)^{\mathbf{Sim}(C)}$ contains at most two elements. The $\mathscr{L}$-structure $G$ is stable with respect to $K$ if and only if $K^G \sim K$ and $G \to K$ if and only if $\mathbf{Sim}(K^G) = \mathring{1}$. Hence if $G \not\to K$ then $G$ is not stable with respect to $K$ if and only if the equivalence class $\mathbf{Sim}(K^G)$ is not the class $\mathbf{Sim}(K)$ or the class $\mathring{1}$. This implies that in a certain sense the size of the Boolean lattice $\mathbf{Sim}(K)^C$ is an indication of the extent to which $K$ is multiplicative. This leads us to investigate the Boolean lattice $x^D$ for a distributive lattice $D$ with exponentiation and element $x \in D$. Remember that the $\mathscr{L}$-structure $K$ of the proper $\mathscr{L}$-category $\boldsymbol{C}$ is multiplicative if and only if $\mathbf{Sim}(K)$ is meet irreducible in the distributive lattice $\mathbf{Sim}(\boldsymbol{C})$.

Let $D$ be a distributive lattice with exponentiation. Then $M(D)$ denotes the set of meet irreducible elements of $D$. If $L$ is a lattice then $C(L)$ denotes the set of coatoms of the lattice $L$.

**Theorem 6.** *Let $x \in D$, a distributive lattice admitting exponentiation and let $y > x$ in $D$. Then*

$$y \in C(x^D) \Leftrightarrow x^y > x \quad \textit{and} \quad y \in M(D)\,.$$

**Proof.** Assume that $x^y > x$ and $y \in M(D)$. We show first that $y \in x^D$ and suppose for a contradiction that $y \notin x^D$. As $y \wedge x^y = x$ and both $y > x$ and $x^y > x$, we know that $y$ and $x^y$ are incomparable in $D$. Then we obtain from assertions (8) and (19), that $y < x^{x^y}$ and from assertion (14), with $x^y$ in place of $y$, that $x^y \wedge x^{x^y} = x$. Note that $x^{x^y} \not\geqslant x^y$ because $x^{x^y} \wedge x^y = x < x^y$. This in turn implies that $y \vee x^y \not\leqslant x^{x^y}$. If $x^{x^y} \leqslant y \vee x^y$ then $x^{x^y} < y \vee x^y$ and the elements $x, y, x^{x^y}, x^y, y \vee x^y$ would form a 5-element nonmodular lattice. Thus, we have the situation as depicted in Fig. 2, with $x^{x^y}$ and $y \vee x^y$ incomparable.

Since $y \in M(D)$,

$$t = x^{x^y} \wedge (y \vee x^y) = (x^{x^y} \wedge y) \vee (x^{x^y} \wedge x^y) = y \vee x > y\,,$$

which is not possible because $y > x$.

Thus $y \in x^D$. As $y \in M(D)$, $y \in M(x^D)$, (remember that the meet operations coincide). But in any Boolean algebra relative complementation shows that the existing meet irreducibles must be coatoms.

Assume that $y \in C(x^D)$. We first prove that $y \in M(D)$. If not then there are elements $u > y$ and $v > y$ such that $y = u \wedge v$. Then

$$x \leqslant x^u \leqslant x^y \quad \text{and} \quad x \leqslant x^v \leqslant x^y .$$

If $x^u = x^y$ then because $y \in x^D$,

$$y = x^{x^y} = x^{x^u} \geqslant u \geqslant y ,$$

which is not possible. Thus $x^u < x^y$ and $x^v < x^y$. Using the fact that $y$ is a coatom of $x^D$ we get that $x^u = x = x^v$. This means,

$$x^y = x^{u \wedge v} \leqslant (x^u)^v = x^v = x ,$$

which again is impossible. The inequality above follows from assertion (20). Thus, we have $y \in M(D)$.

It is immediate that $x^y > x$, as it is an atom of $x^D$. □

**Theorem 7.** *Let $x \in D$, a distributive lattice admitting exponentiation. If $z > y > x$ and $y \in M(D)$ then $x^z = x$ and hence $z$ is stable with respect to $x$.*

**Proof.** If not, $x^z > x$ and $z \wedge x^z = x$. If $x^y = x$ then $x^z \leqslant x^y = x$, an immediate contradiction. If $x^y > x$ then by Theorem 6, $y \in C(x^D)$ and so $y < z \leqslant x^{x^z} \in x^D$ means $x^{x^z} = \mathring{1}$ and hence $x^z = x$, which is again a contradiction. (Note that $\mathring{1} \in x^D$ and if $x^{x^z} > y$ a coatom then $x^{x^z} = \mathring{1}$.) □

## Section 3

We consider now the special case when $\mathscr{L}$ is a unary language. If the cardinality of $\mathscr{L}$ is $\kappa$ we may assume without loss that $\mathscr{L} = \kappa$. In this case $\mathscr{L}$-models consist of sets $S$ so that for each $x \in S$ there is some $C(x) \subseteq \kappa$, a set of colours. So, an object ($\mathscr{L}$-model) is actually a (multi)-set of subsets of $\kappa$. If $x, y \in S$ and $C(x) \subseteq C(y)$ then there is a homomorphism $\varphi\colon S \to S - \{x\}$, where

$$\varphi(t) = \begin{cases} t & \text{if } t \neq x, \\ y & \text{otherwise.} \end{cases}$$

Evidently each equivalence class of objects can be represented by an order ideal in $\underline{2}^\kappa$ where the ordering in the distributive lattice is given by $\subseteq$, meet by $\cap$ and the join by $\cup$. (A subset $S$ of $\underline{2}^\kappa$ is an *order ideal* if whenever $x \in S$ and $x \geqslant y$, then $y \in S$.) From this it follows that the collection $\mathscr{D}_\kappa$ of similarity classes so ordered satisfies

$$\mathscr{D}_\kappa = \underline{2}^{\underline{2}^\kappa} ,$$

the free distributive lattice on $\kappa$ generators. (See [5, p. 80 ff.].) Anyway, it follows from our general discussion about $\mathscr{L}$-structures that $\mathscr{D}_\kappa$ is a distributive lattice which admits exponentiation. Given two elements $A$ and $B$ in $\mathscr{D}_\kappa$ it is actually not difficult to see that $A^B$ is the order ideal of all those subsets $x \in \underline{2}^\kappa$ such that if $x \notin A$ then $x \notin B$. In other words $A^B = \{x \in \underline{2}^\kappa\colon ((y \leqslant x) \wedge (y \in B)) \Rightarrow y \in A\} \cap B = A \Leftrightarrow B \subseteq A$. It is immediate that $\{x \in \underline{2}^\kappa\colon (y \leqslant x \vee y \in B) \Rightarrow y \in A)$ is an order ideal, that $(\{x \in \underline{2}^\kappa\colon (y \leqslant x \vee y \in B) \Rightarrow y \in A\}) \cap B = A$ and if $Z \cap B \leqslant A$, then $Z \subseteq \{x \in \underline{2}^\kappa\colon (y \leqslant x \vee y \in B) \Rightarrow y \in A\}$. Remember that $M(\mathscr{D}_\kappa)$ is the set of meet irreducible elements of $\mathscr{D}_k$.

A subset $S$ of $\underline{2}^\kappa$ is an *order filter* if for every element $x \in S$ and every $y \geqslant x$ also $y$ is an element of $S$. An order filter $S$ of $\underline{2}^\kappa$ is a *filter* of $\underline{2}^\kappa$ if for any two elements $x$ and $y$ of $S$ also $x \cap y$ is an element of $S$. If $S$ is a subset of $\underline{2}^\kappa$ then the complement of $S$ is the set $\bar{S} = \underline{2}^\kappa - S$. Observe that $S$ is an order ideal of $\underline{2}^\kappa$ if and only if $\bar{S}$ is an order filter of $\underline{2}^\kappa$.

**Theorem 8.** *The element $S$ of $\mathscr{D}_\kappa$ is in $M(\mathscr{D}_\kappa)$ if and only if $\bar{S}$ is a filter of $\underline{2}^\kappa$ .*

**Proof.** Assume for a contradiction that $\bar{S}$ is a filter and that there are two ideals $A$ and $B$ with $A \nsubseteq B$ and $B \nsubseteq A$ but $A \cap B = S$ and hence $\bar{A} \cup \bar{B} = \bar{S}$. There is then an element $a \in \bar{A} - \bar{B}$ and an element $b \in \bar{B} - \bar{A}$. If $a \cap b \in \bar{A}$ then $b \in \bar{A}$ and if $a \cap b \in \bar{B}$ then $a \in \bar{B}$. But $a \cap b$ would then not be an element of $\bar{A} \cup \bar{B} = \bar{S}$ in contradiction to $\bar{S}$ being a filter.

If $S$ is meet irreducible and $\bar{S}$ is not a filter then there are elements $a$ and $b$ in $\bar{S}$ such that $a \cap b \notin \bar{S}$. Let $A = \{x\colon \exists y \in \bar{S} \text{ with } y \leqslant a \text{ and } x \geqslant y\}$ and $B = \bar{S} - \{y\colon y \leqslant a\}$. Clearly $A$ and $B$ are order filters. But $b \notin A$ because if there is $y \in \bar{S}$ with $y \leqslant a$ and $b \geqslant y$ then $y \leqslant a \cap b$ and hence $a \cap b \in \bar{S}$. Also $a \notin B$ and clearly $\bar{S} = A \cup B$. Hence $S = \bar{A} \cap \bar{B}$ in contradiction to our assumption that $S$ is meet irreducible. $\square$

Note that if $\kappa$ is finite then every filter of $\underline{2}^\kappa$ is a principal filter. We obtain then the following corollary.

**Corollary 1.** *If $\mathscr{L}$ is a finite unary language then the model $S$ of $\mathscr{L}$ is multiplicative if and only if for every element $x$ of $S$ there is a unary relation $R \in \mathscr{L}$ and an element $y \in S$ such that $C(x) \subseteq C(y) = \mathscr{L} - R$.*

We assume now that $\mathscr{L}$ is a finite unary language and contains $r$ elements. We represent the similarity classes of $\mathscr{L}$-structures by the set $\mathscr{D}_r$ of order ideals in $\overline{2^r}$ . Let $K$ be an element of $\mathscr{D}_r$. Because $\overline{2^r}$ is finite also $K^{\mathscr{D}_r}$ is finite and hence atomic. All of the coatoms of $\mathscr{D}_r$ are multiplicative elements of $\mathscr{D}_r$ by Theorem 6 and no element of $\mathscr{D}_r$ which is strictly larger than some multiplicative element which in turn is larger than or equal to $K$ is an element of $\mathscr{D}_r$ by Theorem 7. Hence we obtain the following corollary.

**Corollary 2.** *If $\mathscr{L}$ is a finite unary language containing r elements and K is an element of* $\mathbf{FU}_{\mathscr{L}}$ *then the coatoms of the Boolean lattice* $K^{\mathbf{FU}_{\mathscr{L}}}$ *are the minimal multiplicative elements of* $\mathbf{FU}_{\mathscr{L}}$ *which are larger than or equal to K. This completely determines the Boolean lattice* $K^{\mathbf{FU}_{\mathscr{L}}}$ *because the meet in the Boolean lattice* $K^{\mathbf{FU}_{\mathscr{L}}}$ *is the same as the meet in the distributive lattice* $\mathscr{D}_r$ *which corresponds to the product of* $\mathscr{L}$*-structures.*

## Section 4: A problem

Remember that two graphs $G$ and $H$ are similar if there is a homomorphism from $G$ to $H$ and a homomorphism from $H$ to $G$. If $K$ is a graph then $\mathbf{Sim}(K)$ denotes the similarity class of the graph $K$. The similarity classes of graphs form a distributive lattice where the join corresponds to the disjoint union, the meet to the product and the order relation to homomorphic embedding. The distributive lattice of similarity classes of finite graphs is denoted by $\mathbf{Sim(FG)}$. We have seen that Hedetniemi's conjecture is equivalent to the statement that the Boolean lattice $\mathbf{Sim}(K_n)^{\mathbf{Sim(FG)}}$ contains only two elements for every complete graph $K_n$ and that if the Boolean lattice $\mathbf{Sim}(K_n)^{\mathbf{Sim(FG)}}$ is finite then there is a number $f(n)$ such that if the two graphs $G$ and $H$ have chromatic number larger than $f(n)$ then the chromatic number of $G \times H$ is larger than $n$.

**Problem.** For which graphs $K$ is the Boolean lattice $\mathbf{Sim}(K)^{\mathbf{Sim(FG)}}$ finite?

Note that if the Boolean lattice $\mathbf{Sim}(K_n)^{\mathbf{Sim(FG)}}$ is infinite then there are infinitely many graphs which are pairwise not homomorphic to each other their infinite product is homomorphic to $K_n$ but every finite subproduct has chromatic number strictly larger than $n$.

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